GEOMETRY HW 11

CLAY SHONKWILER

1

Show that a compact orientable manifold of dimension $4n+2$ has even Euler characteristic. What can you say for other dimensions?

Proof. First, note that, since $H^i(M) \simeq H_i(M) \oplus T_{i-i}$, where T_{i-1} denotes the torsion in $H_{i-1}(M)$, the free rank of $H^i(M)$ is equal to the free rank of $H_i(M)$. If we denote the free rank of $H_i(M)$ by rk $H_i(M)$, then

$$\chi(M) = \sum_{i=0}^{4n+2} \text{rk } H_i(M).$$

Now, since M is compact and orientable, we know, by Poincaré Duality, that $H^i(M) \simeq H_{4n+2-i}(M)$ and, hence, that rk $H_i(M) = $ rk $H^i(M) = $ rk $H_{4n+2-i}(M)$. Therefore,

$$\chi(M) = \sum_{i=0}^{4n+2} \text{rk } (-1)^i H_i(M) = \text{rk } H^{2n+1}(M) + 2\sum_{i=0}^{2n} \text{rk } H_i(M).$$

$2\sum \text{rk } H_i(M)$ is certainly even, so the parity of $\chi(M)$ is equal to the parity of rk $H_{2n+1}(M) = $ rk $H^{2n+1}(M)$.

Consider the pairing

$$P : H^{2n+1}(M) \times H^{2n+1}(M) \to H^{4n+2}(M)$$

given by

$$(\alpha, \beta) \mapsto \alpha \cup \beta.$$

Since all we care about is the free rank of H^{2n+1}, we may as well consider this in cohomology with real coefficients where there is no torsion and, thus, this is a well-defined, non-degenerate pairing. Since $H^{2n+1}(M)$ is of odd grade in $H^*(M)$, $\alpha \cup \alpha = 0$ for all $\alpha \in H^{2n+1}(M)$ and $\alpha \cup \beta = -\beta \cup \alpha$ for all $\alpha, \beta \in H^{2n+1}(M)$. Now, since M is compact and orientable, $H^{4n+2}(M, \mathbb{R}) \simeq \mathbb{R}$ and since there is no torsion with real coefficients, $H^{2n+1}(M, \mathbb{R}) \simeq \mathbb{R}^k$ where $k = $ rk $H^{2n+1}(M)$. Hence, P simply defines an antisymmetric, bilinear form $\phi : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$, which is to say, a vector space homomorphism $\Phi : \mathbb{R}^k \otimes \mathbb{R}^k \to \mathbb{R}$. Hence, we can represent the map Φ with a non-singular $k \times k$ matrix A such that $A = -A^t$. Hence,

$$\det A = \det -A^t = (-1)^k \det A,$$

so, since $\det A \neq 0$, $(-1)^k = 1$. Therefore, k is even, which implies that rk $H^{2n+1}(M)$ is even and, hence, $\chi(M)$ is even.

Now, in other dimensions we can say some things. If M has dimension $2n+1$, then our above argument about free ranks of the homology groups still holds, so rk $H_i(M) =$ rk $H_{2n+1-i}(M)$, and hence, since $(-1)^i$ and $(-1)^{2n+1-i}$ have opposite parities,

$$\chi(M) = \sum_{i=0}^{2n+1} (-1)^i \text{rk } H_i(M) = 0.$$

Finally, if M has dimension $4n$, the we can't say anything in general about the Euler characteristic. Even when $n = 1$, we have the examples

$$\chi(S^4) = 2, \quad \chi(\mathbb{CP}^2) = 1 + 1 + 1 = 3.$$

□

2

You can now complete the problem about the cohomology of the connected sum of two n-dimensional manifolds. You only need to discuss the cases that you could not do before, i.e. in dimensions n and $n-1$. You can assume that they are compact, but not that they are orientable.

Answer: There are a few different cases to consider. If M and N are two compact n-dimensional manifolds, then we could have both orientable, only one orientable, or neither orientable. Now, if both M and N are orientable, then $M\#N$ is also orientable, so $H^n(M\#N, \mathbb{Z}) = \mathbb{Z} = H^n(M) = H^n(N)$. If either M or N is non-orientable (or both), then $M\#N$ is non-orientable, so $H^n(M\#N) = \mathbb{Z}/2$. Now, as before, we have the following piece of the exact sequence derived from Mayer-Vietoris:

(1)

$$\begin{array}{ccccccc} H^{n-2}(S^{n-1}) & \longrightarrow & H^{n-1}(M\#N) & \longrightarrow & H^{n-1}(M-\{p\}) \oplus H^{n-1}(N-\{q\}) & \longrightarrow & H^{n-1}(S^{n-1}) \\ & & & & & & \downarrow \\ & & 0 = H^n(S^{n-1}) & \longleftarrow & H^n(M-\{p\}) \oplus H^n(N-\{q\}) & \longleftarrow & H^n(M\#N) \end{array}$$

where $p \in M$ and $q \in N$. Since $M - \{p\}$ and $N - \{q\}$ are not compact, $H^n(M - \{p\}) = H^n(N - \{q\}) = 0$. Also, the proof given in PS5#5 demonstrates that $H^{n-1}(M - \{p\}) = H^{n-1}(M)$ for all compact, orientable M. Note also that $H^{n-1}(S^{n-1}) = \mathbb{Z}$. Hence, if M and N are both orientable, (1) reduces to

$$0 \to H^{n-1}(M\#N) \to H^{n-1}(M) \oplus H^{n-1}(N) \to \mathbb{Z} \to \mathbb{Z} \to 0.$$

Since $\ker \mathbb{Z} \to 0$ is all of $\mathbb{Z}$, the image of $\mathbb{Z} \to \mathbb{Z}$ is all of $\mathbb{Z}$, so $\mathbb{Z} \to \mathbb{Z}$ must be injective, meaning the image of the map into $\mathbb{Z}$ is zero. Thus, we can further reduce to

$$0 \to H^{n-1}(M\#N) \to H^{n-1}(M) \oplus H^{n-1}(N) \to 0.$$

Since this sequence splits, we see that, for M and N compact and orientable,

$$H^{n-1}(M\#N) \simeq H^{n-1}(M) \oplus H^{n-1}(N).$$

On the other hand, if M is a compact, non-orientable manifold, then, again using the same argument as in PS5#5, $H^{n-1}(M) \simeq H^{n-1}(M-\{p\}) \oplus \mathbb{Z}$. Therefore, if one of M and N is orientable and the other is not, then (1) reduces to

$$0 \to H^{n-1}(M\#N) \to H^{n-1}(M) \oplus H^{n-1}(N) \oplus \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2 \to 0.$$

Since this sequence is exact, $\mathbb{Z} \to \mathbb{Z}/2$ must be surjective; hence, the kernel of this map must be $2\mathbb{Z} \simeq \mathbb{Z}$ as a group. Therefore, the middle term surjects onto $2\mathbb{Z}$; this will account for one copy of $\mathbb{Z}$ in this middle term (of which there is obviously at least one), so the kernel of this map will simply be $H^{n-1}(M) \oplus H^{n-1}(N)$. Since $H^{n-1}(M\#N)$ must inject onto this kernel, we see immediately that

$$H^{n-1}(M\#N) \simeq H^{n-1}(M) \oplus H^{n-1}(N).$$

Finally, suppose both M and N are non-orientable. Then, (1) reduces to

$$0 \to H^{n-1}(M\#N) \to H^{n-1}(M) \oplus H^{n-1}(N) \oplus \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2 \to 0.$$

Again, the kernel of $\mathbb{Z} \to \mathbb{Z}/2$ is $2\mathbb{Z}$, which must also be the image of the middle term. This accounts for one of the $\mathbb{Z}$ terms in the middle term (again, this might not be one of the $\mathbb{Z}$'s that appears explicitly in the middle term; it might come from one of the other summands). Hence, $H^{n-1}(M\#N)$ surjects onto the kernel, which must be isomorphic toi

$$H^{n-1}(M) \oplus H^{n-1}(N) \oplus \mathbb{Z}.$$

Since $H^{n-1}(M\#N)$ must also inject into this kernel, we conclude that

$$H^{n-1}(M\#N) \simeq H^{n-1}(M) \oplus H^{n-1}(N) \oplus \mathbb{Z}$$

when M and N are both non-orientable.

♣

3

Let D act properly discontinuously on a topological oriented manifold M. Define what it means for the group action to be orientation-preserving, and show that the quotient is orientable. Indicate how the orientation homomorphism $O : \pi_1(M) \to \mathbb{Z}/2$ we discussed earlier for differentiable manifolds also works for topological manifolds.

Proof. First, recall the definition of an orientation on a topological manifold; that is, that there exists a map $\phi : M \to \bigsqcup_{p\in M} H_n(M, M-\{p\})$ such that

$$p \mapsto \alpha_p$$

where α_p is a generator of $H_n(M, M-\{p\})$ and, for each $p \in M$, there exists a neighborhood U of p homeomorphic to $\mathbb{R}^n$ such that there is an $\alpha_U \in H_n(M, M-U)$ that maps to α_p under the natural map $i_* : H_n(M, M_U) \to$

$H_n(M, M - \{p\})$ induced by the inclusion. First, note that, for $g \in D$, the map $L_g : M \to M$ given by $L_g(p) = g \cdot p$ induces a map $L_g : (M, M - \{p\}) \to (M, M - \{g \cdot p\})$ and, hence, a homomorphism

$$(L_g)_* : H_n(M, M - \{p\}) \to H_n(M, M - \{g \cdot p\}).$$

Then we will say that the action of D is orientation-preserving if the following diagram commutes for all $g \in D$ and all $p \in M$:

$$\begin{array}{ccc} H_n(M, M - \{p\}) & \xrightarrow{(L_g)_*} & H_n(M, M - \{g \cdot p\}) \\ \uparrow{\scriptstyle \phi} & & \uparrow{\scriptstyle \phi} \\ M & \xrightarrow{\quad L_g \quad} & M \end{array}$$

That is, that $(L_g)_* \circ \phi = \phi \circ L_g$. Now, suppose the action of D is indeed orientation-preserving. Then we know, since D acts properly discontinuously, that M/D is a manifold. Let $\pi : M \to M/D$ be the quotient map. Define $\psi : M/D \to \bigsqcup_{\bar{p} \in M/D} H_n(M, M - \{p\})$ by

$$\bar{p} \mapsto \pi_* \circ \phi(p)$$

where $\pi(p) = \bar{p} \in M/D$. Since the action of D is orientation-preserving, ψ is well-defined, since, if $p, p' \in \pi^{-1}(\bar{p})$, $p' = g \cdot p = L_g(p)$ for some $g \in D$. Also, since π is the quotient map, $\pi = \pi \circ L_g$. Hence,

$$(\pi_* \circ \phi)(p') = (\pi_* \circ \phi \circ L_g)(p) = (\pi_* \circ (L_g)_* \circ \phi)(p) = (\pi \circ L_g)_*(\phi(p)) = (\pi_* \circ \phi)(p).$$

Now, since D acts properly discontinuously, M/D looks locally exactly like M, so $\pi_* : H_n(M, M - \{p\}) \to H_n(M/D, M/D - \{\pi(p)\})$ is an isomorphism taking the generator $\alpha_p \in H_n(M, M - \{p\})$ to the generator $\alpha_{\pi(p)}$ of $H_n(M/D, M/D - \{\pi(p)\})$. Hence, ψ really does map $\bar{p}$ to $\alpha_{\bar{p}}$ for each $\bar{p} \in M/D$. Finally, for $\bar{p} \in M/D$, there exists a neighborhood V' of $\bar{p}$ that is homeomorphic to a neighborhood V of p for each $p \in \pi^{-1}(p)$. Hence, if U is the neighborhood of p homeomorphic to $\mathbb{R}^n$ such that the generator α_U in $H_n(M, M - U)$ maps to α_q for each $q \in U$, then, for each $\bar{q} \in \pi(U \cap V) =: U'$, $\alpha_{U'} \mapsto \alpha_q$. Therefore, ψ really does define an orientation on M/D, so M/D is orientable.

Now, to define the orientation homomorphism, we use the same procedure we used to define it for differentiable manifolds. If γ_p is a closed loop based at $p \in M$, then, for each point q on γ_p, there exists a neighborhood U_q of q such that U_q is homeomorphic to $\mathbb{R}^n$, which is orientable, so there exists an orientation ϕ_q on U_q. The collection $\{U_q\}_{q \in \gamma_p}$ defines an open cover of γ_p, so we can take a finite subcover, $\{U_0, \ldots, U_n\}$, where $p \in U_0$ and $p \in U_n$. Now, take the orientation ϕ_0 on U_0; $U_0 \cap U_1 \neq \emptyset$; for $q \in U_0 \cap U_1$, either $\phi_0(q) = \phi_1(q)$ or $\phi_0(q) = -\phi_1(q)$. If the former, let $\phi_1' = \phi_1$ and if the latter, let $\phi_1' = -\phi_1$. Then ϕ_1' defines an orientation on U_1 that agrees with ϕ_0 on $U_0 \cap U_1$. Iterating this process, we see that we can define ϕ_k' for each $k = 1, \ldots, n$ such that ϕ_k' agrees with ϕ_j' on $U_k \cap U_j$ when the intersection is non-empty.

Since $U_0 \cap U_n \ni p$, ϕ_0 and ϕ'_n define orientations on $U_0 \cap U_n$; if they agree, then we say that the loop γ_p is orientation-preserving. If they disagree, we say that they are orientation-reversing. Just as in the differentiable case, this property will be invariant under homotopies, so the notion of orientation-preserving or -reversing is well-defined on homotopy classes. Hence, we define $O : \pi_1(M) \to \mathbb{Z}/2$ by

$$[\gamma] \mapsto \begin{cases} 0 & \gamma \text{ is orientation-preserving} \\ 1 & \gamma \text{ is orientation-reversing} \end{cases}$$

This map will be a homomorphism by the identical argument given in the differentiable case, and we again have the result that O is trivial if and only if M is orientable. □

4

Show that if a topological manifold in $\mathbb{Z}/p$ orientable with $p \neq 2$ a prime, then M is also $\mathbb{Z}$ orientable.

Proof. We generalize the notion of the two-sheeted orientable cover in the following way: let

$$\widetilde{M}_R = \{\alpha_x | x \in M \text{ and } \alpha_x \text{ a generator of } H_n(M, M - \{x\}; R)\}.$$

The map $\alpha_x \mapsto x$ is certainly a surjection from $\widetilde{M}_R$ to M. Now, to make $\widetilde{M}_R$ a covering space, we put a topology on it by taking, for each neighborhood B homeomorphic to $\mathbb{R}^n$ and each generator α_B of $H_n(M, M-B; R)$, the set $\widetilde{B}(\alpha_B)$ consisting of all $\alpha_x \in \widetilde{M}_R$ such that $x \in B$ and $\alpha_x = i_*(\alpha_B)$ where $i : (M, M - B) \to (M, M - \{x\})$ is the usual inclusion. Then these $\widetilde{B}(\alpha_B)$ form a basis for a topology on $\widetilde{M}_R$ and, under this topology, $\widetilde{M}_R \to M$ is a covering map (here we follow the construction in Hatcher, pp. 234-235). This is an m-sheeted covering, where m is the order of R^*, where R^* is the group of units of R.

Now, $H_n(M, M - \{x\}; R) \simeq H_n(M, M - \{x\}; \mathbb{Z}) \otimes R$ by universal coefficients, since $H_{n-1} = 0$ (since $(M, M - \{x\})$ has the same homology as a sphere). Hence, we can identify elements of $H_n(M, M-\{x\}; R)$, which is to say elements of $\widetilde{M}_R$, by $\alpha_x \otimes r$ where α_x is a $\mathbb{Z}$-orientation at x and $r \in R^*$. Then, for each $r \in R^*$, the subspace

$$\widetilde{M}_r = \{\pm\alpha_x \otimes r | \alpha_x \in \widetilde{M}_{\mathbb{Z}}\}$$

is also a covering of M. Now, if $r \neq -r$ (that is, $2r \neq 0$), then $\widetilde{M}_r$ will simply be a copy of the usual orientable double cover, $\widetilde{M}_{\mathbb{Z}}$, whereas if $r = -r$, then $\widetilde{M}_r$ is just a copy of M.

If $R = \mathbb{Z}/p$, then, since every non-zero element of $\mathbb{Z}/p$ is a unit, a $\mathbb{Z}/p$-orientation of M corresponds to a global section of $\widetilde{M}_{\mathbb{Z}/p} \to M$. Since $r \neq -r$ for any $r \in \mathbb{Z}/p^*$, $\widetilde{M}_r$ is just $\widetilde{M}_{\mathbb{Z}}$ for each $r \in \mathbb{Z}/p^*$ and so a global section of $\widetilde{M}_{\mathbb{Z}/p}$ can only exist if there is a global section of $\widetilde{M}_{\mathbb{Z}} \to M$. However, such a global section of $\widetilde{M_{\mathbb{Z}}} \to M$ constitutes precisely a $\mathbb{Z}$-orientation on M, so if M is not $\mathbb{Z}$-orientable, it cannot be $\mathbb{Z}/p$-orientable. Therefore, by contrapositive, if M is $\mathbb{Z}/p$-orientable, then M is $\mathbb{Z}$-orientable. $\square$

DRL 3E3A, University of Pennsylvania
E-mail address: shonkwil@math.upenn.edu