ALGEBRA HW 10

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1

(a): If A is a square matrix satisfying $A^3 - A^2 + A - I = 0$, find A^{-1} in terms of A.

Answer: If $A^3 - A^2 + A - I = 0$, then

$$(A^2 - A + I)A = A^3 - A^2 + A = I = A^3 - A^2 + A = A(A^2 - A + I)$$

so $A^{-1} = A^2 - A + I$.

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(b): If A is a square matrix satisfying $A^7 = 0$, find $(I - A)^{-1}$ in terms of A.

Answer: Note, first of all, that

$$(1 - x)(1 + x + x^2 + x^3 + x^4 + x^5 + x^6) = 1 - x^7.$$

Hence, if $A^7 = 0$, then

$$(I - A)(I + A + A^2 + A^3 + A^4 + A^5 + A^6) = I - A^7 = I,$$

so $(I - A)^{-1} = I + A + A^2 + A^3 + A^4 + A^5 + A^6$.

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2

(a): Let $V = \{\text{differentiable functions on } \mathbb{R}\}$. Prove that the functions e^x, e^{2x}, e^{3x} are linearly independent in the vector space V.

Proof. Suppose that these three functions are not linearly independent. Then there exist $a, b \in \mathbb{R}$ such that

$$e^x = ae^{2x} + be^{3x}.$$

Differentiating, we see that

$$e^x = 2ae^{2x} + 3be^{3x} = 2(ae^{2x} + be^{3x}) + be^{3x} = 2e^x + be^{3x},$$

which is to say that $e^x + be^{3x} = 0$. Differentiating to get the second line, we obtain the system

$$\begin{aligned} e^x + be^{3x} &= 0 \\ e^x + 3be^{3x} &= 0 \end{aligned}$$

Subtracting yields: $2be^{3x} = 0$, which implies that $b = 0$. Hence, the original equation and the first derivative tell us that

$$ae^{2x} = e^x = 2ae^{2x}$$

which means $ae^{2x} = 0$, so $a = 0$. However, since $e^x \neq 0$ for any x, this is clearly impossible. From this contradiction, then, we conclude that e^x, e^{2x}, e^{3x} are linearly independent. $\square$

(b): Let W be the set of solutions to the differential equation $f'' - f = 0$, and let V be the set of solutions to $f''' - f' = 0$. Show that W is a vector subspace of V, find a basis for W, and extend this basis to a basis of V.

Answer: Suppose $f \in W$. Then $f'' - f = 0$. Differentiating both sides, we see that $f''' - f' = 0$, so $f \in V$. Hence, W is a subset of V. Now, if $f, g \in W$ and $a, b \in \mathbb{R}$, then

$$(af + bg)'' - (af + bg) = af'' + bg'' - af - bg = a(f'' - f) + b(g'' - g) = 0,$$

so W is closed under addition and scalar multiplication. Similarly, if $f, g \in V$ and $a, b \in \mathbb{R}$, then

$$(af+bg)''' - (af+bg)' = af''' + bg''' - af' - bg' = a(f''' - f') + b(g''' - g') = 0,$$

so V is closed under addition and scalar multiplication. Since V and W inherit all the other requisites of a vector space from the space of all differentiable functions, we see that, indeed, W is a subspace of the vector space V.

Now, if $f \in W$, then $f'' - f = 0$. This homogeneous differential equation has characteristic equation

$$0 = r^2 - 1 = (r + 1)(r - 1),$$

and so solutions to the differential equation are of the form $f(x) = c_1e^{-x} + c_2e^x$. Since our choice of f was arbitrary, we see that $\{e^{-x}, e^x\}$ gives a basis for W.

Now, an element of V is a solution of the equation $f''' - f' = 0$, which has characteristic equation

$$0 = r^3 - r = r(r^2 - 1) = r(r + 1)(r - 1),$$

and so solutions are of the form $c_1e^{-x} + c_2e^x + c_3$; hence, the basis for W can be expanded to the basis $\{e^{-x}, e^x, 1\}$ for V.

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3

Let n be an integer, and let $\alpha_1, \ldots, \alpha_{n+1}$ be distinct real numbers. Let $P_n \subset \mathbb{R}[x]$ be the vector space of polynomials of degree $\leq n$. Define $F : P_n \to \mathbb{R}^{n+1}$ by $f \mapsto (f(\alpha_1), \ldots, f(\alpha_{n+1}))$.

(a): Show that F is an isomorphism.

Proof. First, we need to show that F is linear. Suppose $f, g \in P_n$ and $a, b \in \mathbb{R}$. Then

$$\begin{aligned} F(af+bg) &= ((af+bg)(\alpha_1), \ldots, (af+bg)(\alpha_{n+1})) \\ &= (af(\alpha_1)+bg(\alpha_1), \ldots, af(\alpha_{n+1})+bg(\alpha_{n+1})) \\ &= (af(\alpha_1), \ldots, af(\alpha_{n+1})) + (bg(\alpha_1, \ldots, bg(\alpha_{n+1})) \\ &= a(f(\alpha_1), \ldots, f(\alpha_{n+1})) + b(g(\alpha_1), \ldots, g(\alpha_{n+1})) \\ &= aF(f) + bF(g), \end{aligned}$$

so F is linear.

Now, since $1, x, x^2, \ldots, x^n$ is a basis for P_n, $\dim P_n = n+1$. Thus, we need only show that F is injective to be assured that it is an isomorphism. To that end, suppose $f \in \ker F$. Then

$$(0, \ldots, 0) = F(f) = (f(\alpha_1), \ldots, f(\alpha_{n+1}),$$

so $\alpha_1, \ldots, \alpha_{n+1}$ are roots of f. Since f is of degree $\leq n$, f can have at most n roots if it is nonzero. Thus, we conclude that f must be the zero polynomial. Since our choice of $f \in \ker F$ was arbitrary, we conclude that $\ker F = \{0\}$, so F is injective and, thus, an isomorphism. $\square$

(b): *Explicitly* find $F^{-1}(e_1), \ldots, F^{-1}(e_{n+1})$ in the case $n = 3$, $\alpha_j = j$ for $j = 1, 2, 3, 4$.

Answer: We know that, if a_1, a_2, a_3 are the roots of a degree 3 polynomial f, then

$$f(x) = c(x-a_1)(x-a_2)(x-a_3)$$

for some constant $c \in \mathbb{R}$. Hence, if $F(f_1) = e_1 = (1, 0, 0, 0)$, then f_1 has roots $2, 3, 4$, so

$$f_1(x) = c(x-2)(x-3)(x-4).$$

Furthermore, $f_1(1) = 1$, so

$$1 = f_1(1) = c(1-2)(1-3)(1-4) = c(-6),$$

so $c = -1/6$. Hence,

$$f_1(x) = \frac{-1}{6}(x-2)(x-3)(x-4) = \frac{-1}{6}x^3 + \frac{3}{2}x^2 - \frac{13}{3}x + 4.$$

Similarly, we see that

$$\begin{aligned} F^{-1}(e_2) = f_2(x) &= \tfrac{1}{2}(x-1)(x-3)(x-4) = \tfrac{1}{2}x^3 - 4x^2 + \tfrac{19}{2}x - 6 \\ F^{-1}(e_3) = f_3(x) &= \tfrac{-1}{2}(x-1)(x-2)(x-4) = \tfrac{-1}{2}x^3 + \tfrac{7}{2}x^2 - 7x + 4 \\ F^{-1}(e_4) = f_4(x) &= \tfrac{1}{6}(x-1)(x-2)(x-3). = \tfrac{1}{6}x^3 - x^2 + \tfrac{11}{6}x - 1 \end{aligned}$$

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(c): Deduce that $F^{-1}(e_1), \ldots, F^{-1}(e_{n+1})$ form a basis of P_n. In the case considered in (b), express x as a linear combination of them.

Proof. Since F is an isomorphism, F^{-1} is an isomorphism as well and, thus, must map basis elements of P_n to basis elements of $\mathbb{R}^{n+1}$. Since $e_1, \dots, e_{n+1}$ form a basis for $\mathbb{R}^{n+1}$, it must be the case that $F^{-1}(e_1), \dots, F^{-1}(e_{n+1})$ form a basis for P_n.

Now, in the case considered in (b) above, we see that

$$F(x) = (1, 2, 3, 4) = 1e_1 + 2e_2 + 3e_3 + 4e_4.$$

Therefore,

$$\begin{aligned} x = F^{-1}(1,2,3,4) &= F^{-1}(e_1 + 2e_2 + 3e_3 + 4e_4) \\ &= F^{-1}(e_1) + 2F^{-1}(e_2) + 3F^{-1}(e_3) + 4F^{-1}(e_4) \\ &= f_1 + 2f_2 + 3f_3 + 4f_4 \end{aligned}$$

Double-checking with the solutions for f_1, f_2, f_3, f_4, we see that

$$\begin{aligned} f_1 + 2f_2 + 3f_3 + 4f_4 &= (\tfrac{-1}{6}x^3 + \tfrac{3}{2}x^2 - \tfrac{13}{3}x + 4) + 2(\tfrac{1}{2}x^3 - 4x^2 + \tfrac{19}{2}x - 6) \\ &\quad +3(\tfrac{-1}{2}x^3 + \tfrac{7}{2}x^2 - 7x + 4) + 4(\tfrac{1}{6}x^3 - x^2 + \tfrac{11}{6}x - 1) \\ &= (\tfrac{-1}{6}x^3 + \tfrac{3}{2}x^2 - \tfrac{13}{3}x + 4) + (x^3 - 8x^2 + 19x - 12) \\ &\quad +(\tfrac{-3}{2}x^3 + \tfrac{21}{2}x^2 - 21x + 12) + (\tfrac{2}{3}x^3 - 4x^2 + \tfrac{22}{2}x - 4) \\ &= (\tfrac{-1}{6} + 1 - \tfrac{3}{2} + \tfrac{2}{3})x^3 + (\tfrac{3}{2} - 8 + \tfrac{21}{2} - 4)x^2 \\ &\quad +(-\tfrac{13}{3} + 19 - 21 + \tfrac{22}{2})x + (4 - 12 + 12 - 4) \\ &= x, \end{aligned}$$

just as we suspected. □

4

(a): If V and W are vector spaces over a field K, and if $F : V \to W$ is a homomorphism, let $F^* : W^* \to V^*$ be the map on dual spaces given by $F^*(\phi) = \phi \circ F$. Show that $F \mapsto F^*$ defines a homomorphism $\text{Hom}(V, W) to \text{Hom}(W^*, V^*)$, satisfying $(F \circ G)^* = G^* \circ F^*$ if $F : V \to W$ and $G : U \to V$.

Proof. Let $F, G \in \text{Hom}(V, W)$. Then, if $a, b \in K$

$$aF + bG \mapsto (aF + bG)^*$$

and, if $\phi \in W^*$, then

$$(aF+bG)^*(\phi) = \phi\circ(aF+bG) = \phi\circ aF+\phi\circ bG = a(\phi\circ F)+b(\phi\circ G) = aF^*(\phi)+bG^*(\phi),$$

since ϕ is linear, so $F \mapsto F^*$ defines a homomorphism.

Now, if $F : V \to W$ and $G : U \to V$ are homomorphisms and $\phi \in W^*$, then

$$(G^*\circ F^*)(\phi) = G^*(F^*(\phi)) = G^*(\phi\circ F) = (\phi\circ F)\circ G = \phi\circ(F\circ G) = (F\circ G)^*(\phi),$$

so we see that $G^* \circ F^* = (F \circ G)^*$. □

(b): Show that the above map $\text{Hom}(V, W) \to \text{Hom}(W^*, V^*)$ is an isomorphism if V and W are finite dimensional.

Proof. Let us denote the map defined in (a) by $H : \mathrm{Hom}(V,W) \to \mathrm{Hom}(W^*,V^*)$. Now, suppose $F \in \mathrm{Hom}(V,W)$ and $H(F) = 0$. Then, for $\phi \in W^*$

$$0 = H(F)(\phi) = F^*(\phi) = \phi \circ F.$$

Since our choice of ϕ was arbitrary, we see that $\phi \circ F = 0$ for all $\phi \in W^*$. Now, if $F \neq 0$, then $F(v) \neq 0$ for some $v \in V$. Let $w_1, \ldots, w_n$ be a basis for W. Then

$$F(v) = a_1 w_1 + \ldots + a_n w_n$$

for some $a_1, \ldots, a_n \in K$ such that $a_k \neq 0$ for at least one $k \in \{1, \ldots, n\}$. Define $\phi : W \to K$ such that $\phi(w_k) = 1$ and $\phi(w_i) = 0$ for all $i \neq k$. Then $\phi \in W^*$ and $\phi(F(v)) = a_k$. However, this contradicts the fact that $\phi \circ F = 0$ for all $\phi \in W^*$. From this contradiction, then, we conclude that $F = 0$.

Hence, ker $H = \{0\}$. Since H is an injective homomorphism and $\mathrm{Hom}(V,W)$ and $\mathrm{Hom}(W^*,V^*)$ have the same dimension, H is also surjective and, therefore, an isomorphism. □

(c): Show that if $0 \to U \to V \to W \to 0$ is exact, then so is $0 \to W^* \to V^* \to U^* \to 0$.

Proof. First, name the maps as follows:

$$\begin{tikzcd}
0 \arrow[r] & U \arrow[r, "G"] \arrow[d] & V \arrow[r, "F"] \arrow[d] & W \arrow[r] \arrow[d] & 0 \\
0 & U^* \arrow[l] & V^* \arrow[l, "G^*"'] & W^* \arrow[l, "F^*"'] & 0 \arrow[l]
\end{tikzcd}$$

In order to show that $0 \to W^* \to V^* \to U^* \to 0$ is exact, we need to show that following three facts:

$$\ker F^* = 0; \quad \ker G^* = \text{image } F^*; \quad \text{image } G^* = U^*.$$

First, suppose $\phi \in W^*$ such that $F^*(\phi) = 0$. Then

$$0 = F^*(\phi) = \phi \circ F;$$

since F is surjective (because the first sequence is exact), the image of F is all of W, so $\phi(w) = 0$ for all $w \in W$. This, in turn, implies that $\phi = 0$, so we see that ker $F^* = 0$.

To show that ker $G^* =$ image F^*, suppose $\phi \in W^*$. Then, since the first sequence is exact, $F \circ G = 0$, so

$$(G^* \circ F^*)(\phi) = G^*(F^*(\phi)) = G^*(\phi \circ F) = \phi \circ (F \circ G) = \phi \circ 0 = 0.$$

Hence, image $F^* \subset$ ker G^*. On the other hand, suppose $\phi \in$ ker G^*. Then

$$0 = G^*(\phi)(u) = \phi \circ G(u)$$

for all $u \in U$. Now, since the first sequence is exact, for all $v \in \ker F$, there exists $uinU$ such that $G(u) = v$. Hence, we see that

$$\phi(\ker F) = 0.$$

Now, let $w \in W$. Then $w = F(v)$ for some $v \in V$. Define $\psi : W \to K$ by

$$\psi(w) = \phi(v).$$

To see that this is well-defined, suppose $w \in W$ and $w = F(v_1) = F(v_2)$. Then

$$0 = F(v_1) - F(v_2) = F(v_1 - v_2),$$

so $v_1 - v_2 \in \ker F$. Now, since $\phi(\ker F) = 0$, we see that

$$0 = \phi(v_1 - v_2) = \phi(v_1) - \phi(v_2),$$

so $\phi(v_1) = \phi(v_2)$, so $\psi(w)$ is well-defined. Additionally, if $w_1, w_2 \in W$ and $a, b \in K$, then there exist $v_1, v_2 \in V$ such that $F(v_1) = w_1$, $F(v_2) = w_2$), so

$$aw_1 + bw_2 = aF(v_1) + bF(v_2) = F(av_1 + bv_2),$$

so

$$\psi(aw_1 + bw_2) = \phi(av_1 + bv_2) = a\phi(v_1) + b\phi(v_2) = a\psi(w_1) + b\psi(w_2),$$

so $\psi \in W^*$. Finally, we see that

$$F^*(\psi)(v) = \psi \circ F(v) = \phi(v),$$

so $F^*(\psi) = \phi$ and, hence, $\ker G^* \subset \text{image } F^*$. Since containment goes both way, we conclude that $\ker G^* = \text{image } F^*$.

Finally, to show that image $G^* = U^*$, let $\phi \in U^*$. If $v \in V$, then $v = v_1 + v_2$ for $v_1 \in \text{image } G$ and $v_2 \in (\text{image } G)^\perp$ and this decomposition is unique. Let $\psi : V \to K$ be defined by

$$\psi(v) = \phi(u_1)$$

where $v = v_1 + v_2$ and $v_1 = G(u_1)$. Now, suppose $v = v_1 + v_2$ and $G(u_1) = G(u_1') = v_1$. Then

$$0 = G(u_1) - G(u_1') = G(u_1 - u_1'),$$

so $u_1 - u_1' \in \ker G = \{0\}$; hence, $u_1 = u_1'$, so we see that, since the decomposition $v = v_1 + v_2$ is unique, ψ is well-defined. Now, if $a, b \in K$ and $v, v' \in V$ such that $v = v_1 + v_2 = G(u_1) + v_2$ and $v' = v_1' + v_2' = G(u_1') + v_2'$, then

$$\begin{aligned} \psi(av + bv') &= \psi(a(v_1 + v_2) + b(v_1' + v_2')) \\ &= \psi((av_1 + bv_1') + (av_2 + bv_2')) \\ &= \phi(au_1 + bu_1') \\ &= a\phi(u_1) + b\phi(u_1') \\ &= a\psi(v) + b\psi(v'), \end{aligned}$$

so $\psi \in V^*$. Now, given how we've defined ψ, we see that, if $u \in U$,

$$G^*(\psi)(u) = \psi \circ G(u) = \psi(G(u)) = \phi(u),$$

so we see that, indeed, image $G^* = U^*$.

Therefore, we conclude that, indeed, $0 \to W^* \to V^* \to U^* \to 0$ is exact. □

(d): What if instead we consider modules over rings?

Answer: It's clear that the proof given in (a) works just as well for modules M and N over a ring R as for vector spaces. Furthermore, the proofs in (b) and (c) still work if M and N are free modules of finite rank. To see that the proof in (b) does not work for general modules, consider the $\mathbb{Z}$-modules $M = 2\mathbb{Z}$ and $N = \mathbb{Z}$. Let $F : 2\mathbb{Z} \to \mathbb{Z}$ be the inclusion map. Then we want to show that $F^* : \mathbb{Z}^* \to (2\mathbb{Z})^*$ is not surjective. To that end, let $f : 2\mathbb{Z} \to \mathbb{Z}$ be defined by

$$f(a) = \frac{1}{2}a.$$

Then, for $a, b \in 2\mathbb{Z}$ and $c, d \in \mathbb{Z}$,

$$f(ca + db) = \frac{1}{2}(ca + db) = \frac{1}{2}ca + \frac{1}{2}db = c\left(\frac{1}{2}a\right) + d\left(\frac{1}{2}b\right) = cf(a) + df(b),$$

so, in fact, $f \in (2\mathbb{Z})^*$. Now, if $g \in \mathbb{Z}^*$, then $g(1) = n$ for some $n \in \mathbb{Z}$ and, since g is linear, $g(a) = na$ for all $a \in \mathbb{Z}$. Hence,

$$F^*(g)(2) = (g \circ F)(2) = g(F(2)) = g(2) = 2n \neq 1$$

for any $n \in \mathbb{Z}$, so we see that $F^*(g) \neq f$ no matter what $g \in \mathbb{Z}^*$ we choose, so $F^* : \mathbb{Z}^* \to (2\mathbb{Z})^*$ is not an isomorphism.

As for exact sequences, consider the sequence of $\mathbb{Z}$-modules

$$0 \longrightarrow 2\mathbb{Z} \xrightarrow{G} \mathbb{Z} \xrightarrow{F} \mathbb{Z}/2 \longrightarrow 0.$$

This sequence is certainly exact if G is the inclusion map and F is the standard projection. We want to show that the induced sequence

$$0 \longrightarrow (\mathbb{Z}/2)^* \xrightarrow{F^*} \mathbb{Z}^* \xrightarrow{G^*} (2\mathbb{Z})^* \longrightarrow 0$$

is not exact. Now, the only homomorphism $\mathbb{Z}/2 \to \mathbb{Z}$ is the trivial homomorphism, so $(\mathbb{Z}/2)^* = 0$ so, in order for this sequence to be exact, it must be the case that $G^* : \mathbb{Z}^* \to (2\mathbb{Z})^*$ is an isomorphism. However, we just showed that this is not an isomorphism, so we see that the sequence is not exact.

♣

5

For any finite dimensional vector space V with basis $B = \{e_1, \dots, e_n\}$, and dual basis $B^* = \{\delta_1, \dots, \delta_n\}$ of V^*, define $\phi_{V,B} : V \to V^*$ by $\sum_1^n a_i e_i \mapsto \sum_1^n a_i \delta_i$, and let $\psi_{V,B} = \phi_{V^*,B^*} \circ \phi_{V,B}$.

(a): Show that $\phi_{V,B} : V \to V^*$ is an isomorphism, but that it depends on the choice of B.

Proof. To see that $\phi_{V,B}$ is a homomorphism, suppose $v, w \in V$. Then

$$v = \sum_{i=1}^n a_i e_i$$

and

$$w = \sum_{j=1}^n b_j e_j$$

for $a_1, \dots, a_n, b_1, \dots, b_n \in K$. Then

$$\begin{aligned} \phi_{V,B}(v+w) &= \phi_{V,B}\left(\sum a_i e_i + \sum b_j e_j\right) \\ &= \phi_{V,B}\left(\sum (a_i + b_i) e_i\right) \\ &= \sum (a_i + b_i)\delta_i \\ &= \sum a_i \delta_i + \sum b_j \delta_j \\ &= \phi_{V,B}\left(\sum a_i e_i\right) + \phi_{V,B}\left(\sum b_j e_j\right) \\ &= \phi_{V,B}(v) + \phi_{V,B}(w), \end{aligned}$$

so $\phi_{V,B}$ is linear. Furthermore, if $f \in V^*$, then $f = \sum a_i \delta_i$ for some $a_i \in K$. Then

$$f = \sum a_i \delta_i = \phi_{V,B}\left(\sum a_i e_i\right),$$

so $\phi_{V,B}$ is surjective. Therefore, since $\phi_{V,B}$ is a surjective homomorphism between finite-dimensional vector spaces of the same dimension, $\phi_{V,B}$ must be an isomorphism.

To see that this isomorphism depends on the choice of basis, let $B' = \{f_1, \dots, f_n\}$ be another basis for V and let $\{\beta_1, \dots, \beta_n\}$ be the associated dual basis. Then $e_1 = \sum a_i f_i$ for $a_1, \dots, a_n \in K$ and we may assume that $\sum a_i^2 \neq 1$ (if $\sum a_i^2 = 1$, then just scale each f_i by some $c \neq 1$ such that $c^2 \neq 1$ and then $e_1 = \sum a_i c^{-1}(c f_i)$ and $\sum (a_i c^{-1})^2 = c^{-2} \neq 1$). Now, certainly

$$\phi_{V,B}(e_1) = \delta_1$$

and

$$\phi_{V,B'}(e_1) = \phi_{V,B'}\left(\sum a_i f_i\right) = \sum a_i \beta_i.$$

Now, $\delta_1(e_1) = 1$, but

$$\sum a_i \beta_i(e_1) = \sum a_i \beta_i \left(\sum a_j f_j\right) = \sum a_i^2 \neq 1,$$

so we see that $\phi_{V,B} \neq \phi_{V,B'}$; hence, the isomorphism $\phi_{V,B}$ depends on the choice of basis B. $\square$

(b): Show that $\psi_{V,B} : V \to V^{**}$ is an isomorphism which is independent of the choice of B (so we may denote it ψ_V). For $v \in V$, show that $\psi_V(v)$ is the element of V^{**} taking $f \in V^*$ to $f(v)$.

Proof. Let $B^{**} = \{\gamma_1, \ldots, \gamma_n\}$ be the double dual basis for V^{**} associated with the dual basis $B^* = \{\delta_1, \ldots, \delta_n\}$. Then, for $a_1, \ldots, a_n \in K$,

$$\psi_{V,B}\left(\sum a_i e_i\right) = \phi_{V^*,B^*} \circ \phi_{V,B}\left(\sum a_i e_i\right) = \phi_{V^*,B^*}\left(\sum a_i \delta_i\right) = \sum a_i \gamma_i.$$

Clearly, then, if $\sum b_j \gamma_j \in V^{**}$, it's clear that

$$\sum b_j \gamma_j = \psi_{V,B}\left(\sum b_j e_j\right),$$

so $\psi_{V,B}$ is surjective. Furthermore, if $v, w \in V$, then $v = \sum a_i e_i$ and $w = \sum b_j e_j$ for $a_1, \ldots, a_n, b_1, \ldots, b_n \in K$. Hence,

$$\begin{aligned}
\psi_{V,B}(v+w) &= \psi_{V,B}\left(\textstyle\sum a_i e_i + \sum b_j e_j\right) \\
&= \psi_{V,B}\left(\textstyle\sum (a_i + b_i) e_i\right) \\
&= \textstyle\sum (a_i + b_i)\gamma_i \\
&= \textstyle\sum a_i \gamma_i + \sum b_j \gamma_j \\
&= \psi_{V,B}\left(\textstyle\sum a_i e_i\right) + \psi_{V,B}\left(\textstyle\sum b_j e_j\right) \\
&= \psi_{V,B}(v) + \psi_{V,B}(w),
\end{aligned}$$

so $\psi_{V,B}$ is a homomorphism. Hence, since $\psi_{V,B}$ is a surjective homomorphism between vector spaces of the same dimension, $\psi_{V,B}$ is an isomorphism.

Now, suppose $B' = \{f_1, \ldots, f_n\}$ is another basis for V, with associated dual basis $B'^* = \{\beta_1, \ldots, \beta_n\}$ and double dual basis $B'^{**} = \{\alpha_1, \ldots, \alpha_n\}$. Now, $e_1 = \sum a_i f_i$ for some $a_1, \ldots, a_n \in K$. Then

$$\psi_{V,B}(e_1) = \gamma_1$$

and

$$\psi_{V,B'}(e_1) = \psi_{V,B'}\left(\sum a_i f_i\right) = \sum a_i \alpha_i.$$

Now, if $g \in V^*$, then $g = \sum b_j \delta_j$ for $b_1, \ldots, b_n \in K$. Hence,

$$\psi_{V,B}(e_1)(g) = \gamma_1(g) = \gamma_1\left(\sum b_j \delta_j\right) = \sum b_j \gamma_1(\delta_j) = b_1 = \left(\sum b_j \delta_j\right)(e_1) = g(e_1).$$

On the other hand, $g = \sum c_k \beta_k$ for $c_1, \ldots, c_n \in K$, so

$$g(e_1) = \sum c_k \beta_k(e_1) = \sum c_k \beta_k\left(\sum a_i f_i\right) = \sum c_k a_k.$$

Therefore,

$$\begin{aligned}
\psi_{V,B'}(e_1)(g) &= \textstyle\sum a_i \alpha_i(g) \\
&= \textstyle\sum a_i \alpha_i\left(\sum c_k \beta_k\right) \\
&= \textstyle\sum a_i c_i \\
&= g(e_1) \\
&=
\end{aligned}$$

$$psi_{V,B}(e_1)(g).$$

By the same argument, we see immediately that, for all $i = 1, \dots, n$,

$$\psi_{V,B}(e_i)(g) = g(e_1) = \psi_{V,B'}(e_i)(g),$$

so, since $\psi_{V,B}$ and $\psi_{V,B'}$ are both linear, we see that

$$\psi_{V,B}(v)(g) = g(v) = \psi_{V,B'}(v)(g)$$

for all $v \in V$ and $g \in V^*$. Therefore, we conclude that $\psi_{V,B}$ is independent of the choice of basis and that $\psi_V(v)$ is the element of V^{**} taking $g \in V^*$ to $g(v)$. $\square$

(c): Show that if $F : V \to W$ is a vector space homomorphism with induced homomorphism $F^{**} : V^{**} \to W^{**}$, then $\psi_W \circ F = F^{**} \circ \psi_V$.

Proof. Let $v \in V$. Then $v = \sum a_i e_i$ for $a_1, \dots, a_n \in K$ and $\{e_1, \dots, e_n\}$ a basis for V. Thus, if $f \in W^*$,

$$(\psi_W \circ F)(v)(f) = \psi_W(F(v))(f) = f(F(v)) = f \circ F(v).$$

On the other hand,

$$(F^{**} \circ \psi_V)(v)(f) = (\psi_V(v) \circ F)(f) = \psi_V(v)(F^*(f)) = \psi_V(f \circ F) = f \circ F(v).$$

Therefore, since our choices of $v \in V$ and $f \in V^*$ were arbitrary, we conclude that

$$\psi_W \circ F = F^{**} \circ \psi_V.$$

$\square$

6

If U is a subspace of a vector space V, then the *annihilator* of U is defined to be Ann $U = \{f \in V^* | f|_U = 0\}$.

(a): Show that Ann U is a subspace of V^*. When can it be all of V^*? When can it be 0?

Proof. Suppose $f, g \in$ Ann U. Let $u \in U$. Then

$$(f + g)(u) = f(u) + g(u) = 0 + 0 = 0;$$

since our choice of u was arbitrary, we see that $(f + g)|_U = 0$, so $f + g \in$ Ann U. If $c \in K$, then, if $u \in U$,

$$(cf)(u) = cf(u) = c(0) = 0,$$

so $cf \in$ Ann U. Thus, we see that Ann U is closed under addition and scalar multiplication; since Ann U inherits the rest of the properties of a vector space from V^*, we see that Ann U is a subspace of V^*.

Now, suppose Ann $U = V^*$. Then, for all $f \in V^*$, $f|_U = 0$. Therefore, it must be the case that $U = \{0\}$.

On the other hand, suppose Ann $U = 0$. Then there is no non-zero $f \in V^*$ such that $f|_U = 0$. Let $u_1, \dots, u_k$ be a basis for U and expand it to a basis $u_1, \dots, u_n$ for V. Let $f : V \to K$ be defined by $f(u_n) = 1$ and $f(u_i) = 0$ for $i \neq n$. Then, if $n \neq k$, we see that $f|_U = 0$ and $f \neq 0$. Therefore, we conclude that $k = n$, so $U = V$. □

(b): Let V be a finite dimensional vector space and let U, W be subspaces of V. If $V = U \times W$, show that $V^* = \text{Ann } U \times \text{Ann } W$.

Proof. Let $\phi \in V^*$ and $v \in V$. Then, since $V = U \times W$ and U and W are subspaces of V, it must be the case that v can be uniquely decomposed as $v = u + w$ for $u \in U$ and $w \in W$. Hence,

$$\phi(v) = \phi(u + w) = \phi(u) + \phi(w).$$

Now, define $\phi_1 : V \to K$ and $\phi_2 : W \to K$ be defined such that, if $v = u + w$, then

$$\phi_1(v) = \phi(u)$$

and

$$\phi_2(v) = \phi(w).$$

Now, suppose $v, v' \in V$ and $a, b \in K$. Then $v = u+w$ and $v' = u'+w'$ uniquely for $u, u' \in U$ and $w, w' \in W$. Then

$$\begin{aligned} \phi_1(av + bv') &= \phi_1(a(u + w) + b(u' + w')) \\ &= \phi_1((au + bu') + (aw + bw')) \\ &= \phi(au + bu') \\ &= a\phi(u) + b\phi(u') \\ &= a\phi_1(v) + b\phi_1(v'), \end{aligned}$$

so ϕ_1 is linear and, hence, $\phi_1 \in V^*$. Similarly,

$$\begin{aligned} \phi_2(av + bv') &= \phi_2((au + bu') + (aw + bw')) \\ &= \phi(aw + bw') \\ &= a\phi(w) + b\phi(w') \\ &= a\phi_2(v) + b\phi_2(v'), \end{aligned}$$

so $\phi_2 \in V^*$.

Furthermore, if $w \in W$, then $\phi_1(w) = \phi_1(0+w) = \phi(0) = 0$ and, if $u \in U$, then $\phi_2(u) = \phi_2(u+0) = \phi(0) = 0$. Hence, $\phi_1 \in \text{Ann } W$ and $\phi_2 \in \text{Ann } U$. Furthermore, note that for $v \in V$, v can be uniquely decomposed as $v = u + w$ and

$$\phi(v) = \phi(u + w) = \phi(u) + \phi(w) = \phi_1(v) + \phi_2(v),$$

so $\phi = \phi_1 + \phi_2$. Hence, since our choice of ϕ was arbitrary, we see that for all $\phi \in V^*$, $\phi = \phi_1 + \phi_2$ for $\phi_1 \in \text{Ann } W$ and $\phi_2 \in \text{Ann} U$.

To see that this decomposition is unique, suppose $\phi_1, \phi_1' \in \text{Ann } W$ and $\phi_2, \phi_2' \in \text{Ann } U$ such that $\phi_1 + \phi_2 = \phi_1' + \phi_2'$, then, for $v = u + w$

$$\phi_1(v) = \phi_1(u) = \phi_1(u) + \phi_2(u) = \phi_1'(u) + \phi_2'(u) = \phi_1'(u) = \phi_1'(v),$$

so $\phi_1 = \phi_1'$ Similarly,

$$\phi_2(v) = \phi_2(w) = \phi_1(w) + \phi_2(w) = \phi_1'(w) + \phi_2'(w) = \phi_2'(w) = \phi_2'(v),$$

so $\phi_2 = \phi_2'$. Hence, we conclude that the above decomposition is unique and, therefore, that

$$V^* = \text{Ann } U \times \text{Ann } W.$$

□

7

Call $T \in \text{End}(V)$ an *idempotent* if $T^2 = R$. Show that if V is finite dimensional and T is an idempotent, then there are subspaces $X, Y \subset V$ such that $V = X \times Y$, $T|_X = 0$, $T|_Y =$ identity. Deduce that with respect to some basis of V, the idempotent map T is given by a diagonal matrix whose diagonal entries are of the form $(1, 1, \dots, 1, 0, 0, \dots, 0)$.

Proof. Let T be an idempotent, and let $v \in V$. Let $w = Tv$. Then

$$Tw = T^2 v = Tv = w,$$

so

$$T(v - w) = Tv - Tw = w - w = 0,$$

so $v - w \in \ker T$. Then $v = w + (v - w)$ where $w \in$ image T and $v - w \in \ker T$. Now, suppose $v = w_1 + x_1 = w_2 + x_2$ for $w_1, w_2 \in$ image T and $x_1, x_2 \in \ker T$. Then

$$T(w_1) = T(w_1) + T(x_1) = T(w_1 + x_1) = T(v) = T(w_2 + x_2) = T(w_2) + T(x_2) = T(w_2).$$

Since T is idempotent,

$$w_1 = T(w_1) = T(v) = T(w_2) = w_2.$$

Furthermore,

$$x_1 = v - w_1 = v - w_2 = x_2,$$

so the decomposition is unique.

Hence, since our choice of $v \in V$ was arbitrary, we see that all vectors $v \in V$ can be decomposed uniquely as the sum of an element of image T and an element of $\ker T$ and so $V = X \times Y$ where $Y =$ image T and $X = \ker T$. Clearly, $T|_X = 0$ and, since T is idempotent, $T|_Y =$ identity.

Now, if $x_1, \dots, x_k$ is a basis of X and $y_1, \dots, y_l$ is a basis for Y, then $y_1, \dots, y_l, x_1, \dots, x_k$ is a basis for V. Then $T(y_i) = y_i$ and $T(x_j) = 0$, so the matrix of T with respect to this basis is given by a diagonal matrix with diagonal entries of the form

$$(\underbrace{1, \dots, 1}_{l}, \underbrace{0, \dots, 0}_{k})$$

□

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