

## GEOMETRY HW 9

CLAY SHONKWILER

1

Show that the Klein bottle  $K$  and  $\mathbb{RP}^2 \vee S^1$  have the same cohomology ring with  $\mathbb{Z}$  coefficients, but not with  $\mathbb{Z}/2$  coefficients.

**Answer:** The following is a  $\Delta$  complex of  $K$ :

Then the boundaries of each of the simplices is as follows:

$$\begin{aligned}\partial(p) &= 0 \\ \partial(a) = \partial(b) = \partial(c) &= 0 \\ \partial(\sigma_1) &= a + b - c \\ \partial(\sigma_2) &= c + a - b.\end{aligned}$$

Now, in cohomology, if  $\Delta^*$  denotes the dual of  $\Delta$  for any simplex  $\Delta$ , then

$$\begin{aligned}\delta(p^*) &= 0 \\ \delta(a^*) &= \sigma_1^* + \sigma_2^* \\ \delta(b^*) &= \sigma_1^* - \sigma_2^* \\ \delta(c^*) &= \sigma_2^* - \sigma_1^* \\ \delta(\sigma_1^*) &= \delta(\sigma_2^*) = 0.\end{aligned}$$

Then  $p^*$  is the generator of  $H^0(K, \mathbb{Z}) = \mathbb{Z}$ .  $b^* + c^*$  is a generator of  $H^1(K, \mathbb{Z})$  and, since there's no linear combination of  $\delta(a^*)$ ,  $\delta(b^*)$  and  $\delta(c^*)$  yielding 0 with non-zero coefficient on  $\delta(a^*)$ , this is the only generator. There are no relations on the 1 cochains, so  $H^1(K, \mathbb{Z}) = \mathbb{Z}$ . Now,  $\sigma_1^*$  is a generator of  $H^2$  and, since  $\sigma_1^* - \sigma_2^*$  and  $\sigma_1^* + \sigma_2^*$  are coboundaries,  $\sigma_1^* = \sigma_2^*$  and  $2\sigma_1^* = 0$  in cohomology, so  $H^2(K, \mathbb{Z}) = \mathbb{Z}/2$ .

In computing the ring structure with  $\mathbb{Z}$  coefficients, the only possible non-trivial product is  $(b^* + c^*) \cup (b^* + c^*)$ . Now,

$$\begin{aligned} (b^* + c^*) \cup (b^* + c^*)(\sigma_1) &= (b^* + c^*)(a) \cdot (b^* + c^*)(b) = 0 \cdot 1 = 0 \\ (b^* + c^*) \cup (b^* + c^*)(\sigma_2) &= (b^* + c^*)(c) \cdot (b^* + c^*)(a) = 1 \cdot 0 = 0, \end{aligned}$$

so  $(b^* + c^*) \cup (b^* + c^*) = 0$ . Hence, the ring structure with  $\mathbb{Z}$  coefficients is simply  $\mathbb{Z}[a, b]/(a^2, b^2, ab)$ .

On the other hand, in  $\mathbb{Z}/2$  coefficients,  $p^*$  is still a generator of  $\mathbb{Z}/2$  and  $\sigma_1^* = \sigma_2^*$  is still 2-torsion and generates  $H^2$ , but in  $H^1$  we pick up another generator. Modulo 2,  $\delta(a^*) + \delta(b^*) = 0$ , so  $a^* + b^*$  is also a generator of  $H^1$ , along with  $b^* + c^*$ . Since  $(a^* + b^*) + (b^* + c^*) = a^* + c^*$  modulo 2, these are the only generators, so  $H^1(K, \mathbb{Z}/2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ . In the ring structure, the only possible non-trivial products are the following:

$$\begin{aligned} (a^* + b^*) \cup (a^* + b^*) &= \sigma_1^* \\ (a^* + b^*) \cup (b^* + c^*) &= \sigma_1^* \\ (b^* + c^*) \cup (a^* + b^*) &= \sigma_2^* = \sigma_1^* \\ (b^* + c^*) \cup (b^* + c^*) &= 0 \end{aligned}$$

Hence, the ring structure on  $H^*(K, \mathbb{Z}/2)$  is  $\mathbb{Z}/2[a, b]/(a^3, b^2, a^3 - ab)$ .

On the other hand, we can make the following  $\Delta$  complex for  $\mathbb{RP}^2 \vee S^1$ :

Then the boundaries are as follows:

$$\begin{aligned} \partial(p) &= \partial(q) = 0 \\ \partial(a) &= \partial(b) = q - p \\ \partial(c) &= \partial(d) = 0 \\ \partial(\sigma_1) &= c + b - a \\ \partial(\sigma_2) &= c + a - b. \end{aligned}$$

To compute cohomology, we see that the coboundaries are:

$$\begin{aligned}\delta(p^*) &= -a^* - b^* \\ \delta(q^*) &= a^* + b^* \\ \delta(a^*) &= \sigma_2^* - \sigma_1^* \\ \delta(b^*) &= \sigma_1^* - \sigma_2^* \\ \delta(c^*) &= \sigma_1^* + \sigma_2^* \\ \delta(d^*) &= 0 \\ \delta(\sigma_1^*) &= \delta(\sigma_2^*) = 0.\end{aligned}$$

Hence,  $p^* + q^*$  is the generator of  $H^0(\mathbb{RP}^2 \vee S^1, \mathbb{Z}) = \mathbb{Z}$ . Now,  $\delta(a^* + b^*) = 0$ , but  $a^* + b^*$  is a coboundary. We can't get any other linear combinations of  $\delta(a^*)$ ,  $\delta(b^*)$  and  $\delta(c^*)$  that is zero that isn't already a multiple of  $a^* + b^*$ . Hence, the only generator of  $H^1(\mathbb{RP}^2 \vee S^1, \mathbb{Z}) = \mathbb{Z}$  is  $d^*$ . Finally,  $\sigma_1^* = \sigma_2^*$  is 2-torsion and is a generator of  $H^2(\mathbb{RP}^2 \vee S^1, \mathbb{Z}) = \mathbb{Z}/2$ .

Now, in terms of a ring structure on  $H^*(\mathbb{RP}^2 \vee S^1, \mathbb{Z})$ , the only possible non-trivial product is  $d^* \cup d^*$ . However,

$$\begin{aligned}(d^* \cup d^*)(\sigma_1) &= d^*(c) \cdot d^*(b) = 0 \\ (d^* \cup d^*)(\sigma_2) &= d^*(c) \cdot d^*(a) = 0,\end{aligned}$$

so  $d^* \cup d^* = 0$ . Hence, the ring structure on  $H^*(\mathbb{RP}^2 \vee S^1, \mathbb{Z})$  is simply  $\mathbb{Z}[a, b]/(a^2, b^2, ab)$ , which is precisely what we computed to be the ring structure on  $H^*(K, \mathbb{Z})$ .

On the other hand, in  $\mathbb{Z}/2$  coefficients, we still have most of the same generators, except that now  $\delta(b^* + c^*) = 0$ ; since  $b^* + c^*$  is not a coboundary,  $b^* + c^*$  joins  $d^*$  as a generator for  $H^1$ . Since  $a^* + b^*$  is a coboundary and  $(b^* + c^*) + (a^* + b^*) = a^* + c^*$ , we see that these are the only generators, so  $H^1(\mathbb{RP}^2 \vee S^1, \mathbb{Z}/2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ . In the other dimensions, we have the same generators, so  $H^0(\mathbb{RP}^2 \vee S^1, \mathbb{Z}/2) = \mathbb{Z}/2$  and  $H^2(\mathbb{RP}^2 \vee S^1, \mathbb{Z}/2) = \mathbb{Z}/2$ . Now, as to ring structure, the only possible non-trivial products are the following:

$$\begin{aligned}(b^* + c^*) \cup (b^* + c^*) &= \sigma_1^* \\ (b^* + c^*) \cup d^* &= 0 \\ d^* \cup (b^* + c^*) &= 0 \\ d^* \cup d^* &= 0,\end{aligned}$$

so the ring structure on  $H^*(\mathbb{RP}^2 \vee S^1, \mathbb{Z}/2)$  is

$$\mathbb{Z}/2[a, b]/(a^3, b^2, ab).$$

This is different than the ring structure on  $H^*(K, \mathbb{Z}/2)$  which, recall, was  $\mathbb{Z}/2[a, b]/(a^3, b^2, a^2 - ab)$ .



## 2

Let  $f : S^n \rightarrow T^n$  be a continuous map. Show that  $f_*$  and  $f^*$  are 0 in positive degrees. How about  $f : T^n \rightarrow S^n$ ?

*Proof.* We need to assume  $n > 1$ , since when  $n = 1$  we have a map  $S^1 \rightarrow S^1$ , which need not induce the zero map on homology/cohomology (for example, the identity map induces an isomorphism in homology/cohomology). So assume  $n > 1$  and let  $f : S^n \rightarrow T^n$  be continuous. Then, since

$$T^n = \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{n \text{ times}},$$

we can view  $f : S^n \rightarrow T^n$  as a map  $f : S^n \rightarrow S^1 \times \cdots \times S^1$  where

$$f(x) = (f_1(x), \dots, f_n(x)),$$

where each  $f_i : S^n \rightarrow S^1$  is continuous. Now, for any such  $i$ , since  $n > 1$ ,  $S^n$  is simply connected so we can lift  $f_i$  to  $\tilde{f}_i : S^n \rightarrow \mathbb{R}$ , the universal cover of  $S^1$ :

$$\begin{array}{ccc} & & \mathbb{R} \\ & \nearrow \tilde{f}_i & \downarrow \sigma \\ S^n & \xrightarrow{f_i} & S^1 \end{array}$$

Now,  $\text{im } \tilde{f}_i$  is connected and, hence, contractible in  $\mathbb{R}$ , so  $\tilde{f}_i$  is homotopic to a constant map by some homotopy  $H_{i,t}$  such that  $H_{i,0}(x) = \tilde{f}_i(x)$  and  $H_{i,1}(x) = c$  for all  $x \in S^n$  and some  $c \in \mathbb{R}$ . Now,

$$\sigma \circ H_{i,0}(x) = \sigma \circ \tilde{f}_i(x) = f_i(x), \quad \sigma \circ H_{i,1}(x) = \sigma(c),$$

and  $\sigma \circ H_{i,t}$  is continuous and, therefore, a homotopy of  $f_i$  with the constant map  $x \mapsto \sigma(c)$ . Putting these homotopies together, we see that  $f$  itself must be homotopic to a constant map from  $S^n$  to  $T^n$ . Since the constant map induces zero maps in homology and cohomology, this in turn implies that  $f_* = 0$  and  $f^* = 0$ .

Turning to the case where we have a continuous  $f : S^n \rightarrow T^n$ , note, first, that  $H_i(S^n) = 0$  and  $H^i(S^n) = 0$  for  $0 < i < n$ , so the only possible dimension in which  $f_*$  or  $f^*$  could be non-zero is in dimension  $n$ . In order to find an example of a map  $f : T^n \rightarrow S^n$ , let us consider the CW complexes of  $T^n$  and  $S^n$ . The CW complex of  $S^n$  is easy, simply an  $n$ -cell attached to a 0-cell. The CW complex for  $T^n$  is the product of the CW complexes for  $n$  copies of  $S^1$ ; hence, there are  $\binom{n}{i}$   $i$ -cells in the CW complex (since there are  $\binom{n}{i}$  ways to choose  $i$  1-cells to product together). Furthermore, the boundary of each  $i$ -cell is simply the product of the boundaries of the 1-cells that went into the product producing that  $i$ -cell; since the boundary of the 1-cell in the CW complex of  $S^1$  is the 0-cell, we see that the boundary of each  $i$ -cell is just the 0-cell and so the boundary and coboundary maps are just the zero maps.

Therefore, the single  $n$ -cell in the CW complex of  $T^n$  is the generator of  $H_n(T^n)$  and its dual is the generator of  $H^n(T^n)$ . Let us define the map  $f$  on the CW complexes that sends the  $(n-1)$ -skeleton of the CW complex for  $T^n$  to the zero cell and maps the  $n$ -cell onto the  $n$ -cell in the CW complex of  $S^n$  by the identity map. Since all the boundaries in the CW complex of  $T^n$  are 0, this map really is continuous and so induces a continuous  $f : T^n \rightarrow S^n$ . Since  $f$  acts as the identity on the  $n$ -cell,  $f_* : H_n(T^n) \rightarrow H_n(S^n)$  maps the generator to the generator, so  $f_* \neq 0$  (in fact,  $f_* : H_n(T^n) \rightarrow H_n(S^n)$  is an isomorphism). Similarly,  $f^* : H^n(S^n) \rightarrow H^n(T^n)$  also maps generator to generator, and so  $f^* \neq 0$  either.  $\square$

3

Can you find a  $\Delta$  complex which computes the ring structure of  $\mathbb{C}\mathbb{P}^2$ ?  
No.

4

Say something smart about the cup products in a connected sum  $M\#N$  in terms of those of  $M$  and  $N$ . Complete answers are not necessary.

**Answer:** We use the Mayer-Vietoris sequence:

$$\dots \rightarrow H^i(U \cup V) \rightarrow H^i(U) \oplus H^i(V) \rightarrow H^i(U \cap V) \rightarrow H^{i+1}(U \cup V) \rightarrow \dots$$

where  $U \approx M$  ( $U$  is a  $M$  plus a tiny strip at the point of attachment of the connected sum),  $V \approx N$  and  $U \cap V$  can be deformation retracted to  $S^{n-1}$ . Then for  $0 < i < n-1$ , we know that  $H^i(S^{n-1}) = 0$ ,  $U$  is homotopy equivalent to  $M - \{pt\}$  and  $V$  is homotopy equivalent to  $N - \{pt\}$ . Since  $\mathbb{R}^n - \{pt\}$  can be deformation retracted to  $S^{n-1}$  and  $H^i(S^{n-1}) = 0$  for  $0 < i < n-1$ ,  $H^i(M - \{p\}) = H^i(M)$  and  $H^i(N - \{pt\}) = H^i(N)$ . Also, for these  $i$ ,  $H^i(U \cap V) = H^i(S^{n-1}) = 0$ , we see that the Mayer-Vietoris sequence reduces to

$$\dots \rightarrow 0 \rightarrow H^i(M\#N) \rightarrow H^i(M) \oplus H^i(N) \rightarrow 0 \rightarrow \dots,$$

so for these  $i$ ,  $H^i(M\#N) = H^i(M) \oplus H^i(N)$ . We're not, in general, going to be able to say very much about  $H^n$  and  $H^{n-1}$ .

Now, if  $\alpha \in H^i(M\#N)$  and  $\beta \in H^j(M\#N)$  where  $i+j < n-1$ , then, since  $H^i(M\#N) \simeq H^i(M) \oplus H^i(N)$  and  $H^j(N) \simeq H^j(M) \oplus H^j(N)$  we can think of  $\alpha = \gamma + \delta$  for  $\gamma \in H^i(M)$  and  $\delta \in H^i(N)$  and  $\beta = \zeta + \xi$  for  $\zeta \in H^j(M)$  and  $\xi \in H^j(N)$ . Also, recall from the proof of Mayer-Vietoris that  $C_{i+j}(M\#N) \simeq C_{i+j}(M+N)$ , so, and for any  $\sum_i \sigma_i \in C_{i+j}(M\#N)$ ,  $\sum_k \sigma_k = \sum_k \rho_k + \sum_k \tau_k$  for  $\rho_k \in C_{i+j}(M)$  and  $\tau_k \in C_{i+j}(N)$ . Hence,

$$\alpha \cup \beta = (\gamma + \delta) \cup (\zeta + \xi) = \gamma \cup \zeta + \delta \cup \zeta + \gamma \cup \xi + \delta \cup \xi.$$

Now, an element in  $H^k(M)$  applied to some element of  $C_k(N)$  is going to be zero, and *vice versa*, so

$$\begin{aligned} \delta \cup \zeta \left( \sum_k \sigma_k \right) &= \delta \cup \zeta \left( \sum_k \rho_k + \sum_k \tau_k \right) \\ &= \sum_k (\delta \cup \zeta)(\rho_k) + \sum_k (\delta \cup \zeta)(\tau_k) \\ &= 0. \end{aligned}$$

Hence,  $\delta \cup \zeta = 0$ . A similar argument shows that  $\gamma \cup \xi = 0$ , so we see that

$$(\gamma + \delta) \cup (\zeta + \xi) = \gamma \cup \zeta + \delta \cup \xi.$$

Hence, for elements of grades  $i$  and  $j$  with  $i + j < n - 1$ , the cup product structure is just given by coordinate-wise products. Obviously, for elements of grades  $i$  and  $j$  with  $i + j > n$ , their products will be zero, so the only problem arises when  $i + j = n$  or  $i + j = n - 1$ . In these cases, I don't know, since we don't even know what the cohomology groups are in these dimensions.



DRL 3E3A, UNIVERSITY OF PENNSYLVANIA  
*E-mail address:* shonkwil@math.upenn.edu