ALGEBRA HW 11

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1

Let V, W, Y be finite dimensional vector spaces over K.

(a): Show that there are natural isomorphisms $(V \otimes W)^* = V^* \otimes W^* = \operatorname{Hom}(V, W^*) = \operatorname{Hom}(W, V^*)$.

Proof. $(V \otimes W)^* = V^* \otimes W^*$: Define the map $\phi : V^* \times W^* \to (V \otimes W)^*$ by

$$(f, g) \mapsto (v \otimes w \mapsto f(v)g(w)).$$

Then, if $f_1, f_2 \in V^*$, $g_1, g_2 \in W^*$ and $a, b \in K$, then

$$\begin{aligned}
\phi(af_1 + bf_2, g_1) &= (v \otimes w \mapsto (af_1 + bf_2)(v)g_1(w)) \\
&= v \otimes w \mapsto (af_1(v) + bf_2(v))g_1(w) \\
&= v \otimes w \mapsto (af_1(v)g_1(w) + bf_2(v)g_1(w)) \\
&= (v \otimes w \mapsto af_1(v)g_1(w)) + (v \otimes w \mapsto bf_2(v)g_1(w)) \\
&= a(v \otimes w \mapsto f_1(v)g_1(w)) + b(v \otimes w \mapsto f_2(v)g_1(w)) \\
&= a\phi(f_1, g_1) + b\phi(f_2, g_1)
\end{aligned}$$

and

$$\begin{aligned}
\phi(f_1, ag_1 + bg_2) &= v \otimes w \mapsto f_1(v)(ag_1 + bg_2)(w) \\
&= v \otimes w \mapsto f_1(v)(ag_1(w) + bg_2(w)) \\
&= v \otimes w \mapsto (af_1(v)g_1(w) + bf_1(v)g_2(w)) \\
&= (v \otimes w \mapsto af_1(v)g_1(w)) + (v \otimes w \mapsto bf_1(v)g_2(w)) \\
&= a(v \otimes w \mapsto f_1(v)g_1(w)) + b(v \otimes w \mapsto f_1(v)g_2(w)) \\
&= a\phi(f_1, g_1) + b\phi(f_1, g_2),
\end{aligned}$$

so ϕ is bilinear. Therefore, by the universal property of the tensor product, ϕ induces a unique linear map $\Phi : V^* \otimes W^* \to (V \otimes W)^*$. To show that Φ is an isomorphism, then, it suffices merely to show that Φ is injective, since $V^* \otimes W^*$ and $(V \otimes W)^*$ have the same dimension (namely $\dim V \cdot \dim W$). Now, suppose $f \in V^*$ and $g \in W^*$ such that $\Phi(f \otimes g) = 0$. Then

$$0 = \Phi(f \otimes g) = (v \otimes w \mapsto f(v)g(w)),$$

Suppose $f \neq 0$. Then $f(v_0) \neq 0$ for some $v_0 \in V$. Then, since

$$0 = (v_0 \otimes w \mapsto f(v_0)g(w))$$

for all w, we see that $g(w) = 0$ for all $w \in W$. Alternatively, if $g \neq 0$ then $f = 0$. Therefore, if $f \otimes g \in \ker \Phi$, then

$$f \otimes g = \left\{ \begin{array}{c} f \otimes 0 \\ 0 \otimes g \end{array} \right\} = 0$$

so $\ker \Phi = 0$ and so Φ is injective and, therefore, an isomorphism.

$V^* \otimes W^* = \operatorname{Hom}(V, W^*)$: Define the map $\phi : V^* \times W^* \to \operatorname{Hom}(V, W^*)$ by

$$\phi : (x, y) \mapsto (v \mapsto x(v)y.$$

If $x_1, x_2 \in V^*$, $y_1, y_2 \in W^*$ and $a, b \in K$, then

$$\begin{aligned} \phi(ax_1 + bx_2, y_1) &= v \mapsto (ax_1 + bx_2)(v)y_1 \\ &= v \mapsto (ax_1(v) + bx_2(v))y_1 \\ &= v \mapsto ax_1(v)y_1 + bx_2(v)y_1 \\ &= (v \mapsto ax_1(v)y_1) + (v \mapsto bx_2(v)y_1) \\ &= a\phi(x_1, y_1) + b\phi(x_2, y_1) \end{aligned}$$

and

$$\begin{aligned} \phi(x_1, ay_1 + by_2) &= v \mapsto (x_1(v)(ay_1 + by_2) \\ &= v \mapsto ax_1(v)y_1 + bx_1(v)y_2 \\ &= (v \mapsto ax_1(v)y_1) + (v \mapsto bx_1(v)y_2) \\ &= a\phi(x_1, y_1) + b\phi(x_1, y_2) \end{aligned}$$

so ϕ is bilinear. Therefore, by the universal property of the tensor product, it induces a unique linear map $\Phi : V^* \otimes W^* \to \operatorname{Hom}(V, W^*)$. We want to show that Φ is an isomorphism; to that end, since $V^* \otimes W^*$ and $\operatorname{Hom}(V, W^*)$ have the same dimension (namely $\dim(V) \cdot \dim(W)$), we need only show that Φ is injective. Suppose

$$\Phi(f \otimes g) = 0$$

for $f \in V^*$ and $g \in W^*$. Then

$$0 = \Phi(f \otimes g) = (v \mapsto f(v)g),$$

which is to say that $f(v)g(w) = 0$ for all $v \in V$ and $w \in W$. Clearly, this occurs only if $f = 0$ or $g = 0$, so

$$f \otimes g = \left\{ \begin{array}{c} f \otimes 0 \\ 0 \otimes g \end{array} \right\} = 0,$$

so $\ker \Phi = 0$ and, thus, Φ is injective and, therefore, is an isomorphism.

$\operatorname{Hom}(V, W^*) = \operatorname{Hom}(W, V^*)$: Define $\Phi : \operatorname{Hom}(V, W^*) \to \operatorname{Hom}(W, V^*)$ by

$$(v \mapsto g) \mapsto (w \mapsto (v \mapsto g(w)))$$

and $\Psi : \operatorname{Hom}(W, V^*) \to \operatorname{Hom}(V, W^*)$ by

$$(w \mapsto f) \mapsto (v \mapsto (w \mapsto f(v))).$$

Then, if $(v \mapsto g_1), (v \mapsto g_2) \in \mathrm{Hom}(V, W^*)$, then

$$\begin{aligned}
\Phi((v \mapsto g_1) + (v \mapsto g_2)) &= \Phi(v \mapsto (g_1 + g_2)) \\
&= w \mapsto (v \mapsto (g_1 + g_2)(w)) \\
&= w \mapsto (v \mapsto (g_1(w) + g_2(w))) \\
&= w \mapsto ((v \mapsto g_1(w)) + (v \mapsto g_2(w))) \\
&= (w \mapsto (v \mapsto g_1(w))) + (w \mapsto (v \mapsto g_2(w))) \\
&= \Phi(v \mapsto g_1) + \Phi(v \mapsto g_2),
\end{aligned}$$

so Φ is a homomorphism. If $(w \mapsto f_1), (w \mapsto f_2) \in \mathrm{Hom}(W, V^*)$, then

$$\begin{aligned}
\Psi((w \mapsto f_1) + (w \mapsto f_2)) &= \Psi(w \mapsto (f_1 + f_2)) \\
&= v \mapsto (w \mapsto (f_1 + f_2)(v)) \\
&= v \mapsto (w \mapsto (f_1(v) + f_2(v))) \\
&= v \mapsto ((w \mapsto f_1(v)) + (w \mapsto f_2(v))) \\
&= (v \mapsto (w \mapsto f_1(v))) + (v \mapsto (w \mapsto f_2(v))) \\
&= \Psi(w \mapsto f_1) + \Psi(w \mapsto f_2),
\end{aligned}$$

so Ψ is a homomorphism. Now, if $(v \mapsto g) \in \mathrm{Hom}(V, W^*)$ and we let $f = (v \mapsto g(w))$, then

$$\begin{aligned}
\Psi \circ \Phi(v \mapsto g) &= \Psi(w \mapsto (v \mapsto g(w))) \\
&= \Psi(w \mapsto f) \\
&= v \mapsto (w \mapsto f(v)) \\
&= v \mapsto (w \mapsto g(w)) \\
&= v \mapsto g.
\end{aligned}$$

On the other hand, if $(w \mapsto f) \in \mathrm{Hom}(W, V^*)$ and we let $g = (w \mapsto f(v))$, then

$$\begin{aligned}
\Phi \circ \Psi(w \mapsto f) &= \Phi(v \mapsto (w \mapsto f(v))) \\
&= \Phi(v \mapsto g) \\
&= w \mapsto (v \mapsto g(w)) \\
&= w \mapsto (v \mapsto f(v)) \\
&= w \mapsto f.
\end{aligned}$$

Therefore, we see that

$$\Psi \circ \Phi = Id_{\mathrm{Hom}(V, W^*)} \qquad \Phi \circ \Psi = Id_{\mathrm{Hom}(W, V^*)},$$

so, in fact, Φ and Ψ are isomorphisms.

$\square$

(b): Show that there is a natural isomorphism $\mathrm{Hom}(V \otimes W, Y) = \mathrm{Hom}(V, \mathrm{Hom}(W, Y))$.

Proof. Let $\pi : V \times W \to V \otimes W$ be the standard projection and suppose $f \in \text{Hom}(V \otimes W, Y)$. Then $f \circ \pi$ is a bilinear map from $V \times W$ to Y. Define $\Phi : \text{Hom}(V \otimes W, Y) \to \text{Hom}(V, \text{Hom}(W, Y))$ by

$$f \mapsto (v \mapsto (w \mapsto f \circ \pi(v, w))).$$

Then, since $f \circ \pi$ is bilinear, $w \mapsto f \circ \pi(v, w)$ is a linear map. Now, if $f_1, f_2 \in \text{Hom}(V \otimes W, Y)$ and $a, b \in K$, then

$$\begin{aligned}
\Phi(af_1 + bf_2) &= v \mapsto (w \mapsto (af_1 + bf_2) \circ \pi(v, w)) \\
&= v \mapsto (w \mapsto (af_1 \circ \pi(v, w) + af_2 \circ \pi(v, w))) \\
&= v \mapsto ((w \mapsto af_1 \circ \pi(v, w)) + (w \mapsto bf_2 \circ \pi(v, w))) \\
&= v \mapsto (a(w \mapsto f_1 \circ \pi(v, w)) + b(w \mapsto f_2 \circ \pi(v, w))) \\
&= (v \mapsto a(w \mapsto f_1 \circ \pi(v, w))) + (v \mapsto b(w \mapsto f_2 \circ \pi(v, w))) \\
&= a(v \mapsto (w \mapsto f_1 \circ \pi(v, w))) + b(v \mapsto (w \mapsto f_2 \circ \pi(v, w))) \\
&= a\Phi(f_1) + b\Phi(f_2),
\end{aligned}$$

so Φ is a homomorphism. Since the dimensions of $\text{Hom}(V \otimes W, Y)$ and $\text{Hom}(V, \text{Hom}(W, Y))$ are the same (namely the product of the dimensions of V, W and Y), we need only show that Φ is injective. Suppose, then, that $\Phi(f) = 0$ for $f \in \text{Hom}(V \otimes W, Y)$. Then

$$0 = \Phi(f) = (v \mapsto (w \mapsto f \circ \pi(v, w))),$$

so $f \circ \pi(v, w) = 0$ for all $v \in V$ and $w \in W$. Thus, for $v \in V$ and $w \in W$,

$$0 = f \circ \pi(v, w) = f(v \otimes w),$$

so $f = 0$. Therefore, we see that $\ker \Phi = 0$, so Φ is an isomorphism. □

(c): Show that $\text{Hom}(V \otimes W, Y)$ is naturally isomorphic to the vector space of bilinear maps $V \times W \to Y$.

Proof. Let $B =$ the space of bilinear maps $V \times W \to Y$. By the universal property of tensor products, if $f \in \text{Hom}(V \otimes W, Y)$, then $f \circ \pi$ is a bilinear map from $V \times W$ to Y (where $\pi : V \times W \to V \otimes W$ is the standard projection). Therefore, define $\Phi : \text{Hom}(V \otimes W, Y) \to B$ by

$$\Phi(f) = f \circ \pi.$$

Then, if $f_1, f_2 \in \text{Hom}(V \otimes W, Y)$ and $a, b \in K$, then

$$\Phi(af_1 + bf_2) = (af_1 + bf_2) \circ \pi = af_1 \circ \pi + af_2 \circ \pi = a\Phi(f_1) + b\Phi(f_2),$$

so Φ is a homomorphism. On the other hand, if $g \in B$, then, by the universal property of the tensor product, g induces a unique linear map $\overline{g} \in \text{Hom}(V \otimes W, Y)$ such that $g = \bar{g} \circ \pi$. Define $\Psi : B \to \text{Hom}(V \otimes W, Y)$ by $\Psi(g) = \bar{g}$. Now, suppose $g_1, g_2 \in B$, $a, b \in K$. Then

$$ag_1 + bg_2 = \Psi(ag_1 + bg_2) \circ \pi = \overline{ag_1 + bg_2} \circ \pi.$$

Therefore,

$$\begin{aligned}(a\Psi(g_1)+b\Psi(g_2))\circ\pi &= (a\overline{g_1}+b\overline{g_2})\circ\pi\\ &= a\overline{g_1}\circ\pi+b\overline{g_2}\circ\pi\\ &= ag_1+bg_2\\ &= \Psi(ag_1+bg_2)\circ\pi;\end{aligned}$$

since the induced map is unique, we see that $a\Psi(g_1)+b\Psi(g_2)=\Psi(ag_1+bg_2)$, so Ψ is a homomorphism.

Now, if $f\in\text{Hom}(V\otimes W,Y)$, then

$$\Psi\circ\Phi(f)=\Psi(f\circ\pi)=f$$

and, if $g\in B$, then

$$\Phi\circ\Psi(g)=\Psi(g)\circ\pi=g,$$

so $\Psi\circ\Phi=Id_{\text{Hom}(V\otimes W,Y)}$ and $\Phi\circ\Psi=Id_B$, so we see that Φ and Ψ are isomorphisms. □

2

Let R be the ring of polynomial functions on the unit sphere $S^2\subset\mathbb{R}^3$. Thus this ring is given by $R=\mathbb{R}[x,y,z]/(x^2+y^2+z^2-1)$.

(a): Let $P=(0,0,1)\in S^2$, and let $R_P=\{\frac{f}{g}|f,g\in R;g(P)\neq 0\}$. Show directly that R_P is a local ring, and find a set of generators for I.

Proof. Let $U\subset R_P$ be the set of units in R_P. Let $\frac{f}{g}\in U$. Then $\frac{g}{f}\in R_P$, which means that $f(P)\neq 0$. On the other hand, if $\frac{f}{g}\in R_P$ such that $f(P)\neq 0$, then $\frac{g}{f}\in R_P$ and so $\frac{f}{g}\in U$. Therefore,

$$U=\left\{\frac{f}{g}|f,g\in R,f(P)\neq 0,g(P)\neq 0\right\}.$$

Therefore, if $J\subset R_P$ is an ideal, $J\subset R_P-U$. However, R_P-U is itself an ideal: if $\frac{f}{g}\in R_P-U$ and $\frac{f'}{g'}\in R_P$, then

$$\frac{f}{g}\cdot\frac{f'}{g'}=\frac{ff'}{gg'}$$

and $f(P)f'(P)=0\cdot f'(P)=0$, so $\frac{f}{g}\cdot\frac{f'}{g'}\in R_P-U$; call this ideal I. Then I is clearly the unique maximal ideal, since we've just seen that all other proper ideals must be contained in I. Specifically,

$$I=\left\{\frac{f}{g}|f,g\in R,f(P)=0,g(P)\neq 0\right\}.$$

Now, certainly $x,y,z-1\in I$; we claim that these three elements generate I. Now, if $f(P)=0$, then

$$f(x,y,z)=x^\alpha y^\beta(z-1)^\gamma h(x,y,z)$$

for $\alpha, \beta, \gamma \geq 0$ and at least one of the α, β, γ strictly positive and $h \in R_P$, so we see that the set $\{x, y, z-1\}$ generates I. □

(b): Show that $I^2 \subset I$ but that $I^2 \neq I$. Let I/I^2 be the image of I under the ring homomorphism $R_P \to R_P/I^2$. Show that I/I^2 is a 2-dimensional vector space over $\mathbb{R}$.

Proof. Clearly, since I is an ideal, $I^2 \subset I$. Let $f, g \in I$. Then, in $\mathbb{R}[x,y,z]$,

$$\deg(fg) = \deg(f) + \deg(g).$$

The only relation in R is

$$x^2 + y^2 + z^2 = 1,$$

which gives the relations $x^2 = 1 - y^2 - z^2$, $y^2 = 1 - x^2 - z^2$ and $z^2 = 1 - x^2 - y^2$. Note that none of these relations (except the original one) are degree-reducing. Hence, viewed in R, $\deg(fg)$ is equal to the degree of fg in $\mathbb{R}[x,y,z]$ unless fg is divisible by $x^2+y^2+z^2$. Since $x^2+y^2+z^2$ is irreducible in $\mathbb{R}[x,y,z]$, fg cannot be so divisible unless either f or g is divisible by $x^2+y^2+z^2$. However, since $x^2+y^2+z^2 = 1$ in R, we can make this substitution prior to multiplying and so, if we take $\deg(f)$ and $\deg(g)$ to be the minimal degrees of elements of the equivalence classes of f and g in $\mathbb{R}[x,y,z]$, we see that $\deg(fg) = \deg(f) + \deg(g)$ in R.

Now, suppose $x \in I^2$. Then there exist $f, g \in I$ such that

$$f(x,y,z)g(x,y,z) = x.$$

Hence,

$$1 = \deg(x) = \deg(fg) = \deg(f) + \deg(g),$$

so either $\deg(f) = 0$ or $\deg(g) = 0$. This, however, implies that either f or g is constant in R; since there are no constant terms in I, this is impossible. Therefore, we conclude that $x \notin I^2$ and so $I^2 \neq I$.

Now, let $f \in I/I^2$. Then, since $x, y, z-1$ generate I,

$$f(x,y,z) = f_1(x,y,z)x^\alpha + f_2(x,y,z)y^\beta + f_3(x,y,z)(z-1)^\gamma$$

for $f_1, f_2, f_3 \in R_P$ and $\alpha, \beta, \gamma \in \mathbb{N} - \{0\}$. Now,

$$z - 1 = \frac{(z+1)(z-1)}{z+1} = \frac{z^2-1}{z+1} = \frac{-(x^2+y^2)}{z+1} = \frac{-1}{z+1}(x^2+y^2) \in I^2,$$

since $x^2 + y^2 \in I^2$, so, in fact,

$$f(x,y,z) = f_1(x,y,z)x^\alpha + f_2(x,y,z)y^\beta$$

in I/I^2. Now, if $\alpha > 1$, then

$$x^\alpha = x^{\alpha-1}x \in I^2$$

and, similarly, if $\beta > 1$, $y^\beta \in I^2$, so we see that $\alpha = \beta = 1$. Since $f_1, f_2 \in R_P$, $f_1 = \frac{g_1}{h_1}$ and $f_2 = \frac{g_2}{h_2}$ for $g_1, g_2, h_1, h_2 \in R$ and $h_1(P) \neq 0$, $h_2(P) \neq 0$. Hence,

$$f(x,y,z) = \frac{g_1(x,y,z)}{h_1(x,y,z)}x + \frac{g_2(x,y,z)}{h_2(x,y,z)}y.$$

Let

$$\widetilde{h_i}(x,y,z) = h_i(x,y,z) - h_i(0,0,1).$$

Then $\widetilde{h_i}(P) = 0$ and so $\widetilde{h_i} \in I^2$. Thus, in I/I^2, $\widetilde{h_1}\widetilde{h_2} = 0$ and $f\widetilde{h_i} = 0$, so

$$\begin{aligned}
& h_2(x,y,z)g_1(x,y,z)x + h_1(x,y,z)g_2(x,y,z)y \\
= \;& f(x,y,z)(h_1(x,y,z)h_2(x,y,z)) \\
= \;& f(x,y,z)((h_1(x,y,z) - h_1(0,0,1) + h_1(0,0,1))(h_2(x,y,z) - h_2(0,0,1) + h_2(0,0,1))) \\
= \;& f(x,y,z)((\widetilde{h_1}(x,y,z) + h_1(0,0,1))(\widetilde{h_2}(x,y,z) + h_2(x,y,z))) \\
= \;& f(x,y,z)(\widetilde{h_1}(x,y,z)\widetilde{h_2}(x,y,z) + \widetilde{h_1}(x,y,z)h_2(0,0,1) \\
& + \widetilde{h_2}(x,y,z)h_1(0,0,1) + h_1(0,0,1)h_2(0,0,1)) \\
= \;& f(x,y,z)h_1(0,0,1)h_2(0,0,1).
\end{aligned}$$

Therefore, since $g_1' = \frac{h_2 g_1}{h_1(0,0,1)h_2(0,0,1)}$ and $g_2' = \frac{h_1 g_2}{h_1(0,0,1)h_2(0,0,1)} \in R$,

$$f(x,y,z) = g_1'(x,y,z)x + g_2'(x,y,z)y$$

for $g_1', g_2' \in R$.

Let

$$\widetilde{g_i'}(x,y,z) = g_i'(x,y,z) - g_i'(0,0,1).$$

Then $\widetilde{g_i'}(P) = 0$, so $\widetilde{g_i'} \in I$ and hence $\widetilde{g_i'}(x,y,z)x \in I^2$. Thus,

$$g_1'(x,y,z)x = (g_1'(x,y,z) - g_1'(0,0,1) + g_1'(0,0,1))x = \widetilde{g_1'}(x,y,z)x + g_1'(0,0,1)x,$$

which is equal to simply $g_1'(0,0,1)x$ in I/I^2. A similar argument for g_2' and y yields the result

$$f(x,y,z) = g_1'(0,0,1)x + g_2'(0,1,0)y.$$

Therefore, since $g_1'(0,0,1)$ and $g_2'(0,1,0)$ are simply scalars, we see that I/I^2 is a 2-dimensional vector space over $\mathbb{R}$ with basis $\{x, y\}$. □

3

In the situation of problem 2:

(a): Let $T \subset \mathbb{R}^3$ be the tangent plane to S^2 at P. Thus, $T = \{(x,y,1) | x, y \in \mathbb{R}\}$. Show that T is a 2-dimensional vector space over $\mathbb{R}$, under the addition $(x,y,1) + (x',y',1) = (x+x', y+y', 1)$ and scalar multiplication $c(x,y,1) = (cx, cy, 1)$. What is the 0-vector of T?

Proof. Let $(x, y, 1), (x', y', 1) \in T$. Then

$$(x, y, 1) + (x', y', 1) = (x + x', y + y', 1) \in T$$

and, if $c \in \mathbb{R}$,

$$c(x, y, 1) = (cx, cy, 1) \in T,$$

so T is closed under addition and scalar multiplication. Also,

$$\begin{aligned} c((x, y, 1) + (x', y', 1)) = c(x + x', y + y', 1) &= (c(x + x'), c(y + y'), 1) \\ &= (cx + cx', cy + cy', 1) \\ &= c(x, y, 1) + c(x', y', 1) \end{aligned}$$

and, for $d \in \mathbb{R}$,

$$(c+d)(x, y, 1) = ((c+d)x, (c+d)y, 1) = (cx+dx, cy+dy, 1) = c(x, y, 1)+d(x, y, 1),$$

so the distributive laws hold and so T is a vector space. Note that $(0, 0, 1)$ is the 0 vector of T. If $(x, y, 1) \in T$, then

$$(x, y, 1) = x(1, 0, 1) + y(0, 1, 1)$$

and, if $c, d \in \mathbb{R}$ such that

$$c(1, 0, 1) + d(0, 1, 1) = 0,$$

then

$$0 = c(1, 0, 1) + d(0, 1, 1) = (c, d, 1),$$

so $c = d = 0$. Therefore, we see that $\{(1, 0, 1), (0, 1, 1)\}$ is a basis for T, and so we conclude that T is two-dimensional. □

(b): Let $f \in T^*$, the dual space of T. Show that $f : T \to R$ extends to a unique linear functional $f' : \mathbb{R}^3 \to \mathbb{R}$. Let $\bar{f} : S^2 \to \mathbb{R}$ be the restriction of f' to S^2. Show that $\bar{f} \in R$, and moreover $\bar{f} \in I$.

Proof. Let $f \in T^*$. Then define $f' : \mathbb{R}^3 \to \mathbb{R}$ by

$$f'(x, y, z) = f(x, y, 1).$$

Let $(x, y, z), (x', y', z') \in \mathbb{R}^3$ and let $a, b \in \mathbb{R}$. Then

$$\begin{aligned} f'(a(x, y, z) + b(x', y', z')) &= f'(ax + bx', ay + by', az + bz') \\ &= f(ax + bx', ay + by', 1) \\ &= af(x, y, 1) + bf(x', y', 1) \\ &= af'(x, y, z) + bf'(x', y', z'), \end{aligned}$$

so we see that f' is linear. Furthermore, if there exists linear $g : \mathbb{R}^3 \to \mathbb{R}$ such that $g|_T = f$, then, for $c \in \mathbb{R}$,

$$g(0, 0, c) = g(c(0, 0, 1)) = cg(0, 0, 1) = c \cdot 0 = 0.$$

Hence, for any $(x, y, z) \in \mathbb{R}^3$,

$$\begin{aligned} f'(x,y,z) - g(x,y,z) &= f(x,y,1) - g((x,y,1) + (0,0,z-1)) \\ &= f(x,y,1) - (g(x,y,1) + g(0,0,z-1)) \\ &= f(x,y,1) - f(x,y,1) - g(0,0,z-1) \\ &= -g(0,0,z-1) \\ &= 0; \end{aligned}$$

since our choice of (x, y, z) was arbitrary, we see that $g = f'$, so f' is the unique linear extension of f to $\mathbb{R}^3$. Let $\bar{f} : S^2 \to \mathbb{R}$ be the restriction of f to S^2.

Now, let $c_1 = \bar{f}(1,0,0)$, $c_2 = \bar{f}(0,1,0)$ and $c_3 = \bar{f}(0,0,1)$. Then $c_3 = 0$ since $(0,0,1)$ is the 0 element of T and, for $(x, y, z) \in \mathbb{R}^3$,

$$\begin{aligned} \bar{f}(x,y,z) &= \bar{f}((x,0,0) + (0,y,0) + (0,0,z)) \\ &= \bar{f}(x,0,0) + \bar{f}(0,y,0) + \bar{f}(0,0,z) \\ &= x\bar{f}(1,0,0) + y\bar{f}(0,1,0) + z\bar{f}(0,0,1) \\ &= c_1x + c_2y + c_3z \\ &= c_1x + c_2y, \end{aligned}$$

so $\bar{f} \in R$. Furthermore,

$$\bar{f}(P) = \bar{f}(0,0,1) = 0,$$

so $f \in I$. □

(c): If $f \in T^*$, let $\phi(f) \in I/I^2$ be the image of $\bar{f} \in I$ under $I \to I/I^2$. Show that $\phi : T^* \to I/I^2$ is an isomorphism of vector spaces.

Proof. Note, first, that if $f \in T^*$,

$$\phi(f) = f'$$

where f' is as defined in (b) above.

Let $e_1 = (1,0,1)$ and $e_2 = (0,1,1)$. Then e_1, e_2 defines a basis on T. Let f_1, f_2 be the corresponding dual basis for T^*. Now, since $\overline{f_1}(x,y,z) = x$ extends f_1 and $\overline{f_2}(x,y,z) = y$ extends f_2 and we saw in (b) that such extensions are unique, we see that $\overline{f_1}$ and $\overline{f_2}$ are the unique linear extensions of f_1 and f_2 to maps $\mathbb{R}^3 \to \mathbb{R}$. Let f_1' and f_2' be their restrictions to S^2. Then

$$\phi(f_1) = f_1' = x \text{ and } \phi(f_2) = f_2' = y,$$

the basis on I/I^2. Now, let $f \in T^*$. Then

$$f = af_1 + bf_2$$

for $a, b \in \mathbb{R}$. Furthermore, if $\overline{f}(x,y,z) = ax + by$, then

$$\overline{f}(x,y,1) = ax + by = a\overline{f_1}(x,y,1) + b\overline{f}(x,y,1) = f(x,y,1),$$

so $\overline{f}$ is the unique extension of f to a linear functional on $\mathbb{R}^3$. Hence, if f' is the restriction of $\overline{f}$ to S^2, then

$$\phi(af_1+bf_2)(x,y,z) = \phi(f)(x,y,z) = f'(x,y,z) = ax+by = a\phi(f_1)(x,y,z)+b\phi(f_2)(x,y,z),$$

so we see that ϕ is linear on the basis elements of T^*. Thus, ϕ is linear; since ϕ maps the basis of T^* bijectively onto the basis of I/I^2, we see that ϕ is, in fact, an isomorphism. ☐

(d): Conclude that T is isomorphic to $(I/I^2)^*$ via ϕ.

Proof. Since $\phi : T^* \to I/I^2$ is an isomorphism, the following sequence is exact:

$$\begin{tikzcd} 0 \arrow[r] & T^* \arrow[r, "\phi"] & I/I^2 \arrow[r] & 0. \end{tikzcd}$$

In PS10#4, we showed that induced maps on duals preserve exact sequences, so

$$\begin{tikzcd} 0 & T^{**} \arrow[l] & (I/I^2)^* \arrow[l, "\phi^*"'] & 0 \arrow[l] \end{tikzcd}$$

is exact, and so $\phi^* : (I/I^2)^* \to T^{**}$ is an isomorphism. Now, T^{**} is naturally isomorphic to T by the map $\psi : T^{**} \to T$ defined in PS10#5, so we see that the composition

$$\psi \circ \phi^* : (I/I^2)^* \to T$$

is an isomorphism. ☐

4

Let V be a K-vector space and let $0 \to W' \to W \to W'' \to 0$ be an exact sequence of K-vector spaces. Show that the induced sequences $0 \to V \otimes W' \to V \otimes W \to V \otimes W'' \to 0$; $0 \to \mathrm{Hom}(V,W') \to \mathrm{Hom}(V,W) \to \mathrm{Hom}(V,W'') \to 0$; and $0 \to \mathrm{Hom}(W'',V) \to \mathrm{Hom}(W,V) \to \mathrm{Hom}(W',V) \to 0$ are also exact.

Proof. Let us denote the maps as follows:

$$\begin{tikzcd} 0 \arrow[r] & W' \arrow[r, "G"] & W \arrow[r, "F"] & W'' \arrow[r] & 0 \end{tikzcd} \tag{1}$$

$$\begin{tikzcd} 0 \arrow[r] & V \otimes W' \arrow[r, "G^{\otimes}"] & V \otimes W \arrow[r, "F^{\otimes}"] & V \otimes W'' \arrow[r] & 0 \end{tikzcd} \tag{2}$$

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}(V,W') \arrow[r, "G_*"] & \mathrm{Hom}(V,W) \arrow[r, "F_*"] & \mathrm{Hom}(V,W'') \arrow[r] & 0 \end{tikzcd} \tag{3}$$

$$\begin{tikzcd} 0 & \mathrm{Hom}(W',V) \arrow[l] & \mathrm{Hom}(W,V) \arrow[l, "G^*"'] & \mathrm{Hom}(W'',V) \arrow[l, "F^*"'] & 0 \arrow[l] \end{tikzcd} \tag{4}$$

Let $e_1, \ldots, e_n$ be a basis for V.

(2): To see that (2) is exact, we need to show:

$$\ker G^{\otimes} = 0 \qquad \operatorname{im} G^{\otimes} = \ker F^{\otimes} \qquad \operatorname{im} F^{\otimes} = V \otimes W''.$$

To see that $\ker G^{\otimes} = 0$, suppose $v \otimes w' \in \ker G^{\otimes}$. Then

$$0 = G^{\otimes}(v \otimes w') = v \otimes G(w'),$$

so either $v = 0$ or $G(w') = 0$. Since G is injective, this implies that either $v = 0$ or $w' = 0$. Since $0 \otimes w' = 0 = v \otimes 0$, we see that $v \otimes w \in \ker G^{\otimes}$ implies $v \otimes w' = 0$. Hence, $G^{\otimes}$ is injective.

To see $\operatorname{im} F^{\otimes} = V \otimes W''$, let $\sum_{i=1}^{k} v_i \otimes w_i'' \in V \otimes W''$. Then, since F is surjective, there exist $w_1, \ldots, w_k \in W$ such that $F(w_i) = w_i''$. Therefore,

$$F^{\otimes}\left(\sum_{i=1}^{k} v_i \otimes w_i\right) = \sum_{i=1}^{k} F^{\otimes}(v_i \otimes w_i) = \sum_{i=1}^{k} v_i \otimes F(w_i) = \sum_{i=1}^{k} v_i \otimes w_i'',$$

so we see that $F^{\otimes}$ is surjective.

Finally, to see that $\operatorname{im} G^{\otimes} = \ker F^{\otimes}$, first let $\sum_{i=1}^{k} v_i \otimes w_i \in \operatorname{im} G^{\otimes}$. Then there exist $w_i' \in W'$ such that $G(w_i') = w_i$ for all $i = 1, \ldots, k$. Now,

$$\begin{aligned} F^{\otimes}\left(\sum_{i=1}^{k} v_i \otimes w_i\right) &= \sum_{i=1}^{k} F^{\otimes}(v_i \otimes w_i) \\ &= \sum_{i=1}^{k} v_i \otimes F(w_i) \\ &= \sum_{i=1}^{k} v_i \otimes F(G(w_i')) \\ &= \sum_{i=1}^{k} v_i \otimes 0 \\ &= 0, \end{aligned}$$

since im $G = \ker F$, so we see that im $G^{\otimes} \subset \ker F^{\otimes}$. On the other hand, if $\sum_{i=1}^{k} v_i \otimes w_i \in \ker F^{\otimes}$, then, for each $i = 1, \ldots, k$, $v_i = \sum_{j=1}^{n} a_{i_j} e_j$ and

$$\begin{aligned}
0 &= F^{\otimes}\left(\sum_{i=1}^{k} v_i \otimes w_i\right) \\
&= \sum_{i=1}^{k} v_i \otimes F(w_i) \\
&= \sum_{i=1}^{k} \left(\sum_{j=1}^{n} a_{i_j} e_j\right) \otimes F(w_i) \\
&= \sum_{j=1}^{n} e_j \otimes \left(\sum_{i=1}^{k} a_{i_j} F(w_i)\right) \\
&= \sum_{j=1}^{n} e_j \otimes \left(\sum_{i=1}^{k} F(a_{i_j} w_i)\right) \\
&= \sum_{j=1}^{n} e_j \otimes F\left(\sum_{i=1}^{k} a_{i_j} w_i\right);
\end{aligned}$$

since $e_i \otimes w$ is linearly independent of $e_j \otimes w'$ for any $w, w' \in W''$ and $i \neq j$, we see that this implies that

$$F\left(\sum_{i=1}^{k} a_{i_j} w_i\right) = 0$$

for all $j = 1, \ldots, n$. Therefore, since (1) is exact, there exists $w_j' \in W'$ such that $G(w_j') = \sum_{i=1}^{k} a_{i_j} w_i$ for each $j = 1, \ldots, n$.

Now,

$$\begin{aligned}
G^{\otimes}\left(\sum_{j=1}^{n} e_j \otimes w'_j\right) &= \sum_{j=1}^{n} G^{\otimes}(e_j \otimes w'_j) \\
&= \sum_{j=1}^{n} e_j \otimes G(w'_j) \\
&= \sum_{j=1}^{n} e_j \otimes \left(\sum_{i=1}^{k} a_{i_j} w_i\right) \\
&= \sum_{j=1}^{n}\sum_{i=1}^{k} e_j \otimes a_{i_j} w_i \\
&= \sum_{j=1}^{n}\sum_{i=1}^{k} a_{i_j} e_j \otimes w_i \\
&= \sum_{i=1}^{k}\sum_{j=1}^{n} a_{i_j} e_j \otimes w_i \\
&= \sum_{i=1}^{k}\left(\sum_{j=1}^{n} a_{i_j} e_j\right) \otimes w_i \\
&= \sum_{i=1}^{k} v_i \otimes w_i,
\end{aligned}$$

so ker $F^{\otimes} \subset$ im $G^{\otimes}$. Having proved containment both ways, we conclude that ker $F^{\otimes} =$ im $G^{\otimes}$, and so conclude that (2) is exact.

(3): To see that (3) is exact, we must show that

$$\ker G_* = 0 \qquad \text{im } G_* = \ker F_* \qquad \text{im } F_* = \mathrm{Hom}(V, W'').$$

Note that for $\phi \in \mathrm{Hom}(V, U)$ and $H : U \to U'$, $H_* : \mathrm{Hom}(V, U) \to \mathrm{Hom}(V, U')$ is defined by

$$H_*(\phi)(v) = (H \circ \phi)(v)$$

for any vector space U.

Now, let $\phi \in \ker G_*$. Then

$$0 = G_*(\phi)(v) = (G \circ \phi)(v) = G(\phi(v))$$

for all $v \in V$. Since G is injective, this in turn means that $\phi(v) = 0$ for all $v \in V$, so $\phi = 0$. Therefore, we see that ker $G_* = 0$.

Now, let $\phi \in \mathrm{Hom}(V, W'')$. Let $v \in V$. Then there exists $w \in W$ such that $F(w) = \phi(v)$, since F is surjective. Therefore, define $\psi : V \to W$ by

$$\psi(v) = w.$$

To see that ψ is linear, let $v, v' \in V$ and $a, b \in K$. Then $\psi(v) = w$ and $\psi(v') = w'$ for $w, w' \in W$. Then $F(w) = \phi(v)$ and $F(w') = \phi(v')$ and, since F is linear, $F(aw + bw') = aF(w) + bF(w')$. Hence,

$$\psi(av + bv') = aw + bw' = a\psi(v) + b\psi(v'),$$

so $\psi \in \text{Hom}(V, W)$. Furthermore, for all $v \in V$, there exists $w \in W$ such that $F(w) = \phi(v)$ and

$$F_*(\psi)(v) = (F \circ \psi)(v) = F(\psi(v)) = F(w) = \phi(v),$$

so we see that $F_*(\psi) = \phi$, so F_* is surjective.

Now, to see that im G_* = ker F_*, let $\phi \in$ im G_*. Then there exists $\psi \in \text{Hom}(V, W')$ such that $G_*(\psi) = \phi$. Hence, for $v \in V$,

$$F_*(\phi)(v) = (F \circ \phi)(v) = F(\phi(v)) = F(G_*(\psi)(v)) = F(G(\psi(v))) = (F \circ G)(\psi(v)) = 0,$$

since $F \circ G = 0$. Since our choice of v was arbitrary, we see that $F_*(\phi) = 0$, so im $G_* \subset$ ker F_*. On the other hand, if $\phi \in$ ker F_*, then

$$0 = F_*(\phi)(v) = (F \circ \phi)(v) = F(\phi(v))$$

for all $v \in V$. Since (1) is exact, there exists $w \in W'$ such that $G(w) = \phi(v)$ for all $v \in V$. Hence, define $\psi : V \to W'$ by $\psi(v) = w$. Then, if $v, v' \in V$ and $a, b \in K$, then $\psi(v) = w$ and $\psi(v') = w'$ for some $w, w' \in W'$. Furthermore, $G(w) = \phi(v)$ and $G(w') = \phi(v')$ and, since G is linear, $G(aw + bw') = aG(w) + bG(w')$. Hence,

$$\psi(av + bv') = aw + bw' = a\psi(v) + b\psi(v'),$$

so $\psi \in \text{Hom}(V, W')$. Now, for $v \in V$, $\psi(v) = w$ for some $w \in W'$ and

$$G_*(\psi)(v) = (G \circ \psi)(v) = G(\psi(v)) = G(w) = \phi(v),$$

so $G_*(\psi) = \phi$ and so we see that ker $F_* \subset$ im G_*. Therefore, since containment has been shown both ways, we see that im G_* = ker F_* and so (3) is exact.

(4): To see that (4) is exact, we must show that

$$\ker F^* = 0 \qquad \text{im } F^* = \ker G^* \qquad \text{im } G^* = \text{Hom}(W', V).$$

Note that for $\phi \in \text{Hom}(U', V)$ and $H : U \to U'$, $H^* : \text{Hom}(U', V) \to \text{Hom}(U, V)$ is defined by

$$H^*(\phi)(v) = (\phi \circ H)(u)$$

for any vector space U.

Now, let $\phi \in$ ker F^*. Then, for all $w \in W$,

$$0 = F^*(\phi)(w) = (\phi \circ F)(w) = \phi(F(w)).$$

Since F is surjective, this implies that $\phi(w'') = 0$ for all $w'' \in W''$, so $\phi = 0$. Hence, ker $F^* = 0$.

Let $\phi \in \text{Hom}(W', V)$. $\pi_G : W \to W$ be the orthogonal projection onto the image of G in W. Define $\psi : W \to V$ by

$$\psi(w) = (\phi \circ G^{-1} \circ \pi_G)(w).$$

Then ψ is well-defined, since $G^{-1} : \text{im } G \to W'$ is well-defined (because G is injective). Note that, if $w_1, w_2 \in \text{im } G$ and $a, b \in K$, then $G(w_1') = w_1$ and $G(w_2') = w_2$ for some $w_1', w_2' \in W'$ and

$$G(aw_1' + bw_2') = aG(w_1') + bG(w_2') = aw_1 + bw_2,$$

so $G^{-1}(aw_1 + bw_2) = aw_1' + bw_2'$. Therefore, if $w, w' \in W$ and $a, b \in K$, then

$$\begin{aligned} \psi(aw + bw') &= (\phi \circ G^{-1} \circ \pi_G)(aw + bw') \\ &= \phi(G^{-1}(\pi_G(aw + bw'))) \\ &= \phi(G^{-1}(a\pi_G(w) + b\pi_G(w'))) \\ &= \phi(a(G^{-1} \circ \pi_G)(w) + b(G^{-1} \circ \pi_G)(w')) \\ &= a(\phi \circ G^{-1} \circ \pi_G)(w) + b(\phi \circ G^{-1} \circ \pi_G)(w') \\ &= a\psi(w) + b\psi(w') \end{aligned}$$

since ϕ and π_G are linear. Therefore, we see that $\psi \in \text{Hom}(W, V)$. Furthermore, for all $w' \in W'$,

$$\begin{aligned} (G^* \circ \psi)(w') = G^*(\psi)(w') = (\psi \circ G)(w') &= \psi(G(w')) \\ &= (\phi \circ G^{-1} \circ \pi_G)(G(w')) \\ &= \phi(G^{-1}(\pi_G(G(w')))) \\ &= \phi(G^{-1}(G(w'))) \\ &= \phi(w'), \end{aligned}$$

so $G^* \circ \psi = \phi$ and thus G^* is surjective.

To see that $\text{im } F^* = \ker G^*$, let $\phi \in \text{im } F^*$. Then $\phi = F^* \circ \psi$ for some $\psi \in \text{Hom}(W'', V)$. Therefore, for $w' \in W'$,

$$(G^* \circ \phi)(w') = (\phi \circ G)(w') = ((F^* \circ \psi)(G(w')) = (\psi \circ F)(G(w')) = \psi(F(G(w'))) = \psi(0) = 0,$$

since $F \circ G = 0$. Therefore, since our choice of w' was arbitrary, we see that $G^* \circ \phi = 0$, so $\text{im } F^* \subset \ker G^*$.

On the other hand, suppose $\phi \in \ker G^*$. Then, for all $w' \in W'$,

$$\phi(G(w')) = (\phi \circ G)(w') = G^*(\phi)(w') = 0.$$

Define $\psi : W'' \to V$ by

$$\psi(w'') = \phi(w),$$

where $F(w) = w''$. Since F is surjective, this ψ is defined for all $w'' \in W''$. Then, if $w_1'', w_2'' \in W''$ and $a, b \in K$, $F(w_1) = w_1''$ and $F(w_2) = w_2''$ for some $w_1, w_2 \in W$. Furthermore,

$$F(aw_1 + bw_2) = aF(w_1) + bF(w_2) = aw_1'' + bw_2''$$

so

$$\psi(aw_1'' + bw_2'') = \phi(aw_1 + bw_2) = a\phi(w_1) + b\phi(w_2) = a\psi(w_1'') + b\psi(w_2'');$$

hence, $\psi \in \text{Hom}(W'', V)$. Now, if $w \in W$, then $\psi(F(w)) = \phi(w_0)$ for some $w_0 \in W$ such that $F(w) = F(w_0)$. Hence,

$$0 = F(w) - F(w_0) = F(w - w_0),$$

so $w - w_0 \in \ker F = \text{im G}$. Hence, $w - w_0 = G(w')$ for some $w' \in W'$. Therefore,

$$\begin{aligned} F^*(\psi)(w) = (\psi \circ F)(w) = \psi(F(w)) &= \phi(w_0) \\ &= \phi(w + (w_0 - w)) \\ &= \phi(w) + \phi(w_0 - w) \\ &= \phi(w) + \phi(G(w')) \\ &= \phi(w), \end{aligned}$$

since $\phi(G(w')) = 0$ for all $w' \in W'$. Since our choice of w was arbitrary, we see that $F^* \circ \psi = \phi$, so $\ker G^* \subset \text{im } F^*$. Having shown containment both ways, we conclude that $\text{im } F^* = \ker G^*$ and thus that (4) is exact. □

5

Call $T \in \text{End } V$ *nilpotent* if $T^m = 0$ for some $m > 0$. Show that if T is nilpotent, then T has no non-zero eigenvalues.

Proof. Suppose $a \in K$ is an eigenvalue of T. Then, for some non-zero $v \in V$,

$$Tv = av.$$

Since $T^m = 0$,

$$0 = T^m v = T^{m-1}(Tv) = T^{m-1}av = aT^{m-1}v = \ldots = a^{m-1}Tv = a^m v.$$

Since $v \neq 0$, this implies that $a^m = 0$, so a is a zero divisor. However, since no field contains any non-zero zero divisors, this in turn implies that $a = 0$.

Therefore, since our choice of eigenvalue a was arbitrary, we conclude that T has no non-zero eigenvalues. □

6

Let V be a vector space. If $T \in \text{End } V$ and $W \subset V$, call W a *T-irreducible* subspace if W is T-invariant and the only T-invariant subspaces of W are 0 and W.

(a): Suppose that V is a finite-dimensional $\mathbb{C}$-vector space, that $T \in \text{End } V$ and its powers form a group of order n under composition, and that W is a T-invariant subspace. Show that W has a T-invariant complement W'.

Proof. Let W'' be an arbitrary complement of W. Then $V = W \times W''$; that is, if $v \in V$, then v can be uniquely decomposed as

$$v = w + w''$$

for $w \in W$ and $w'' \in W''$. Define $P : V \to W$ by

$$P(v) = P(w + w'') = w.$$

Then, if $v_1, v_2 \in V$ and $a, b \in K$, then $v_1 = w_1 + w_1''$ and $v_2 = w_2 + w_2''$ for $w_1, w_2 \in W$ and $w_1'', w_2'' \in W''$ and thus

$$\begin{aligned} P(v_1 + v_2) &= P(a(w_1 + w_1'') + b(w_2 + w_2'')) \\ &= P((aw_1 + bw_2) + (aw_1'' + bw_2'')) \\ &= aw_1 + bw_2 \\ &= aP(v_1) + bP(v_2), \end{aligned}$$

so P is linear. Now, define $S : W \to W$ by

$$S(v) = \frac{1}{n}\sum_{i=0}^{n-1} T^i P T^{-i}(v).$$

Then, if $v_1, v_2 \in V$ and $a, b \in K$,

$$\begin{aligned} S(av_1 + bv_2) &= \frac{1}{n}\sum_{i=0}^{n-1} T^i P T^{-i}(av_1 + bv_2) \\ &= \frac{1}{n}\sum_{i=0}^{n-1} T^i P(aT^{-i}(v_1) + bT^{-i}(v_2)) \\ &= \frac{1}{n}\sum_{i=0}^{n-1} T^i(aPT^{-i}(v_1) + bPT^{-i}(v_2)) \\ &= \frac{1}{n}\sum_{i=0}^{n-1}(aT^i P T^{-i}(v_1) + bT^i P T^{-i}(v_2)) \\ &= a\frac{1}{n}\sum_{i=0}^{n-1} T^i P T^{-i}(v_1) + b\frac{1}{n}\sum_{i=0}^{n-1} T^i P T^{-i}(v_2) \\ &= aS(v_1) + bS(v_2), \end{aligned}$$

so S is linear. Now, since $P(v) \in W$ for all $v \in V$, we see that $P(T^{-i}(v)) \in W$ for all v and i and, since W is T-invariant, $T^i P T^{-i}(v) \in W$ for all $v \in V$ and $i = 1, \ldots, n$. Hence, $S(v) \in W$ for all $v \in V$ and so im $S \subset W$. On the other hand, since P is the identity on W, we see that

$$T^i P T^{-i}(w) = T^i T^{-i}(w) = w$$

for all $w \in W$, so

$$S(w) = \frac{1}{n}\sum_{i=0}^{n-1} T^i P T^{-i}(w) = \frac{1}{n}\sum_{i=0}^{n-1} w = w$$

Therefore, $W \subset$ im S. Having shown containment both ways, we conclude that im $S = W$. Furthermore, since we just showed that $S|_W$ is the identity; since im $S = W$, this implies that $S^2 = S$. Thus, by our work in PS10#7, we know that

$$V = W \times W'$$

where $W' = \ker S$, so W' is a complement of W. Now, let $w' \in W'$. Then

$$\begin{aligned}
S(T(w')) &= \frac{1}{n}\sum_{i=0}^{n-1} T^i P T^{-i}(T(w')) \\
&= \frac{1}{n}\sum_{i=0}^{n-1} T^i P T^{-(i-1)}(w') \\
&= \frac{1}{n}\sum_{i=0}^{n-1} T T^{i-1} P T^{-(i-1)}(w') \\
&= \frac{1}{n} T \left(\sum_{i=0}^{n-1} T^{i-1} P T^{-(i-1)}(w')\right) \\
&= T\left(\frac{1}{n}\sum_{i=0}^{n-1} T^{i-1} P T^{-(i-1)}(w')\right) \\
&= T(S(w')) \\
&= T(0) \\
&= 0,
\end{aligned}$$

so $T(w') \in W'$ and, hence, W' is T-invariant. □

(b): Under the hypotheses of (a), show that V can be written as the direct product of T-irreducible subspaces.

Proof. By induction on $\dim V$. If $\dim V = 0$, then we're done. Similarly, if $\dim V = 1$, then $V = V$ is already the direct product of T-irreducible subspaces.

Now, suppose all vector spaces of dimension $\leq k$ can be written as a direct product of T-irreducible subspaces. Let V be a vector space of dimension $k+1$. Then, by our result in (a), we know that

$$V = W \times W'$$

where W and W' are T-invariant. Furthermore, $\dim W \leq k$ and $\dim W' \leq k$ so, by the induction hypothesis,

$$W = \prod_{i=1}^{n} W_i$$

and

$$W' = \prod_{i=1}^{m} W_i'$$

for T-irreducible subspaces $W_1, \dots, W_n, W_1', \dots, W_m'$. Then

$$V = \left(\prod_{i=1}^{n} W_i \right) \times \left(\prod_{j=1}^{m} W_j' \right) = W_1 \times \ldots \times W_n \times W_1' \times \ldots \times W_m',$$

so V is the product of T-irreducible subspaces.

Having shown the base case and the inductive step, we conclude, by induction, that for any finite-dimensional vector space V, V can be written as the direct product of T-irreducible subspaces. □

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