Consider the surface 
\[ X(u,v) = \left( u + uv^2 - \frac{u^3}{3}, v + u^2v - \frac{v^3}{3}, u^2 - v^2 \right). \]

(a): Calculate the first and second fundamental forms of this surface and show that its mean curvature is zero.

**Answer:** We know that 
\[ E = \langle X_u, X_u \rangle, \quad F = \langle X_u, X_v \rangle \quad \text{and} \quad G = \langle X_v, X_v \rangle, \]
so we need to calculate \( X_u \) and \( X_v \). Now,
\[ X_u = (1 - u^2 + v^2, 2uv, 2u) \]
and
\[ X_v = (2uv, 1 - v^2 + u^2, -2v). \]
Hence,
\[ E = \langle X_u, X_u \rangle = (1 - u^2 + v^2)^2 + 4u^2v^2 + 4u^2 \]
\[ = 1 + 2u^2 + 2v^2 + u^4 + v^4 + 2u^2v^2 \]
\[ = (1 + u^2 + v^2)^2 \]

\[ F = \langle X_u, X_v \rangle = 2uv(1 - u^2 + v^2) + 2uv(1 - v^2 + u^2) - 4uv \]
\[ = 2uv - 2u^2v + 2uv^3 + 2uv - 2uv^3 + 2uv^3 - 4uv \]
\[ = 0 \]
and
\[ G = \langle X_v, X_v \rangle = 4u^2v^2 + (1 - v^2 + u^2)^2 + 4v^2 \]
\[ = 1 + 2u^2 + 2v^2 + u^4 + v^4 + 2u^2v^2 \]
\[ = (1 + u^2 + v^2)^2 \]

Now,
\[ N = \frac{X_u \times X_v}{||X_u \times X_v||} = \frac{X_u \times X_v}{\sqrt{EG - F^2}} \]
\[ = \frac{(-2uv^2 - 2u - 2u^3, 2u^2v + 2 + 2v^3, 1 - 2u^2v^2 - u^4 - v^4)}{\sqrt{(1 + u^2 + v^2)^4}} \]
Hence,

\[
L = \frac{1}{(1+u^2+v^2)^2} \langle \left(-2uv^2-2u-2u^3,2u^2v+2v+2v^3,1-2u^2v^2-u^4-v^4\right),
\left(-2u,2v,2\right)\rangle
= \frac{1}{(1+u^2+v^2)^2} 2(1+2u^2+2v^2+u^4+v^4+2u^2v^2)
= 2 \frac{(1+u^2+v^2)^2}{(1+u^2+v^2)^2}
= 2.
\]

Now,

\[
M = \frac{1}{(1+u^2+v^2)^2} \langle \left(-2uv^2-2u-2u^3,2u^2v+2v+2v^3,1-2u^2v^2-u^4-v^4\right),
\left(2v,2u,0\right)\rangle
= \frac{1}{(1+u^2+v^2)^2} (0)
= 0.
\]

and

\[
N = \frac{1}{(1+u^2+v^2)^2} \langle \left(-2uv^2-2u-2u^3,2u^2v+2v+2v^3,1-2u^2v^2-u^4-v^4\right),
\left(2u,-2v,-2\right)\rangle
= -L
= -2.
\]

That is to say,

\[
L = 2, \quad N = -2, \quad M = 0.
\]

To compute the mean curvature, we calculate the principal curvatures:

\[
k_1 = -a_{11} = \frac{-MF+LG}{EG-F^2}, \quad k_2 = -a_{22} = \frac{-MF+NE}{EG-F^2}.
\]

Since \(M = F = 0\),

\[
k_1 = \frac{LG}{EG} = \frac{L}{E}, \quad k_2 = \frac{NE}{EG} = \frac{N}{G}.
\]

Hence,

\[
k_1 = \frac{2}{(1+u^2+v^2)^2}, \quad k_2 = \frac{-2}{(1+u^2+v^2)^2}.
\]

Therefore,

\[
H = k_1 + k_2 = \frac{2}{(1+u^2+v^2)^2} - \frac{2}{(1+u^2+v^2)^2} = 0.
\]
(b): Find the parametric equations of the conjugate minimal surface of \( X(u, v) \).

**Answer:** Denote Enneper’s surface as parametrized in (a) by

\[
X(u, v) = (x(u, v), y(u, v), z(u, v)).
\]

Then the conjugate minimal surface to Enneper’s surface is

\[
\Xi(u, v) = (\xi(u, v), \eta(u, v), \zeta(u, v))
\]

where

\[
\begin{align*}
\xi_v &= \frac{1}{w} (Gx_u - Fx_v) \\
\xi_u &= \frac{1}{w} (-Fx_u - Ex_v) \\
\eta_v &= \frac{1}{w} (Gy_u - Fy_v) \\
\eta_u &= \frac{1}{w} (-Fy_u - Ey_v) \\
\zeta_v &= \frac{1}{w} (Gz_u - Fz_v) \\
\zeta_u &= \frac{1}{w} (-Fz_u - Ez_v)
\end{align*}
\]

where \( E, F, G \) are as in (a) and \( w = \sqrt{EG - F^2} \). Now, since we saw in (a) that \( E = G = (1 + u^2 + v^2)^2 \) and \( F = 0, \) \( w = E = G \) and the above reduce to

\[
\begin{align*}
\xi_v &= x_u = 1 + v^2 - u^2 \\
\xi_u &= -x_v = -2uv \\
\eta_v &= y_u = 2uv \\
\eta_u &= -y_v = -1 - u^2 + v^2 \\
\zeta_v &= z_u = 2u \\
\zeta_u &= -z_v = 2v,
\end{align*}
\]

so it’s easy to see that

\[
\Xi(u, v) = \left( v - u^2v + \frac{v^3}{3}, -u + uv^2 - \frac{u^3}{3}, 2uv \right).
\]

A simple computation (which I won’t reproduce here) confirms that this surface has the same first fundamental form as Enneper’s surface and mean curvature zero, so it really is a minimal surface.

\[
\star
\]

(c): Prove Enneper’s surface \( \{ X(u, v); -\infty < u < \infty, -\infty < v < \infty \} \) is complete.

**Answer:** I honestly have no idea how to prove this, despite having spent rather a long time trying to figure it out.
2

(a): Prove that developable surfaces \( \{X(u,v) = x(u) + vV(u)\} \) are characterized by the condition (added to the regularity assumptions) \((\dot{x}(u), V(u), \dot{V}(u)) = 0\), identically.

**Proof.** In order to show this, we show that each characterization of developability implies the other.

Suppose \( S \) is a developable surface with parametrization \( X(u,v) = x(u) + vV(u) \). Let \( X(u_0, v_0) \) be a point of \( S \). Then, since \( S \) is developable, the tangent plane \( T_{X(u_0,v_0)}S \) is tangent to \( D \) all along the line \( v \mapsto X(u_0,v_0) + vV(u_0) \). Hence, the normal \( N \) to \( S \) is constant along this line; since this is true for all choices of \( u_0 \) and \( v_0 \), we see that \( N_v \equiv 0 \) on all of \( S \). In particular, this implies that

\[
M = \langle N, X_{uv} \rangle = -\langle N_v, X_u \rangle = 0.
\]

In order to use this information, we need to compute \( N \) and \( X_{uv} \) explicitly. To that end, note that \( X_u = \dot{x}(u) + v\dot{V}(u) \) and \( X_v = V(u) \). Hence,

\[
X_u \times X_v = (\dot{x}(u) + v\dot{V}(u)) \times V(u)
= \dot{x}(u) \times V(u) + v\dot{V}(u) \times V(u),
\]

so

\[
N = \frac{\dot{x}(u) \times V(u) + v\dot{V}(u) \times V(u)}{\|\dot{x}(u) \times V(u) + v\dot{V}(u) \times V(u)\|},
\]

which is well-defined since \( X_u \times X_v \neq 0 \) (by the regularity condition). Note that this calculation depended only on the fact that \( S \) is a ruled surface.

Now,

\[
X_{uv} = \dot{V}(u),
\]

so

\[
0 = M \langle N, X_{uv} \rangle
= \frac{1}{\|\dot{x}(u) \times V(u) + v\dot{V}(u) \times V(u)\|}(\dot{V}(u), \dot{x}(u) \times V(u) + v\dot{V}(u) \times V(u))
= \frac{1}{\|\dot{x}(u) \times V(u) + v\dot{V}(u) \times V(u)\|} (\dot{x}(u), V(u), \dot{V}(u)).
\]

Therefore, it must be the case that \((\dot{x}(u), V(u), \dot{V}(u)) = 0\).

On the other hand, suppose \( S \) is a ruled surface parametrized by \( X(u,v) = x(u) + vV(u) \), satisfies the given regularity conditions and is such that \((\dot{x}(u), V(u), \dot{V}(u)) = 0\). Then, as in (1),

\[
N = \frac{\dot{x}(u) \times V(u) + v\dot{V}(u) \times V(u)}{\|\dot{x}(u) \times V(u) + v\dot{V}(u) \times V(u)\|}
\]
so
\[ \langle N_v, X_u \rangle = -\langle N, X_{uv} \rangle = -\frac{1}{\|\dot{x}(u) \times V(u) + v\dot{V}(u) \times V(u)\|} \langle \dot{x}(u), V(u), \ddot{V}(u) \rangle = 0 \]
and
\[ \langle N_v, X_v \rangle = -\langle N, X_{vv} \rangle = -\langle N, 0 \rangle = 0. \]

Hence, since \( X_u \) and \( X_v \) are linearly independent, we conclude that \( N_v \equiv 0 \) identically, which means that \( N \) is constant along the lines \( v \mapsto X(u_0, v_0) + v\dot{V}(u_0) \), which in turn implies that \( T_{X(u_0, v_0)}S \) is tangent to \( S \) all along this line for any \( u_0, v_0 \). Therefore, we see that \( S \) is a developable surface.

Thus, having shown implications both ways, we conclude that the two conditions for being developable given are equivalent. \( \square \)

(b): We now consider ruled surfaces that are strictly not developable, meaning that, for each \( u \), \( (\dot{x}(u), V(u), \dot{V}(u)) \neq 0 \). Show that a strictly non-developable ruled surface \( S \) has no local singularities and that its Gaussian curvature is everywhere strictly negative.

**Proof.** Note that our computations for \( X_u, X_v \) and \( M \) in part (a) are valid for any ruled surface, so we can import them directly to this problem. Now, \( X_{uv} = (X_v)_u = (V(u))_v = 0 \), so \( N = 0 \). Hence,
\[ K = \frac{LN - M^2}{EG - F^2} = -\frac{M^2}{EG - F^2}, \]
so we don't even need to bother computing \( L \). Using the value for \( M \) computed in (a), we have that
\[ K = \frac{-M^2}{EG - F^2} = -\frac{1}{\|\dot{x}(u) \times V(u) + v\dot{V}(u) \times V(u)\|} \langle \dot{x}(u), V(u), \ddot{V}(u) \rangle \frac{(\dot{x}(u), V(u), \ddot{V}(u))^2}{EG - F^2}. \]
Since \( \|X_u \times X_v\| = \|\dot{x}(u) \times V(u) + v\dot{V}(u) \times V(u)\| \) and \( (\dot{x}(u), V(u), \ddot{V}(u))^2 \) are both strictly positive, if we can show that \( EG - F^2 > 0 \), then that will suffice to show that \( K < 0 \). However,
\[ EG - F^2 = \|X_u\|^2\|X_v\|^2 - \langle X_u, X_v \rangle^2 = \|X_u \times X_v\|^2 > 0. \]
Therefore, we conclude that, indeed, \( K < 0 \). \( \square \)

(c): We shall assume now that the parameter \( u \) of the generating curve \( \Gamma_0 = \{x(u)\} \) of \( S \) describes its oriented arc length \( (\dot{x} \cdot \dot{x} = 1) \) and that the direction \( V(u) \) of the rulings is everywhere normal to \( \Gamma_0 \), i.e. \( V(u) = n(u) \cos \theta(u) + b \sin \theta(u) \), where \( n \) denotes the principal normal and \( b \) the binormal of \( \Gamma_0 \). Prove that \( S \) is strictly non-developable if and only if
\[ \frac{d\theta(u)}{du} + \tau(u) \neq 0, \]
where \( \tau(u) \) denotes the torsion of \( \Gamma_0 \).
Proof. Suppose \( S \) is strictly non-developable. Then \((\dot{x}(u), \dot{V}(u), \dot{V}(u)) \neq 0\). Now,

\[
\dot{V}(u) = \dot{n}(u) \cos \theta(u) + b(u) \sin \theta(u),
\]

so

\[
\dot{V}(u) = \dot{n}(u) \cos \theta(u) - n(u) \dot{\theta}(u) \sin \theta(u) + \dot{b}(u) \sin \theta(u) + b(u) \dot{\theta}(u) \cos \theta(u)
\]

\[
= (-\kappa \dot{x}(u) + \tau b(u)) \cos \theta(u) - n \dot{\theta}(u) \sin \theta(u) - \tau n(u) \sin \theta(u) + b(u) \dot{\theta}(u) \cos \theta(u)
\]

\[
= -\kappa \cos \theta(u) \dot{x}(u) - (\tau + \dot{\theta}(u)) \sin \theta(u) n(u) + (\dot{\theta}(u) + \tau) \cos \theta(u) b(u),
\]

by the Frenet equations. Therefore,

\[
\dot{V}(u) \times \ddot{V}(u) = (\dot{\theta}(u) + \tau) \dot{x}(u) - \kappa \sin \theta(u) \cos \theta(u) n(u) + \kappa \cos^2 \theta(u) b(u),
\]

so

\[
0 \neq (\dot{x}(u), \dot{V}(u), \ddot{V}(u))
\]

\[
= \dot{x}(u) \cdot (\dot{V}(u) \times \ddot{V}(u))
\]

\[
= (\dot{\theta}(u) + \tau) \|\dot{x}(u)\|^2
\]

\[
= \frac{d\theta(u)}{du} + \tau.
\]

Note that all of the equalities in the above expression hold regardless of any assumptions we made about \( S \), so we see that, if \( S \) is a surface satisfying the given hypothesis and such that \( \frac{d\theta(u)}{du} + \tau(u) \neq 0 \), then \( S \) is strictly non-developable. \( \square \)

(d): With the same assumptions as in (c), prove that, if \( \theta(u) = 0 \) (identically) and \( \tau(u) \neq 0 \), then the mean curvature \( H(u, v) \) of \( S \) equals 0 all along \( \Gamma_0 \).

Proof. Suppose this is the case. Then \( \dot{V}(u) = n(u) \), so \( X(u, v) = x(u) + vn(u) \). Hence, along the curve \( \Gamma_0 \) (i.e. where \( v = 0 \)),

\[
X_u = \dot{x}(u) + v \dot{n}(u) = \dot{x}(u)
\]

and

\[
X_v = n(u).
\]

Now, note that \( F = \langle X_u, X_v \rangle = \dot{x}(u), n(u) \rangle = 0 \). Moreover,

\[
X_{uu} = \ddot{x}(u)
\]

\[
X_{uv} = \dot{n}(u)
\]

\[
X_{vv} = 0
\]

and

\[
N = \frac{X_u \times X_v}{\|X_u \times X_v\|} = \frac{\ddot{x}(u) \times n(u)}{\|\dot{x}(u) \times n(u)\|} = b(u).
\]

Therefore, \( N = 0 \) and

\[
L = \langle N, X_{uu} \rangle = \langle b(u), \ddot{x}(u) \rangle = 0,
\]
so

\[ H = \frac{1}{2} \left( \frac{LG - 2MF + NE}{EG - F^2} \right) = 0 \]

since \( L = F = N = 0 \). \( \square \)

(e): (Challenge) Calculate \( H(u, v) \) under the same assumptions of questions (b), (c), (d). In particular, show that, if the curvature \( \kappa(u) \) of \( \Gamma_0 \) is positive, then there is a positive valued function \( f(u) \) such that \( H(u, v) = 0 \), if and only if either \( v = 0 \) or \( v = f(u) \).

Partial Answer: Suppose \( S \) is as in (b), (c) and (d); that is, \( S \) is a strictly non-developable ruled surface with \( \| \dot{\mathbf{x}} \| = 1 \) and \( \mathbf{V}(u) = \mathbf{n}(u) \). Note that, since \( S \) is strictly non-developable, (c) implies that \( \tau(u) \not= 0 \) (since \( \theta(u) = 0 \)). Now, under these assumptions, \( \mathbf{X}(u, v) = \mathbf{x}(u) + v\mathbf{V}(u) = \mathbf{x}(u) + v\mathbf{n}(u) \). Hence,

\[
\mathbf{X}_u = \dot{\mathbf{x}}(u) + v\mathbf{n}(u) = \dot{\mathbf{x}}(u) + v(-\kappa(u)\dot{\mathbf{x}}(u) + \tau(u)\mathbf{b}(u)) = (1 - v\kappa(u))\dot{\mathbf{x}}(u) + v\tau(u)\mathbf{b}(u)
\]

\[
\mathbf{X}_v = \mathbf{n}(u).
\]

Hence,

\[
\mathbf{X}_u \times \mathbf{X}_v = (1 - v\kappa(u))\mathbf{b}(u) - v\tau(u)\dot{\mathbf{x}}(u),
\]

so

\[
\mathbf{N} = \frac{\mathbf{X}_u \times \mathbf{X}_v}{\| \mathbf{X}_u \times \mathbf{X}_v \|} = \frac{1}{v^2\tau^2 + (1 - v\kappa)^2} [-v\tau(u)\dot{\mathbf{x}}(u) + (1 - v\kappa(u))\mathbf{b}(u)].
\]

Also,

\[
\mathbf{X}_{uu} = \ddot{\mathbf{x}}(u) + v(-\dot{\kappa}(u)\dot{\mathbf{x}}(u) - \kappa(u)\ddot{\mathbf{x}}(u) + \dot{\tau}(u)\mathbf{b}(u) + \tau(u)\dot{\mathbf{b}}(u)) = -v\dot{\kappa}(u)\dot{\mathbf{x}}(u) + (\kappa(u) - v\kappa^2(u) - \tau^2(u))\mathbf{n}(u) + v\dot{\tau}(u)\mathbf{b}(u),
\]

while

\[
\mathbf{X}_{uv} = -\kappa(u)\dot{\mathbf{x}}(u) + \tau(u)\mathbf{b}(u)
\]

and

\[
\mathbf{X}_{vv} = 0.
\]

Hence,

\[
L = \langle \mathbf{N}, \mathbf{X}_{uu} \rangle
\]

\[
= \frac{-v\tau(u)\dot{\mathbf{x}}(u) + (1 - v\kappa(u))\mathbf{b}(u), -v\dot{\kappa}(u)\dot{\mathbf{x}}(u) + (\kappa(u) - v\kappa^2(u) - \tau^2(u))\mathbf{n}(u) + v\dot{\tau}(u)\mathbf{b}(u)}{v^2\tau^2 + (1 - v\kappa)^2}
\]

(2)

\[
= \frac{v^2\tau(u)\ddot{\kappa}(u) + v\dot{\tau}(u) - v^2\dot{\tau}(u)\kappa(u)}{v^2\tau^2 + (1 - v\kappa)^2}.
\]

Also, note that \( N = 0, F = \langle \mathbf{X}_u, \mathbf{X}_v \rangle = 0, G = 1 \) and

\[
E = \langle \mathbf{X}_u, \mathbf{X}_u \rangle
\]

\[
= (1 - v\kappa(u))\dot{\mathbf{x}}(u) + v\tau(u)\mathbf{b}(u), (1 - v\kappa(u))\dot{\mathbf{x}}(u) + v\tau(u)\mathbf{b}(u)
\]

\[
= (1 - v\kappa(u))^2 + v^2\tau^2(u).\]
Therefore,
\[ H = \frac{1}{2} \left( \frac{L G - 2 M F + N E}{E G - F^2} \right) = \frac{L G}{2 E G} = \frac{L}{2 E}, \]
which is zero if and only if either \( v = 0 \) (which was part (d) above) or \( L = 0 \), which, given (2), means that
\[ 0 = v^2 \tau(u) \kappa(u) + v \dot{\tau}(u) - v^2 \dot{\tau}(u) \kappa(u). \]
In turn, this equality holds if and only if
\[ v = \frac{\dot{\tau}(u)}{\ddot{\tau}(u) \kappa(u) - \tau(u) \dot{\kappa}(u)}. \]
I believe the righthand side of this equality is a positive function, but I can’t prove it. Assuming it is, this would demonstrate that \( H = 0 \) if and only if either \( v = 0 \) or \( V = f(u) \) for some positive function \( f(u) \).

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