GEOMETRY FINAL

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1. Advanced Calculus

Prove the following result:

Theorem 1.1. *Let $U \subseteq \mathbb{R}^{n+1}$ be a non-empty open set and let $f : U \to \mathbb{R}$ be a smooth function and suppose $S = f^{-1}(c)$ is a non-empty subset such that $\nabla f(q) \neq 0$ for all $q \in S$. Now let $g : U \to \mathbb{R}$ be a smooth map and suppose that $p \in S$ is a local extrema of $g : S \to \mathbb{R}$, then there exists a real number $\lambda \in \mathbb{R}$ such that $\nabla g(p) = \lambda \nabla f(p)$. The number λ is known as a* ***Lagrange multiplier****.*

Proof. Let $\alpha : (-\epsilon, \epsilon)$ be a smooth curve on S such that $\alpha(0) = p$. Define $h(t) = g(\alpha(t))$. Then, since p is a local extremum of g, zero must be a local extremum of h. Now, since $h : \mathbb{R} \to \mathbb{R}$, we know that this implies that

$$0 = h'(0) = (g \circ \alpha)'(0) = dg_{\alpha(0)} \cdot \alpha'(0) = \nabla g(p) \cdot \alpha'(0).$$

Now, for any $v \in T_pS$ there exists a smooth curve $\alpha : (-\epsilon, \epsilon)$ such that $\alpha(0) = p$ and $\alpha'(0) = v$, this means that $\nabla g(p)$ is normal to the tangent plane at p. On the other hand, we know from calculus that, since S is simply a level surface of f, ∇f is normal to S at every point in S. Hence, $\nabla f(p)$ is parallel to $\nabla g(p)$.

Therefore, since $\nabla f(p)$ is non-zero, we see that there exists some $\lambda \in \mathbb{R}$ (possibly zero if $\nabla g(p) = 0$) such that $\nabla g(p) = \lambda \nabla f(p)$. $\square$

2. A Useful Result

Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} e^{\frac{-1}{x^2}} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

(a) Show that f is a smooth function and hence the function $g(x) = f(x-a)f(b-x)$ is smooth, where $a < b$.

Proof. Certainly, the constant function 0 is smooth and the function $e^{\frac{-1}{x^2}}$ is smooth, so problems can only arise if some derivative of $e^{\frac{-1}{x^2}}$ is non-zero at $x = 0$. Now, if we denote $g(x) = e^{\frac{-1}{x^2}}$, then

$$\begin{aligned} g'(x) &= \frac{2e^{\frac{-1}{x^2}}}{x^3} \\ g''(x) &= \frac{4e^{\frac{-1}{x^2}}}{x^6} - \frac{6e^{\frac{-1}{x^2}}}{x^4} \end{aligned}$$

and so, in general, higher derivatives of g will be linear combinations of terms of the form $\frac{e^{\frac{-1}{x^2}}}{x^k}$ for positive integers k. Therefore, we need to show that, when $x \to 0$, terms of this form vanish. To do so, we make the substitution $y = \frac{1}{x}$ and repeatedly apply L'Hopital's rule:

$$\begin{aligned}
\lim_{x\to 0^+} \frac{e^{\frac{-1}{x^2}}}{x^k} &= \lim_{y\to\infty} \frac{e^{-y^2}}{\frac{1}{y^k}} \\
&= \lim_{y\to\infty} y^k e^{-y^2} \\
&= \lim_{y\to\infty} \frac{y^k}{e^{y^2}} \\
&= \lim_{y\to\infty} \frac{ky^{k-1}}{2ye^{y^2}} \\
&= \lim_{y\to\infty} \frac{ky^{k-2}}{2e^{y^2}} \\
&= \lim_{y\to\infty} \frac{k(k-2)y^{k-3}}{4ye^{y^2}} \\
&= \lim_{y\to\infty} \frac{k(k-2)y^{k-4}}{4e^{y^2}} \\
&\vdots
\end{aligned}$$

We see that eventually we get to the point where the numerator is just a constant term and the denominator is either a constant multiplied by e^{y^2} or a constant multiplied by ye^{y^2}. In either case, it's clear that $g^{(j)}(0) = 0$ for all $j = 1, 2, \ldots$. Therefore, we see that all of the derivatives of f from both the right and left vanish at the origin, so f is, indeed, smooth.

Since f is smooth, both $f(x-a)$ and $f(b-x)$ are smooth functions of x, meaning their product, $g(x) = f(x-a)(b-x)$ is also smooth. □

(b) Show that given $0 < a < b$ there exists a smooth function $H : \mathbb{R}^n \to \mathbb{R}$ such that

i. $H(x) = 1$ for $x \in B(0, a) \subset \mathbb{R}^n$
ii. $H(x) = 0$ for $||x|| \geq b$
iii. $0 < H(x) < 1$ for $a < ||x|| < b$.

Proof. Define

$$H(x) = 1 - \frac{\int_{-\infty}^{||x||} g(t)dt}{\int_{-\infty}^{\infty} g(t)dt}.$$

Now, if $t \geq b$, then $b - t \leq 0$, meaning that $g(t) = f(t-a)f(b-t) = f(t-a)(0) = 0$. Similarly, if $t \leq a$, then $g(t) = 0$. Therefore, we see that

$$\int_{-\infty}^{\infty} g(t)dt = \int_a^b g(t)dt.$$

Hence, it's clear that, for all $||x|| \geq b$,

$$H(x) = 1 - \frac{\int_{-\infty}^{x} g(t)dt}{\int_{-\infty}^{\infty} g(t)dt} = 1 - \frac{\int_a^{||x||} g(t)dt}{\int_a^b g(t)dt} = 1 - \frac{\int_a^b g(t)dt}{\int_a^b g(t)dt} = 1 - 1 = 0.$$

Similarly, if $||x|| \leq a$, (i.e., $x \in B(0,a)$),

$$H(x) = 1 - \frac{\int_{-\infty}^{||x||} g(t)dt}{\int_{-\infty}^{\infty} g(t)dt} = 1 - \frac{0}{\int_a^b g(t)dt} = 1 - 0 = 1.$$

Now, since $g(t) \geq 0$ for all $t \in \mathbb{R}$, $\int_{-\infty}^{||x||} g(t)dt$ is strictly increasing as $||x||$ increases. Therefore, $H(x)$ is decreasing as $||x||$ increases, meaning that, for $a < ||x|| < b$,

$$0 < H(x) < 1.$$

Therefore, H is a function fulfilling the above requirements. □

3. Frenet Frame Revisited

Let S be an orientable regular surface with fixed orientation N and let $\alpha : I \to S$ be a curve parametrized by arc length. At the point $p = \alpha 0$ consider the orthonormal basis $\{T(s) = \alpha'(s), N(s) = N_{\alpha(s)}, V(s) = N(s) \times T(s)\}$.

(a) Show that

$$\begin{array}{rcccccc} T'(s) & = & 0 & + & \kappa_g V & + & \kappa_n N \\ V'(s) & = & -\kappa_g T & + & 0 & + & \tau_g N \\ N'(s) & = & -\kappa_n T & - & \tau_g V & + & 0 \end{array}$$

where κ_g and κ_n are the geodesic and normal curvaturers of α, respectively, and $\tau_g = V' \cdot N = -N' \cdot V$ is referred to as the *geodesic torsion*.

Proof. First, let us compute $T'(s)$ for $s = 0$. Now,

$$\begin{aligned} T'(s) = \tfrac{dT}{ds} &= \tfrac{d\alpha'}{ds} \\ &= \tfrac{d\alpha'}{ds} - \langle \tfrac{d\alpha'}{ds}, N\rangle N + \langle \tfrac{d\alpha'}{ds}, N\rangle N \\ &= \tfrac{D\alpha'}{ds} + \langle \alpha''(s), N\rangle N \\ &= \kappa_g (N \times T) + \kappa_n N \\ &= \kappa_g V + \kappa_n N. \end{aligned}$$

Now, turning to $V'(s)$, we see that

$$V'(s) = (N \times T)'(s) = N' \times T + N \times T'.$$

Note that, by the anti-commutativity of the cross product and the fact that, for vectors u, v, w,

$$(u \times v) \times w = \langle u, w\rangle v - \langle v, w\rangle u,$$

we have the following equality.

$$N \times V = N \times (N \times T) = -(N \times T) \times N = -\langle N, N\rangle T + \langle T, N\rangle N = -T,$$

since N is orthogonal to T and $||N|| = 1$. Hence,

$$N' \times T = N' \times -(N \times V) = (N \times V) \times N' = \langle N, N'\rangle V - \langle V, N'\rangle N = -\langle V, N'\rangle N = \tau_g N.$$

On the other hand,

$$N \times T' = N \times (\kappa_g V + \kappa_n N) = \kappa_g N \times V + \kappa_n N \times N = \kappa_g N \times V = -\kappa_g T.$$

Putting these results together, then, we see that

$$V'(s) = N' \times T + N \times T' = \tau_g N - \kappa_g T.$$

In order to compute $N'(s)$, first we note the following:

$$V \times T = (N \times T) \times T = \langle N, T \rangle T - \langle T, T \rangle N = -N.$$

Hence,

$$\begin{aligned} N'(s) = -(V \times T)'(s) &= -V' \times T - V \times T' \\ &= -(\kappa_g T + \tau_g N) \times T - V \times (\kappa_g V + \kappa_n N) \\ &= -\kappa_g T \times T - \tau_g N \times T - \kappa_g V \times V - \kappa_n V \times N \\ &= -\tau_g N \times T - \kappa_n V \times N \\ &= -\tau_g V - \kappa_n T. \end{aligned}$$

Hence, compiling the above results, we see that, in fact,

$$\begin{array}{rcccccc} T'(s) &=& 0 &+& \kappa_g V &+& \kappa_n N \\ V'(s) &=& -\kappa_g T &+& 0 &+& \tau_g N \\ N'(s) &=& -\kappa_n T &-& \tau_g V &+& 0 \end{array}$$

□

(b) Show that α is a line of curvature if and only if $\tau_g \equiv 0$.

Proof. Suppose α is a line of curvature. Then $\alpha'(s)$ is contained in a principal direction at p, then it must be the case that $\alpha'(s) = T(s)$ is an eigenvector of dN. Hence,

$$\gamma(s)T(s) = dN(\alpha'(s)) = N'(s)$$

for some γ. However, from part (a) above, we know that

$$\gamma(s)T(s) = N'(s) = -\kappa_n T - \tau_g V.$$

Since V is orthogonal to T, this can only be true if $\tau_g \equiv 0$.

On the other hand, suppose $\tau_g \equiv 0$. Then

$$-\kappa_n \alpha'(s) = -\kappa_n T = N'(s) = dN(\alpha'(s)),$$

meaning that $\alpha'(s)$ is an eigenvector of dN. Since the directions given by the eigenvectors of dN are the principal directions, this implies that α is a line of curvature. □

4. Singular Points

Let S be an oriented regular surface and X a smooth vector field on S with an isolated singularity at $p \in S$. Can $\text{Ind}_p(X)$ be zero? If your answer is yes, please give an example of such an oriented regular surface S and a smooth vector field X on S with an isolated singularity at some $p \in S$ such that $\text{Ind}_p(X) = 0$. If your answer is no, please provide a proof.

Answer: Yes, it is possible to construct such a surface. Let $S = \mathbb{R}^2$ parametrized by $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ where ϕ is the identity map. Then, at each point p in $\mathbb{R}^2$, $\Phi_1 = (p; 1, 0)$. Let X be given by $X_p = (p; ||p||, ||p||)$. Since $||p|| = 0$ if and only if $p = 0$, it is clear that the only singular point of X

is the origin, so the origin is an isolated singular point. Now, at each point $q \in \mathbb{R}^2$, the angle between X_q and Φ_1 is simply $\pi/4$. Let $\alpha : [0, 2\pi] \to \mathbb{R}^2$ be given by

$$\alpha(t) = (\cos t, \sin t),$$

the unit circle, which is a closed curve around the origin. Then, if we let $\psi(t)$ denote the angle between $\alpha(t)$ and Φ_1, we know that $\psi(t) = \pi/4$ for all $t \in [0, 2\pi]$. Hence,

$$2\pi \mathrm{Ind}_{(0,0)} = \psi(2\pi) - \psi(0) = \pi/4 - \pi/4 = 0.$$

Therefore, the index of X at the origin is zero.

♣

5. Discontinuous Group Actions

(a) Let G act properly discontinuously on a smooth manifold M. Show that M/G has a differentiable structure with respect to which π is a local diffeomorphism.

Proof. Let $p \in M$. Then, since G acts properly discontinuously on M, there exists a neighborhood $\mathcal{O}$ of p such that $g(\mathcal{O}) \cap \mathcal{O} = \emptyset$ for any $g \in G \backslash \{e\}$. Now, let $\psi : V \subset \mathbb{R}^n \to M$ be a coordinate chart containing p and let

$$W = \psi(V) \cap \mathcal{O}.$$

Now, let

$$U = \mathrm{Int}(\psi^{-1}(W))$$

and ϕ be ψ restricted to U. Then, for any coordinate chart $\gamma : U' \to S$ such that $\phi(U) \cap \gamma(U') \neq \emptyset$, $\phi^{-1} \circ \gamma$ and $\gamma^{-1} \circ \phi$ are differentiable precisely because $\psi^{-1} \circ \gamma$ and $\gamma^{-1} \circ \psi$ are differentiable and ϕ is simply a restriction of the domain of ψ. Therefore, by the maximality condition in the definition of manifolds, $\phi : U \to \phi(U) \subset \mathcal{O} \subset M$ is a coordinate chart containing p. Note that this implies that $\phi(U) \cap g(\phi(U)) = \emptyset$.

Since our choice of p above was arbitrary, we see that to each point in M is associated a chart $\phi_\alpha : U_\alpha \to M$ containing it such that $g(\phi_\alpha(U_\alpha)) \cap \phi_\alpha(U) = \emptyset$. Hence, the family $\phi_\alpha(U_\alpha)$ covers M and so

$$\cup_\alpha \pi \circ \phi_\alpha(U_\alpha) = M/G.$$

Now, suppose α and β are such that

$$(\pi \circ \phi_\alpha)(U_\alpha) \cap (\pi \circ \phi_\beta)(U_\beta) = A \neq \emptyset.$$

Let $U_1 = (\pi \circ \phi_\alpha)^{-1}(A)$ and let $U_2 = (\pi \circ \phi_\beta)^{-1}(A)$. Now, it may well be the case that $\phi_\alpha(U_1) \cap \phi_\beta(U_2) = \emptyset$. However, it must be true that $\phi_\alpha(U_1) = g(\phi_\beta(U_2))$ for some $g \in G$. Since the action of G is properly discontinuous, $g(\phi_\beta(U_2)) \cap \phi_\beta(U_2) = \emptyset$ for non-identity $g \in G$. Hence,

$$(\pi \circ \phi_\alpha)^{-1} \circ (\pi \circ \phi_\beta)(U_2) = (\pi \circ \phi_\alpha)^{-1}(A) = \phi_\alpha^{-1}(\pi^{-1}(A)) = \phi_\alpha^{-1} \circ g \circ \phi_\beta(U_2).$$

Since g is a diffeomorphism, this map is differentiable on $\mathbb{R}^n$. A similar calculation ensures that

$$(\pi \circ \phi_\beta)^{-1} \circ (\pi \circ \phi_\alpha) : U_1 \to U_2$$

is differentiable. Therefore, we conclude that, indeed, the family $\{U_\alpha, \pi\circ\phi_\alpha\}$ gives a differentiable structure on M/G.

Now, let $p \in M$ and let $\phi_\alpha : U_\alpha \to M$ be a coordinate chart containing p and let $\pi \circ \phi_\beta : U_\beta \to M/G$ be a coordinate chart containing $\pi(p)$. Then

$$(\pi \circ \phi_\beta)^{-1} \circ \pi \circ \phi_\alpha = (\pi \circ \phi_\beta)^{-1} \circ (\pi \circ \phi_\alpha),$$

which we already know is differentiable at p from our calculations above. On the other hand

$$\phi_\alpha^{-1} \circ \pi^{-1} \circ (\pi \circ \phi_\beta) = (\pi \circ \phi_\alpha)^{-1} \circ (\pi \circ \phi_\beta),$$

which, again, we already know is differentiable at $\pi(p)$. Therefore, since π is differentiable at p and has a differentiable inverse at $\pi(p)$, we conclude that π is a local diffeomorphism in a neighborhood of p. Since our choice of p was arbitrary, we conclude that π is a local diffeomorphism on all of M. □

(b) Let $S^n \subset \mathbb{R}^{n+1}$ be the n-sphere. Let $I : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ denote the identity map. Show that the group $G = \{I, -I\}$ acts properly discontinuously on S^n. The resulting quotient S^n/G is called the **real projective space** and is denoted $\mathbb{RP}^n$.

Proof. First, note that for $p \in S^n$,

$$(-I \circ I)(p) = -I(I(p)) = -I(p) = I(-I(p)) = (I \circ -I)(p)$$

Now, the identity map on S^n is clearly a diffeomorphism, so we need only check that $-I$ is a diffeomorphism to ensure that G acts on S^n. Let $p \in S^n$. Then $p = (x_1, \ldots, x_{n+1})$ where $x_k > 0$ for some $k = 1, \ldots, n+1$. Parametrize S^n by maps $\phi_i : U \to S^n$ where $U = \{q \in \mathbb{R}^n : ||q|| < 1\}$ where

$$\phi_i(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, \sqrt{1 - (x_1^2 + \ldots + x_n)}, x_i, \ldots, x_n).$$

Then each ϕ_i is a differentiable map with differentiable inverse. If $p \in \phi_i(U) \cap \phi_j(U)$, then

$$\phi_i^{-1} \circ \phi_j$$

and

$$\phi_j^{-1} \circ \phi_i$$

are differentiable on this intersection, so this parametrization gives a differentiable structure on S^n. Note that, since ϕ_k is differentiable for all $k = 1, \ldots, n+1$, so is $-\phi_k$. Hence,

$$\phi_i^{-1} \circ -I \circ \phi_j = \phi_i^{-1} \circ -\phi_j$$

and

$$\phi_j^{-1} \circ -I \circ \phi_i = \phi_j^{-1} \circ -\phi_i$$

are differentiable. Since $-I$ is its own inverse and our choice of p was arbitrary, this suffices to show that $-I$ is a diffeomorphism. Hence, we see that G acts on S^n.

Again, let $p \in S^n$. Hence, $p = (x_1, \ldots, x_{n+1})$ where $x_k > 0$ for some $k = 1, \ldots, n+1$. Let $\mathcal{O} = \{(x_1, \ldots, x_{n+1}) \in S^n : x_k > 0\}$. Then $\mathcal{O}$ is certainly a neighborhood of p. Furthermore,

$$-I(\mathcal{O}) \cap \mathcal{O} = \{(x_1, \ldots, x_{n+1}) \in S^n : x_k < 0\} \cap \{(x_1, \ldots, x_{n+1}) \in S^n : x_k > 0\} = \emptyset.$$

Since $-I$ is the only non-identity element of G, this demonstrates that, in fact, G acts properly discontinuously on S^n. □

(c) Suppose G acts properly discontinuously on a manifold M, then one can show that M/G is orientable if and only if there exists an orientation of M which is preserved by all the diffeomorphisms $g = \phi_g : M \to M$. Use this fact to prove that $\mathbb{RP}^n$ is orientable if and only if n is odd.

Proof. ($\Rightarrow$) Suppose $\mathbb{RP}^n$ is orientable. Then, by the given theorem, there exists an orientation of S^n which is preserved by the action of all elements of G. Specifically, there exists an orientation N of S^n which is preserved by $-I$. Hence, there is a parametrization $\{(U_\alpha, \phi_\alpha)\}$ which is compatible with the orientation of S^n such that $\{(U_\alpha, -I \circ \phi_\alpha)\} \cup \{(U_\alpha, \phi_\alpha)\}$ gives an orientation on S^n. Specifically, if

$$\phi_\alpha(U_\alpha) \cap -I \circ \phi_\beta(U_\beta) = W \neq \emptyset$$

then

$$
\begin{aligned}
0 &< \det(d_q((-I \circ \phi_\beta)^{-1} \circ \phi_\alpha)) \\
&= \det(d_q(\phi_\beta^{-1} \circ -I^{-1} \circ \phi_\alpha)) \\
&= \det(d_q(\phi_\beta^{-1} \circ -I \circ \phi_\alpha)) \\
&= \det(d\phi^{-1}_{\beta_{-I(q)}} \cdot -dI_q \cdot d\phi^{-1}_{\alpha_{\phi_\alpha^{-1}(q)}}) \\
&= \det(d\phi^{-1}_{\beta_{-I(q)}} \cdot -\mathrm{Id}_{(n+1)\times(n+1)} \cdot d\phi^{-1}_{\alpha_{\phi_\alpha^{-1}(q)}}) \\
&= (-1)^{n+1} \det(d\phi^{-1}_{\beta_{-I(q)}} \cdot d\phi^{-1}_{\alpha_{\phi_\alpha^{-1}(q)}}) \\
&= (-1)^{n+1} \det(d_q(\phi_\beta^{-1} \circ \phi_\alpha)).
\end{aligned} \tag{1}
$$

Since ϕ_β and ϕ_α are compatible with the orientation, it must be the case that

$$\det(d_q(\phi_\beta^{-1} \circ \phi_\alpha)) > 0.$$

Hence, in order for inequality (1) to hold, it must be the case that $(-1)^{n+1} > 0$, i.e., n is odd.

($\Leftarrow$) On the other hand, if $\mathbb{RP}^n$ is not orientable, then, by the given theorem, it cannot be the case that any orientation on S^n which is preserved by the action of all elements of G. Since all orientations are preserved by the identity map, it must be the case that each orientation fails under the action of $-I$. Now, as we just showed above,

$$\det(d_q((-I \circ \phi_\beta)^{-1} \circ \phi_\alpha)) = (-1)^{n+1} \det(d_q(\phi_\beta^{-1} \circ \phi_\alpha))$$

which must be non-positive for each set of coordinate charts ϕ_α, ϕ_β such that $\phi_\alpha, -I \circ \phi_\beta$ that intersect non-trivially. Now, since S^n is orientable, we know there exist ϕ_α, ϕ_β such that

$$\det(d_q(\phi_\beta^{-1} \circ \phi_\alpha)) > 0,$$

so it must be the case that $(-1)^{n+1} \leq 0$, i.e., n is even. By the contrapositive, then, we see that if n is odd, $\mathbb{RP}^n$ is orientable. □

6. Vector fields on M

(a) Show that for any smooth n-manifold M the tangent bundle TM admits the structure of a smooth $2n$-manifold such that the natural projection $\pi : TM \to M$ is a smooth submersion.

Proof. Let $\{(U_\alpha, \phi_\alpha)\}$ be the differentiable structure on M. Then the coordinates on $U_\alpha \in \mathbb{R}^n$ are given by $(u_1^\alpha, \ldots, u_n^\alpha)$; let $\{\Phi_1^\alpha, \ldots, \Phi_n^\alpha\}$ be the corresponding coordinate vector fields on M. Let $p \in M$ and let $\phi_\alpha : U_\alpha \to M$ be a coordinate chart containing p. Let $v \in T_pM$. Then, in local coordinates, v can be expressed as

$$v = y_1\Phi_{1p}^\alpha + \ldots y_n\Phi_{np}^\alpha.$$

In fact, for all $q \in \phi_\alpha(U_\alpha)$ and $w \in T_qM$,

$$w = y_1\Phi_{1q}^\alpha + \ldots + y_n\Phi_{nq}^\alpha$$

for some $y_1, \ldots, y_n$ depending on w. Now, let

$$y_\alpha : U_\alpha \times \mathbb{R}^n \to T\phi_\alpha(U_\alpha) \subset TM$$

be given by

$$(u_1^\alpha, \ldots, u_n^\alpha, y_1, \ldots, y_n) \mapsto y_1\Phi_{1\phi_\alpha((u_1^\alpha,\ldots,u_n^\alpha))} + \ldots + y_n\Phi_{2\phi_\alpha((u_1^\alpha,\ldots,u_n^\alpha))}.$$

Now, this is clearly surjective, as two elements mapping to the same point requires that the u_i^α's must be equal or else we end up in an entirely different tangent space and the y_i's must be equal or else we get a different linear combination of the Φ_i's.

Now, since $M = \cup_\alpha \phi_\alpha(U_\alpha)$, we see that every tangent plane is hit by such a y_α, so

$$TM = \cup_\alpha y_\alpha(U_\alpha \times \mathbb{R}^n).$$

Suppose that

$$W = y_\alpha(U_\alpha \times \mathbb{R}^n) \cap y_\beta(U_\beta \times \mathbb{R}^n)$$

for some α and β. Then we consider $y_\beta^{-1} \circ y_\alpha : y_\alpha^{-1}(W) \to y_\beta^{-1}(W)$. Let $q = (u_1^\alpha, \ldots, u_n^\alpha)$ and let $v = (y_1, \ldots, y_n)$. Then

$$\begin{aligned} y_\beta^{-1} \circ y_\alpha(u_1^\alpha, \ldots, u_n^\alpha, y_1, \ldots, y_n) &= y_\beta^{-1} \circ y_\alpha(q, v) \\ &= y_\beta^{-1}(y_1\Phi_{1\phi_\alpha((u_1^\alpha,\ldots,u_n^\alpha))} + \ldots + y_n\Phi_{2\phi_\alpha((u_1^\alpha,\ldots,u_n^\alpha))}) \\ &= (\phi_\beta^{-1} \circ \phi_\alpha(q), d(\phi_\beta^{-1} \circ \phi_\alpha)_q(v)) \end{aligned}$$

Now, these coordinate functions are smooth, so we can conclude that, in fact, $y_\beta^{-1} \circ y_\alpha$ is smooth. Therefore, this construction gives a differentiable structure on TM. Since each $U_\alpha \times \mathbb{R}^n$ is an open subset of $\mathbb{R}^{2n}$, we see that TM is a smooth $2n$-manifold.

Now, we need to demonstrate first that π is a smooth map under this structure. Let $q \in TM$, $p = \pi(q) \in M$. Let $\phi_\alpha : U_\alpha \to M$ be a coordinate chart containing p and let $y_\alpha : U_\alpha \times \mathbb{R}^n \to TM$ be the associated coordinate chart containing q. Now, for any

$$v = (u_1^\alpha, \dots, u_n^\alpha, y_1, \dots, y_n) \in U_\alpha \times \mathbb{R}^n,$$

let $p_0 = \phi_\alpha(u_1^\alpha, \dots, u_n^\alpha)$. Then

$$\pi \circ y_\alpha(v) = \pi(y_1\Phi_{1p_0} + \dots + y_n\Phi_{2p_0}) = p_0 = \phi_\alpha(u_1^\alpha, \dots, u_n^\alpha).$$

Hence,

$$\phi_\alpha^{-1} \circ \pi \circ y_\alpha = \phi_\alpha^{-1} \circ \phi_\alpha,$$

which is certainly differentiable. Hence, π is a smooth map.

To see that π is a smooth submersion, let $p \in M$ and let $X_p \in T_pM$. Recall that X_p is a map from $C^\infty(p) \to \mathbb{R}$. Now, define $Y_{X_p} \in T_{X_p}(TM)$ such that, if $f \in C^\infty(X_p)$ is constant on T_pM,

$$Y_{X_p}(f) = X_p(f \circ \pi^{-1}).$$

Then, for any $g \in C^\infty(p)$, we see that

$$d\pi(Y_{X_p})(g) = \pi_*(Y_{X_p})(g) = Y_{X_p}(g \circ \pi) = X_p((g \circ \pi) \circ \pi^{-1}) = X_p(g),$$

where this last two equalities hold because $g \circ \pi$ is constant on T_pM. Therefore, we see that

$$d\pi(Y_{X_p}) = X_p;$$

since our choice of p, and hence X_p, was arbitrary, we see that $d\pi$ is surjective. Since π is a smooth map, this implies that π is a smooth submersion. □

(b) Let M be a smooth n-manifold and let TM have the structure defined in the previous part. Show that $X : M \to TM$ is a smooth vector field if and only if $X : M \to TM$ is a smooth map such that $\pi \circ X = \mathrm{id}_M$.

Proof. (⇒) Suppose $X : M \to TM$ is a smooth vector field. Then, if $p \in M$ and $\phi_\alpha : U_\alpha \to M$ is a coordinate chart containing p,

$$X_p = \sum_{j=1}^{n} y_j(p)\Phi_{jp}$$

for smooth y_i. If $u_1^\alpha, \dots, u_n^\alpha$ are such that $\phi_\alpha(u_1^\alpha, \dots, u_n^\alpha) = p$, and y_α is the coordinate chart on X_p given above, then

$$y_\alpha^{-1} \circ X \circ \phi_\alpha(u_1^\alpha, \dots, u_n^\alpha) = y_\alpha^{-1}(X_p) = y_\alpha^{-1}(\sum_{j=1}^{n} y_j(p)\Phi_{jp}) = (u_1^\alpha, \dots, u_n^\alpha, y_1, \dots, y_n)$$

Each of these coordinates is a smooth function of the $u_1^\alpha, \dots, u_n^\alpha$, so, since our choice of p was arbitrary, we see that this map is smooth and, therefore, X is a smooth map.

Now, choose the same p as before. Then

$$\pi \circ X(p) = \pi(\sum_{j=1}^{n} y_j(p)\Phi_{jp}) = p,$$

so, since our choice of p was arbitrary, we see that $\pi \circ X \equiv \mathrm{id}_M$.

($\Leftarrow$) On the other hand, suppose $X : M \to TM$ is a smooth map such that $\pi \circ X \equiv \mathrm{id}_M$. Let $p \in M$ and let $\phi_\alpha : U_\alpha \to M$ be a coordinate chart containing p; let y_α be the associated coordinate chart on TM containing $X(p)$. Then, since $X(p) \in TM$, we know that $X(p) \in T_qM$ for some $q \in M$. However, since $\pi \circ X \equiv \mathrm{id}_M$, it must be the case that, in fact, $X(p) \in T_pM$. Since this is true, we know that we can express $X(p)$ as a linear combination of the Φ_i's:

$$X(p) = \sum_{j=1}^{n} y_j(p)\Phi_{jp}.$$

Furthermore, since X is smooth,

$$y_\alpha^{-1} \circ X \circ \phi_\alpha(u_1^\alpha, \ldots, u_n^\alpha) = y_\alpha^{-1}(X_p) = y_\alpha^{-1}(\sum_{j=1}^{n} y_j(p)\Phi_{jp}) = (u_1^\alpha, \ldots, u_n^\alpha, y_1, \ldots, y_n)$$

must be smooth. In particular, the coordinate function y_k must be smooth for all $k = 1, \ldots, n$. Hence, we see that

$$X(p) = \sum_{j=1}^{n} y_j(p)\Phi_{jp}$$

where y_k smooth for all $k = 1, \ldots, n$. However, this is precisely the condition that X is a smooth vector field on M. □

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