ALGEBRA HW 8

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1

Find all automorphisms of the ring \( \mathbb{Z}[x] \).

Answer: First, we note that, if \( \phi : \mathbb{Z}[x] \to \mathbb{Z}[x] \) is an automorphism, then \( \phi(1) = 1 \), which in turn means that \( \phi(c) = c \) for all constant terms \( c \). Hence, \( \phi \) is completely determined by \( \phi(x) \). Suppose

\[
\deg(\phi(x)) = d.
\]

Then, for all non-constant polynomials \( f(x) \in \mathbb{Z}[x] \),

\[
\deg(\phi(f(x))) \geq d
\]

since \( f(x) \) is a linear combination of multiples of \( x \). However, since \( \phi \) is an automorphism, it must be surjective, so there exists non-constant \( f(x) \in \mathbb{Z}[x] \) such that \( \phi(f(x)) = x \). Hence, it must be the case that \( d = 1 \), which is to say that

\[
\phi(x) = \alpha x.
\]

Suppose \( \alpha = pq \) for \( p, q \in \mathbb{Z} \), and \( \phi(g(x)) = qx \). Then

\[
\phi(x) = \alpha x = pqx = \phi(p)\phi(g(x)) = \phi(pg(x)).
\]

However, the only values of \( p \) and \( g \) for which this could be satisfied are \( p = \pm 1, g(x) = \pm x \). Hence, there are only two automorphisms of \( \mathbb{Z}[x] \), corresponding to \( \phi_1 : \mathbb{Z}[x] \to \mathbb{Z}[x] \) and \( \phi_2 : \mathbb{Z}[x] \to \mathbb{Z}[x] \), where \( \phi_1(x) = x \) and \( \phi_2(x) = -x \).

♣

2

Determine all maximal ideals of the ring \( \mathbb{Z}[\frac{1}{2}] \), and show that each maximal ideal can be generated by one element.

Answer:

3

Let \( m \) be the ideal of \( \mathbb{Z}[x] \) generated by 5 and \( x \). Prove that \( m \) is a maximal ideal.
Proof. Let $\phi : \mathbb{Z}[x] \rightarrow F_5$ be given by
\[
\sum_{i=0}^{\infty} a_i x^i \mapsto [a_0]
\]
where $[a_0] = a_0 + 5Z \in F_5$. Then, if $\sum a_i x^i, \sum b_i x^i \in \mathbb{Z}[x]$,
\[
\phi \left( \sum a_i x^i + \sum b_i x^i \right) = \phi \left( \sum (a_i + b_i) x^i \right) = a_0 + b_0 = \phi \left( \sum a_i x^i \right) + \phi \left( \sum b_i x^i \right)
\]
and
\[
\phi \left( \left( \sum a_i x^i \right) \left( \sum b_j x^j \right) \right) = \phi \left( \sum a_i b_j x^{i+j} \right) = a_0 b_0 = \phi \left( \sum a_i x^i \right) \phi \left( \sum b_j x^j \right),
\]
so $\phi$ is a ring homomorphism. Hence, by the First Isomorphism Theorem,
\[
\mathbb{Z}[x]/\ker(\phi) \simeq \text{Im}(\phi) = F_5
\]
since $\phi$ is surjective, so $\ker(\phi)$ is a maximal ideal of $\mathbb{Z}[x]$. Now,
\[
\ker(\phi) = \left\{ \sum a_i x^i : a_i = 5x \text{ for some } x \in \mathbb{Z} \right\} = m
\]
the ideal of $\mathbb{Z}[x]$ generated by 5 and $x$. Therefore, since $F_5$ is a field, $m$ is maximal. 

\[
\square
\]

4

Let $\mathbb{Q}[\sqrt{-1}]$ be the subset of $\mathbb{C}$ consisting of all numbers of the form $a + b\sqrt{-1}$, $a, b \in \mathbb{Q}$. The group $\mathbb{Z}/4\mathbb{Z}$ operates on $\mathbb{Q}[\sqrt{-1}]$, such that the element $n + 4\mathbb{Z}$ acts as “multiplication by $\sqrt{-1}^n$” for any $n \in \mathbb{Z}$. The above action induces a ring homomorphism $\alpha : \mathbb{Q}[\mathbb{Z}/4\mathbb{Z}] \rightarrow \text{End}_\mathbb{Q}(\mathbb{Q}[\sqrt{-1}])$, where $\text{End}_\mathbb{Q}(\mathbb{Q}[\sqrt{-1}])$ denotes the ring of all $\mathbb{Q}$-linear endomorphisms of the $\mathbb{Q}$-vector space $\mathbb{Q}[\sqrt{-1}]$. Prove that $\alpha$ is not injective.

Proof. Let $[n] := n + 4\mathbb{Z}$. Then $\alpha : \mathbb{Q}[\mathbb{Z}/4\mathbb{Z}] \rightarrow \text{End}_\mathbb{Q}(\mathbb{Q}[\sqrt{-1}])$ is given by
\[
[n] \mapsto (x \mapsto x \sqrt{-1}^n).
\]
Then
\[
\alpha([1]) = (x \mapsto x \sqrt{-1}) = (x \mapsto -x(-\sqrt{-1})) = (x \mapsto -x \sqrt{-1}^3) = \alpha(-[3]).
\]
Hence, $\alpha$ is not injective. \(\square\)

5

Prove that $\mathbb{R}[x]$ is not finite dimensional over $\mathbb{R}$.

Proof. By contradiction. Suppose $\mathbb{R}[x]$ were finite dimensional over $\mathbb{R}$. Then there exists some finite basis $B = \{b_1, \ldots, b_n\}$ for $\mathbb{R}[x]$ over $\mathbb{R}$. Certainly, $b_i \neq 0$ for all $i = 1, \ldots, n$, so $\deg(b_i)$ is well-defined. Let
\[
m = \max_{i \in \{1, \ldots, n\}} \deg(b_i).
\]
Then \( x^{m+1} \in \mathbb{R}[x] \), but there is no way to write \( x^{m+1} \) as an \( \mathbb{R} \)-linear combination of the \( b_i \)'s. From this contradiction, then, we conclude that, in fact, \( \mathbb{R}[x] \) is not finite dimensional over \( \mathbb{R} \). \( \square \)

6

The three elements of \( \mathbb{Z}/3\mathbb{Z} \) gives an \( \mathbb{R} \)-basis of the group algebra \( \mathbb{R}[\mathbb{Z}/3\mathbb{Z}] \). For each element \( n + 3\mathbb{Z} \) of the group \( \mathbb{Z}/3\mathbb{Z} \), write down the matrix representation of \( n + 3\mathbb{Z} \) acting on the 3-dimensional vector space \( \mathbb{R}[\mathbb{Z}/3\mathbb{Z}] \).

**Answer:** We consider each \( h_n : \mathbb{R}[\mathbb{Z}/3\mathbb{Z}] \to \mathbb{R}[\mathbb{Z}/3\mathbb{Z}] \) given by

\[
x \mapsto (n + 3\mathbb{Z}) + x.
\]

Then,

\[
\begin{align*}
    h_0(0 + 3\mathbb{Z}) &= 0 + 3\mathbb{Z}, \\
    h_0(1 + 3\mathbb{Z}) &= 1 + 3\mathbb{Z}, \\
    h_0(2 + 3\mathbb{Z}) &= 2 + 3\mathbb{Z},
\end{align*}
\]

so the matrix representation is:

\[
\begin{pmatrix}
    1 & 0 & 0 \\
    0 & 1 & 0 \\
    0 & 0 & 1
\end{pmatrix}.
\]

Now,

\[
\begin{align*}
    h_1(0 + 3\mathbb{Z}) &= 1 + 3\mathbb{Z}, \\
    h_1(1 + 3\mathbb{Z}) &= 2 + 3\mathbb{Z}, \\
    h_1(2 + 3\mathbb{Z}) &= 0 + 3\mathbb{Z},
\end{align*}
\]

so the matrix representation is:

\[
\begin{pmatrix}
    0 & 0 & 1 \\
    1 & 0 & 0 \\
    0 & 1 & 0
\end{pmatrix}.
\]

Finally,

\[
\begin{align*}
    h_2(0 + 3\mathbb{Z}) &= 2 + 3\mathbb{Z}, \\
    h_2(1 + 3\mathbb{Z}) &= 0 + 3\mathbb{Z}, \\
    h_2(2 + 3\mathbb{Z}) &= 1 + 3\mathbb{Z},
\end{align*}
\]

so the matrix representation is:

\[
\begin{pmatrix}
    0 & 1 & 0 \\
    0 & 0 & 1 \\
    1 & 0 & 0
\end{pmatrix}.
\]

7

Find a maximal ideal of the group algebra \( \mathbb{F}_7[\mathbb{Z}/3\mathbb{Z}] \).

**Answer:** Denote by \([n]\) the element \( n + 3\mathbb{Z} \). Let \( \phi : \mathbb{F}_7[\mathbb{Z}/3\mathbb{Z}] \to \mathbb{F}_7 \) be given by

\[
a_0[0] + a_1[1] + a_2[2] \mapsto a_0 + a_1 + a_2,
\]

where \( a_0, a_1, a_2 \in \mathbb{F}_7 \). This map is certainly surjective, as, for any \( a \in \mathbb{F}_7 \),

\[
\phi : a[0] + 0[1] + 0[2] \mapsto a + 0 + 0 = a.
\]
Also, for \( a_0[0] + a_1[1] + a_2[2], b_0[0] + b_1[1] + b_2[2] \in \mathbb{F}_7[\mathbb{Z}/3\mathbb{Z}] \),
\[
\phi((a_0[0] + a_1[1] + a_2[2]) + (b_0[0] + b_1[1] + b_2[2]))
= (a_0 + b_0) + (a_1 + b_1) + (a_2 + b_2)
= (a_0 + a_1 + a_2) + (b_0 + b_1 + b_2)
= \phi(a_0[0] + a_1[1] + a_2[2]) + \phi(\phi(b_0[0] + b_1[1] + b_2[2])
\]
and
\[
\phi((a_0[0] + a_1[1] + a_2[2])(b_0[0] + b_1[1] + b_2[2]))
= \phi((a_0b_0 + a_1b_1 + a_2b_2)[0] + (a_0b_1 + a_1b_0 + a_2b_2)[1] + (a_0b_2 + a_2b_2 + a_1b_1)[2])
= (a_0b_0 + a_1b_1 + a_2b_2) + (a_0b_1 + a_1b_0 + a_2b_2) + (a_0b_2 + a_2b_2 + a_1b_1)
= (a_0 + a_1 + a_2)(b_0 + b_1 + b_2)
= \phi(a_0[0] + a_1[1] + a_2[2])\phi(b_0[0] + b_1[1] + b_2[2])
\]
so \( \phi \) is a homomorphism. Hence, by the First Isomorphism Theorem,
\[
\mathbb{F}_7[\mathbb{Z}/3\mathbb{Z}]/\ker(\phi) \cong \mathbb{F}_7
\]
since \( \phi \) is surjective. Since \( \mathbb{F}_7 \) is a field, \( \ker(\phi) \) is a maximal ideal of \( \mathbb{F}_7[\mathbb{Z}/3\mathbb{Z}] \).

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