GEOMETRY HW 4

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3.3.5

Consider the parametrized surface (Enneper's surface)

$$\phi(u,v) = \left(x - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2\right)$$

and show that

(a) The coefficients of the first fundamental form are

$$E = G = (1 + u^2 + v^2)^2, \quad F = 0.$$

Answer: We know that $E = \langle \Phi_1, \Phi_1 \rangle$, $F = \langle \Phi_1, \Phi_2 \rangle$ and $G = \langle \Phi_2, \Phi_2 \rangle$, so we need to calculate Φ_1 and Φ_2. Now,

$$\Phi_1 = (1 - u^2 + v^2, 2uv, 2u)$$

and

$$\Phi_2 = (2uv, 1 - v^2 + u^2, -2v).$$

Hence,

$$\begin{aligned} E = \langle \Phi_1, \Phi_1 \rangle &= (1 - u^2 + v^2)^2 + 4u^2v^2 + 4u^2 \\ &= 1 + 2u^2 + 2v^2 + u^4 + v^4 2u^2v^2 \\ &= (1 + u^2 + v^2)^2 \end{aligned}$$

$$\begin{aligned} F = \langle \Phi_1, \Phi_2 \rangle &= 2uv(1 - u^2 + v^2) + 2uv(1 - v^2 + u^2) - 4uv \\ &= 2uv - 2u^3v + 2uv^3 + 2uv - 2uv^3 + 2u^3v - 4uv \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} G = \langle \Phi_2, \Phi_2 \rangle &= 4u^2v^2 + (1 - v^2 + u^2)^2 + 4v^2 \\ &= 1 + 2u^2 + 2v^2 + u^4 + v^4 2u^2v^2 \\ &= (1 + u^2 + v^2)^2 \end{aligned} .$$

♣

(b) The coefficients of the second fundamental form are

$$e = 2, \quad g = -2, \quad f = 0.$$

Answer: Recall that

$$N = \frac{\Phi_1 \times \Phi_2}{||\Phi_1 \times \Phi_2||} = \frac{\Phi_1 \times \Phi_2}{\sqrt{EG - F^2}}$$

$$= \frac{(-2uv^2 - 2u - 2u^3, 2u^2v + 2v + 2v^3, 1 - 2u^2v^2 - u^4 - v^4)}{\sqrt{(1 + u^2 + v^2)^4}}$$

Hence,

$$\begin{aligned}
e &= \left\langle N, \frac{\partial^2\phi}{\partial u^2}\right\rangle \\
&= \tfrac{1}{(1+u^2+v^2)^2}\langle(-2uv^2-2u-2u^3, 2u^2v+2v+2v^3, 1-2u^2v^2-u^4-v^4), \\
&\quad (-2u, 2v, 2)\rangle \\
&= \tfrac{1}{(1+u^2+v^2)^2}2(1+2u^2+2v^2+u^4+v^4+2u^2v^2) \\
&= 2\tfrac{(1+u^2+v^2)^2}{(1+u^2+v^2)^2} \\
&= 2.
\end{aligned}$$

Now,

$$\begin{aligned}
f &= \langle N, \tfrac{\partial^2\phi}{\partial u\partial v}\rangle \\
&= \tfrac{1}{(1+u^2+v^2)^2}\langle(-2uv^2-2u-2u^3, 2u^2v+2v+2v^3, \\
&\quad 1-2u^2v^2-u^4-v^4), (2v, 2u, 0)\rangle \\
&= \tfrac{1}{(1+u^2+v^2)^2}2(-2uv^3-2uv+2u^3v-2u^3v+2uv+2uv^3) \\
&= \tfrac{1}{(1+u^2+v^2)^2}(0) \\
&= 0.
\end{aligned}$$

and

$$\begin{aligned}
g &= \langle N, \tfrac{\partial^2\phi}{\partial v^2}\rangle \\
&= \tfrac{1}{(1+u^2+v^2)^2}\langle(-2uv^2-2u-2u^3, 2u^2v+2v+2v^3, \\
&\quad 1-2u^2v^2-u^4-v^4), (2u, -2v, -2)\rangle \\
&= -\tfrac{1}{(1+u^2+v^2)^2}\langle(-2uv^2-2u-2u^3, 2u^2v+2v+2v^3, \\
&\quad 1-2u^2v^2-u^4-v^4), (2u, -2v, -2)\rangle \\
&= -e \\
&= -2.
\end{aligned}$$

That is to say,

$$e = 2, \quad g = -2, \quad f = 0.$$

♣

(c) The principal curvatures are

$$k_1 = \frac{2}{(1+u^2+v^2)^2}, \quad k_2 = \frac{-2}{(1+u^2+v^2)^2}.$$

Answer: Now,

$$k_1 = -a_{11} = \frac{-fF+eG}{EG-F^2}, \quad k_2 = -a_{22} = \frac{-fF+gE}{EG-F^2}.$$

Since $f = F = 0$,

$$k_1 = \frac{eG}{EG} = \frac{e}{E} \quad k_2 = \frac{gE}{EG} = \frac{g}{G}.$$

Hence,

$$k_1 = \frac{2}{(1+u^2+v^2)^2} \quad k_2 = \frac{-2}{(1+u^2+v^2)^2}.$$

♣

(d) The lines of curvature are the coordinate curves.

Proof. As given on pg. 161, the differential equation giving the lines of curvature is

$$\begin{vmatrix} (\alpha_1')^2 & -\alpha_1'\alpha_2' & (\alpha_2')^2 \\ E & F & G \\ e & f & g \end{vmatrix} = 0$$

where $\alpha(t) = \phi(\alpha_1(t), \alpha_2(t))$ is a regular curve. Now, recalling that $F = f = 0$, this reduces to the equation

$$0 = -\alpha_1'\alpha_2'(Eg - eG) = -\alpha_1'\alpha_2'(-2(1+u^2+v^2)^2 - 2(1+u^2+v^2)).$$

Since $-4(1+u^2+v^2)^2 \neq 0$, it follows that either $\alpha_1' = 0$ or $\alpha_2' = 0$. However, this occurs precisely when α is a coordinate curve, so we see that the lines of curvature are the coordinate curves. □

(e) The asymptotic curves are $u + v =$ const., $u - v =$ const.

Proof. From pg. 160, we see that a connected regular curve C is an asymptotic curve if and only if, for any parametrization $\alpha(t) = \phi(\alpha_1(t), \alpha_2(t))$,

$$e(\alpha_1')^2 + f\alpha_1'\alpha_2' + g(\alpha_2')^2 = 0.$$

Now, plugging in our values for e, f and g, we see that this reduces to the condition that

$$2(\alpha_1')^2 - 2(\alpha_2')^2 = 0.$$

Reducing, we see that

$$(u')^2 = (v')^2,$$

which is to say that

$$u' = \pm v'.$$

Integrating both sides of the above, we see that,

$$u = \pm v + C.$$

In other words,

$$u \mp v = C.$$

Hence, curves of this form are precisely the asymptotic curves. □

3.3.7

$\phi(u, v) = (\gamma(v)\cos u, \gamma(v)\sin u, \psi(v))$ is given as a surface of revolution with constant Gaussian curvature K. To determine the functions γ and ψ, choose the parameter v in such a way that $(\gamma')^2 + (\psi')^2 = 1$. Show that

(a) γ satisfies $\gamma'' + K\gamma = 0$ and ψ is given by $\psi = \int \sqrt{1 - (\gamma')^2}dv$; thus, $0 < u < 2\pi$, and the domain of v is such that the last integral makes sense.

Proof. We start by computing Φ_1, Φ_2:

$$\Phi_1 = (-\gamma(v)\sin u, \gamma(v)\cos u, 0), \quad \Phi_2 = (\gamma'(v)\cos u, \gamma'(v)\sin u, \psi'(v)).$$

From this, then, we can compute

$$E = \langle \Phi_1, \Phi_1 \rangle = (\gamma(v))^2(\sin^2 u + \cos^2 u) = (\gamma(v))^2,$$
$$F = \langle \Phi_1, \Phi_2 \rangle = -\gamma(v)\gamma'(v)\sin u \cos u + \gamma(v)\gamma'(v)\sin u \cos u = 0$$

and

$$G = \langle \Phi_2, \Phi_2 \rangle = (\gamma'(v))^2(\sin^2 u + \cos^2 u) + (\psi'(v))^2 = (\gamma'(v))^2 + (\psi'(v))^2 = 1.$$

Recall that

$$e = 1/M(\Phi_1, \Phi_2, \frac{\partial^2 \phi}{\partial u^2})$$

$$f = 1/M(\Phi_1, \Phi_2, \frac{\partial^2 \phi}{\partial u \partial v})$$

and

$$g = 1/M(\Phi_1, \Phi_2, \frac{\partial^2 \phi}{\partial v^2})$$

where $M = \sqrt{EG - F^2}$ and (a, b, c) denotes the determinant of the matrix with columns given by a, b and c. Now,

$$\sqrt{EG - F^2} = \sqrt{(\gamma(v)^2} = \phi(v).$$

Hence,

$$\begin{aligned} e &= \tfrac{1}{\gamma(v)} \begin{vmatrix} -\gamma \sin u & \gamma' \cos u & -\gamma \cos u \\ \gamma \cos u & \gamma' \sin u & -\gamma \sin u \\ 0 & \psi' & 0 \end{vmatrix} \\ &= \tfrac{1}{\gamma(v)} - \psi'(\gamma^2 \sin^2 u + \gamma^2 \cos^2 u) \\ &= \tfrac{-\gamma^2 \psi'}{\gamma} \\ &= -\gamma \psi'. \end{aligned}$$

Furthermore,

$$\begin{aligned} f &= \tfrac{1}{\gamma(v)} \begin{vmatrix} -\gamma \sin u & \gamma' \cos u & -\gamma' \sin u \\ \gamma \cos u & \gamma' \sin u & \gamma' \cos u \\ 0 & \psi' & 0 \end{vmatrix} \\ &= \tfrac{1}{\gamma} - \psi'(-\gamma\gamma' \sin u \cos u + \gamma\gamma' \sin u \cos u) \\ &= 0. \end{aligned}$$

and

$$\begin{aligned} g &= \tfrac{1}{\gamma(v)} \begin{vmatrix} -\gamma \sin u & \gamma' \cos u & \gamma'' \cos u \\ \gamma \cos u & \gamma' \sin u & \gamma'' \sin u \\ 0 & \psi' & \psi'' \end{vmatrix} \\ &= \tfrac{1}{\gamma}[-\psi'(-\gamma\gamma'' \sin^2 u + \gamma\gamma'' \cos^2 u) + \psi''(-\gamma\gamma' \sin^2 u - \gamma\gamma' \cos^2 u)] \\ &= \tfrac{1}{\gamma}(\psi\gamma\gamma'' - \psi''\gamma\gamma') \\ &= \psi\gamma'' - \psi''\gamma'. \end{aligned}$$

Now, if we differentiate both sides of the equation $(\gamma')^2 + (\psi')^2 = 1$, we see that $\gamma'\gamma'' = -\psi'\psi''$. Hence,

$$\begin{aligned} K &= \tfrac{eg-f^2}{EG-F^2} \\ &= \tfrac{-\gamma\psi'(\psi'\gamma''-\psi''\gamma')}{\gamma^2} \\ &= \tfrac{-\psi'(\psi'\gamma''-\psi''\gamma')}{\gamma} \\ &= \tfrac{-(\psi')^2\gamma''+\psi'\psi''\gamma'}{\gamma} \\ &= -\tfrac{(\psi')^2\gamma''+(\gamma')^2\gamma''}{\gamma} \\ &= -\tfrac{\gamma''}{\gamma}. \end{aligned}$$

Re-arranging, we see that γ satisfies

$$\gamma'' + k\gamma = 0.$$

Now, on the other hand, since $(\gamma')^2 + (\psi')^2 = 1$, we can re-arrange to see that

$$\psi' = \sqrt{1-(\gamma')^2}.$$

Hence,

$$\psi = \int \sqrt{1-(\gamma')^2}dv,$$

where v is such that this integral makes sense. □

(b) All surfaces of revolution with constant curvature $K = 1$ which intersect perpindicularly the plane xOy are given by

$$\gamma(v) = C\cos v, \quad \psi(v) = \int_0^v \sqrt{1-C^2\sin^2 v}dv,$$

where C is a constant. Determine the domain of v and draw a rough sketch of the profile o f the surface in the xz plane for the cases $C = 1$, $C > 1$, $C < 1$. Observe that $C = 1$ gives the sphere.

Proof. Recall that in part (a) we showed that γ satisfies the equation $\gamma'' + K\gamma = 0$. Plugging in $K = 1$, we see that

$$\gamma'' + \gamma = 0.$$

This has characteristic equation $\lambda^2 + 1 = 0$, meaning the solutions are $\lambda = \pm\sqrt{-1}$. Hence,

$$\gamma(v) = C_1e^{0v}\cos v + C_2e^{0v}\sin v = C_1\cos v + C_2\sin v.$$

After translating the curve, we may assume that at $v = 0$ the curve $(\gamma(v), \psi(v))$ passes through $(C, 0)$ and $(\gamma'(0), \psi'(0)) = (0, z)$. Then

$$0 = \gamma'(0) = -C_1\sin(0) + C_2\cos(0) = C_2.$$

Hence, $\gamma(v) = C_1\cos v$.

Now, using our formula for ψ given in part (a) and plugging in the above for γ, we see that

$$\psi(v) = \int_0^v \sqrt{1-(\gamma')^2}dv = \int_0^v \sqrt{1-C^2\sin^2 v}dv.$$

where $v \in [-\sin^{-1}(1/C), \sin^{-1}(1/C)]$. □

(c) All surfaces of revolution with constant curvature $K = -1$ may be given by one of the following types:

1. $\gamma(v) = C\cosh v, \quad \psi(v) = \int_0^v \sqrt{1 - C^2\sinh^2 v}dv.$
2. $\gamma(v) = C\sinh v, \quad \psi(v) = \int_0^v \sqrt{1 - C^2\cosh^2 v}dv.$
3. $\gamma(v) = e^v, \quad \psi(v) = \int_0^v \sqrt{1 - e^{2v}}dv.$

Determine the domain of v and draw a rough sketch of the profile of the surface in the xz plane.

Proof. Plugging the value $K = -1$ into the expression derived in (a), we see that

$$\gamma'' - \gamma = 0.$$

The characteristic equation, then, is $\lambda^2 - 1 = 0$, so the solution of the differential equation is

$$\gamma(v) = C_1 e^v + C_2 e^{-v}$$

since $\lambda = \pm 1$ is the solution of the characteristic equation. Since there are no initial conditions given, we cannot necessarily reduce γ to one of the above three forms. If we had initial conditions that dictated that $C_1 = \pm C_2$ or $C_1 = 0$ or $C_2 = 0$, then we could clearly reduce to one of the above three cases.

□

(d) The surface of type 3 in part (c) is the pseudosphere of Exercise 6.

Proof. Now, the coordinates of a point $p \in S$ are given by

$$(e^v \cos u, e^v \sin u, \int_0^v \sqrt{1 - e^{2v}}dv).$$

Now, the pseudosphere is given by taking a curve in the xz-plane such that the length of th tractrix is equal to 1 at every point on the curve, and then rotating this curve about the z-axis. So we first demonstrate that the γ and ψ in 3 agree with such a curve in the xz-plane. To that end, let $y = 0$. Since $y = e^v \sin u$, this implies that $u = 0$. Hence, a point a in the plane that lies on this surface is given by

$$(e^v, 0, \int_0^v \sqrt{1 - e^{2v}}dv).$$

We let $x = e^v$, making $z = \int_0^v \sqrt{1 - x^2}dv = -\int_v^0 \sqrt{1 - x^2}$. Since $v \leq 0$ (and so $0 < x \leq 1$). Now, differentiating, we see that the tangent to the curve is given by

$$(e^v, 0, -\sqrt{1 - e^{2v}}) = (x, 0, -\sqrt{1 - x^2}).$$

This tangent, then, clearly has length

$$\sqrt{x^2 + (-\sqrt{1 - x^2})^2} = \sqrt{x^2 + 1 - x^2} = 1.$$

Therefore, the surface agrees with the pseudosphere on the xz-plane.

Now, if we fix v and let u vary, then we see that the coordinates of points on the surface are given by

$$(e^v \cos u, e^v \sin u, \int_0^v \sqrt{1-e^{2v}}).$$

Hence, this describes a circle of radius e^v, so we see that our surface is just the surface of revolution of the xz-plane curve we described above. However, this is precisely the pseudosphere. □

(e) The only surface of revolution with $K \equiv 0$ are the right circular cylinder, the right circular cone, and the plane.

Proof. If $K \equiv 0$, then, using the differential equation established in (a), then

$$\gamma'' = 0.$$

The only possibilities for γ, then, are that γ is a line, a constant not equal to zero, or zero. If γ is a line, then we see that the coordinates give a right circular cone, with its peak at the point where γ intersects the axis. On the other hand, if γ is a nonzero constant, then the coordinates give a right circular cylinder. Finally, if $\gamma \equiv 0$, then the surface is completely determined by z, which is itself constant, so the surface is simply a plane. □

3.3.22

Let $h : S \to \mathbb{R}$ be a differentiable function on a surface S, and let $p \in S$ be a critical point of h (i.e., $dh_p = 0$). Let $w \in T_pS$ and let

$$\alpha : (-\epsilon, \epsilon) \to S$$

be a parametrized curve with $\alpha(0) = p$, $\alpha'(0) = w$. Set

$$H_p h(w) = \frac{d^2(h \circ \alpha)}{dt^2}|_{t=0}.$$

(a) Let $\phi : U \to S$ be a parametrization of S at p, and show that

$$H_p h(u'\Phi_1 + v'\Phi_2) = h_{uu}(p)(u')^2 + 2h_{uv}(p)u'v' + h_{vv}(p)(v')^2.$$

Conclude that $H_p h : T_pS \to \mathbb{R}$ is a well-defined quadratic form on T_pS. $H_p h$ is called the *Hessian* of h at p.

Proof. First, we note that

$$\phi^{-1} \circ \alpha(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$$

and so

$$d(\phi^{-1} \circ \alpha)_t = \begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix}.$$

Note, furthermore, that

$$\begin{aligned} dh_p d\phi_q &= \left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}, \frac{\partial h}{\partial z}\right) \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} \\ &= \left(\frac{\partial h}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial h}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial h}{\partial z}\frac{\partial z}{\partial u}, \frac{\partial h}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial h}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial h}{\partial z}\frac{\partial z}{\partial v}\right) \\ &= (h_u, h_v). \end{aligned}$$

Since p is a critical point, recall that $dh_p = 0$ and so $h_u = h_v = 0$ when evaluated at the values associated with p. Putting all of this together, and recalling that $w = u'\Phi_1 + v'\Phi_2$, we see that

$$\begin{aligned} H_p h(u'\Phi_1 + v'\Phi_2) &= \frac{d^2(h\circ\alpha)}{dt^2}\Big|_{t=0} \\ &= \frac{d^2}{dt^2}(h \circ \alpha) \\ &= \frac{d^2}{dt^2}((h \circ \phi) \circ (\phi^{-1} \circ \alpha)) \\ &= \frac{d}{dt}(dh_p d\phi_q d(\phi^{-1} \circ \alpha)_{t=0}) \\ &= \frac{d}{dt}\left[(h_u, h_v)\begin{pmatrix} u' \\ v' \end{pmatrix}\right] \\ &= \frac{d}{dt}(h_u u' + h_v v') \\ &= \frac{dh_u}{dt}u' + h_u u'' + \frac{dh_v}{dt}v' + h_v v'' \\ &= \left(\frac{\partial h_u}{\partial u}\frac{du}{dt} + \frac{\partial h_u}{\partial v}\frac{dv}{dt}\right)u' + h_u u'' + \left(\frac{\partial h_v}{\partial u}\frac{du}{dt} + \frac{\partial h_v}{\partial v}\frac{dv}{dt}\right) + h_v v'' \\ &= (h_{uu}u' + h_{uv}v')u' + h_u u'' + (h_{vu}u' + h_{vv}v')v' + h_v v'' \\ &= h_{uu}(u')^2 + 2h_{uv}u'v' + h_{vv}(v')^2 + h_u u'' + h_v v'' \\ &= h_{uu}(u')^2 + 2h_{uv}u'v' + h_{vv}(v')^2 + 0u'' + 0v'' \\ &= h_{uu}(u')^2 + 2h_{uv}u'v' + h_{vv}(v')^2. \end{aligned}$$

Since $H_p h : T_p S \to \mathbb{R}$ does not depend on the choice of α, we see that $H_p h$ is indeed a well-defined quadratic form. □

(b) Let $h : S \to \mathbb{R}$ be the height function of S relative to $T_p S$; that is, $h(q) = \langle q - p, N(p)\rangle$, $q \in S$. Verify that p is a critical point of h and thus that the Hessian $H_p h$ is well defined. Show that if $w \in T_p S$, $|w| = 1$, then

$$H_p h(w) = \text{normal curvature at } p \text{ in the direction of } w.$$

Conclude that the Hessian at p of the height function relative to $T_p S$ is the second fundamental form of S at p.

Proof. Denote $q := \phi(u, v)$. Then

$$h(u, v) = \langle \phi(u, v) - p, N(p)\rangle = \langle \phi(u, v), N(p)\rangle - \langle p, N(p)\rangle.$$

Hence,

$$\frac{\partial h}{\partial u} = \langle \Phi_1, N(p)\rangle + 0 = \langle \Phi_1, N(p)\rangle$$

and

$$\frac{\partial h}{\partial v} = \langle \Phi_2, N(p)\rangle + 0 = \langle \Phi_2, N(p)\rangle.$$

However, since $\Phi_1, \Phi_2 \in T_p S$,

$$\langle \Phi_1, N(p)\rangle = \langle \Phi_2, N(p)\rangle = 0.$$

Hence, $dh_p = 0$, so we see that p is indeed a critical point of h.

Now,

$$h_{uu} = \frac{\partial}{\partial u}\langle \frac{\partial \phi}{\partial u}, N(p)\rangle = \langle \frac{\partial^2 \phi}{\partial u^2}, N(p)\rangle = e,$$

$$h_{uv} = \frac{\partial}{\partial v}\langle \frac{\partial \phi}{\partial u}, N(p)\rangle = \langle \frac{\partial^2 \phi}{\partial u \partial v}, N(p)\rangle = f,$$

and

$$h_{vv} = \frac{\partial}{\partial v}\langle \frac{\partial \phi}{\partial v}, N(p)\rangle = \langle \frac{\partial^2 \phi}{\partial v^2}, N(p)\rangle = g.$$

Hence, by the work we did in (a) above,

$$H_p h(w) = e(u')^2 + 2fu'v' + g(v')^2 = II_p(w).$$

Now, since the normal curvature at p in the direction of a unit vector v is simply $II_p(v)$, we see that, in fact

$$H_p h(w) = \text{normal curvature at } p \text{ in the direction of } w.$$

□

3.3.23

A critical point $p \in S$ of a differentiable function $h : S \to \mathbb{R}$ is *nondegenerate* if the self-adjoint linear map $A_p h$ associated to the quadratic form $H_p h$ is nonsingular. Otherwise, p is a *degenerate* critical point. A differentiable function on S is a *Morse function* if all its critical points are nondegenerate. Let $h_r : S \subset \mathbb{R}^3 \to \mathbb{R}$ be the distance function from S to r; i.e.,

$$h_r(q) = \sqrt{\langle q - r, q - r\rangle}, \quad q \in S, r \in \mathbb{R}^3, r \notin S.$$

(a) Show that $p \in S$ is a critical point of h_r if and only if the straight line pr is normal to S at p.

Proof. Let $\alpha(t)$ be a curve on S with $\alpha(0) = p$, $\alpha'(0) = w$. Then p is a critical point if and only if

$$\begin{aligned} 0 = dh_r(w) &= \tfrac{d}{dt}(\langle \alpha(0) - r, \alpha(0) - r\rangle^{1/2})|_{t=0} \\ &= \tfrac{-1}{2\langle \alpha(0)-r, \alpha(0)-r\rangle^{1/2}} \langle \alpha'(0), \alpha(0) - r\rangle \langle \alpha(0) - r, \alpha'(0)\rangle \\ &= \tfrac{-1}{2(\langle p-r,p-r\rangle)^{1/2}} \langle w, p - r\rangle \langle p - r, w\rangle \\ &= \tfrac{-2\langle w,p-r\rangle}{2(\langle p-r,p-r\rangle)^{1/2}} \\ &= \tfrac{-\langle w,p-r\rangle}{\sqrt{\langle p-r,p-r\rangle}} \end{aligned}$$

Hence, we see that it must be the case that $\langle w, p - r\rangle = 0$. Since this implies that the vector on the line joining p and r is orthogonal to w. Since $w \in T_p(S)$ and our choice of α was arbitrary, we see that the line joining p and r is normal to S at p if and only if p is a critical point of h_r. □

(b) Let p be a critical point of $h_r : S \to \mathbb{R}$. Let $w \in T_pS$, $|w| = 1$, and let $\alpha : (-\epsilon, \epsilon) \to S$ be a curve parametrized by arc length with $\alpha(0) = p$, $\alpha'(0) = w$. Prove that

$$H_p h_r(w) = \frac{1}{h_r(p)} - k_n,$$

where k_n is the normal curvature at p along the direction of w. Conclude that the orthonormal basis $\{e_1, e_2\}$, where e_1 and e_2 are alon gthe principal directions of T_pS, diagonalizes the self-adjoint linear map $A_p h_r$. Conclude further that p is a degenerate critical point of h_r if and only if either $h_r(p) = 1/k_1$ or $h_r(p) = 1/k_2$, where k_1 and k_2 are the principal curvatures at p.

Proof. Let ϕ be a coordinate chart at the point p. Then

$$h_r(q) = \sqrt{\langle q-r, q-r\rangle} = \sqrt{\langle \phi(u,v)-r, \phi(u,v)-r\rangle} = \sqrt{\langle \phi,\phi\rangle - 2\langle \phi, r\rangle + \langle r, r\rangle}$$

If we let

$$l(u,v) = \langle \phi,\phi\rangle - 2\langle \phi, r\rangle + \langle r, r\rangle,$$

then

$$h_r = \sqrt{l},$$

$$l_u = 2\langle \phi_u, \phi\rangle - 2\langle \phi_u, r\rangle = \langle \phi_u, \phi - r\rangle$$

and

$$l_v = 2\langle \phi_v, \phi\rangle - 2\langle \phi_v, r\rangle = \langle \phi_u, \phi - r\rangle.$$

Hence,

$$l_{uu} = 2\langle \phi_{uu}, \phi - r\rangle + 2\langle \phi_u, \phi_u\rangle,$$
$$l_{vv} = 2\langle \phi_{vv}, \phi - r\rangle + 2\langle \phi_u, \phi_u\rangle,$$

and

$$l_{uv} = 2\langle \phi_{uv}, \phi\rangle + 2\langle \phi_u, \phi_v\rangle.$$

Hence, since p is a critical point,

$$0 = h_{r_u} = \frac{1}{2} l^{-1/2} l_u$$

and

$$0 = h_{r_v} = \frac{1}{2} l^{-1/2} l_u.$$

Therefore,

$$h_{r_{uu}} = \frac{1}{2} l^{-1/2} l_{uu} - \frac{1}{4} l^{-3/2} (l_u)^2 = \frac{1}{2} l^{-1/2} l_{uu} - \frac{l_u}{4l} h_{r_u} = \frac{1}{2} l^{-1/2} l_{uu},$$
$$h_{r_{vv}} = \frac{1}{2} l^{-1/2} l_{vv} - \frac{1}{4} l^{-3/2} (l_v)^2 = \frac{1}{2} l^{-1/2} l_{vv} - \frac{l_v}{4l} h_{r_v} = \frac{1}{2} l^{-1/2} l_{vv}$$

and

$$h_{r_{uv}} = \frac{1}{2} l^{-1/2} l_{uv} - \frac{1}{4} l^{-3/2} l_u l_v == \frac{1}{2} l^{-1/2} l_{uv} - \frac{l_v}{4l} h_{r_u} == \frac{1}{2} l^{-1/2} l_{uv}.$$

Now, since p is a critical point, $p - r$ is normal to the surface at p, and so is equal to $h_r(p)N_p$, since $h_r(p)$ is definde to be the length of $p - r$. Also, notice that $I_p(w) = 1$ since $|w| = 1$. Hence, by what we showed in 22(a) above,

$$\begin{aligned} H_p h_r(w) &= h_{r_{uu}}(u')^2 + 2h_{r_{uv}}u'v' + h_{r_{vv}}(v')^2 \\ &= \tfrac{1}{2}l^{-1/2}l_{uu}(u')^2 + l^{-1/2}l_{uv}u'v' + \tfrac{1}{2}l^{-1/2}l_{vv}(v')^2 \\ &= \tfrac{1}{h_r(p)}(\tfrac{1}{2}l_{uu}(u')^2 + l_{uv}u'v' + \tfrac{1}{2}l_{vv}(v')^2) \\ &= \tfrac{1}{h_r(p)}(((\langle \phi_{uu}, p-r\rangle + \langle \phi_u, \phi_u\rangle)(u')^2 \\ &\quad +2(\langle \phi_{uv}, p-r\rangle + \langle \phi_u, \phi_v\rangle)u'v' + (\langle \phi_{vv}, p-r\rangle + \langle \phi_u, \phi_v\rangle)(v')^2) \\ &= \tfrac{1}{h_r(p)}((ke+E)(u')^2 + (kf+F)u'v' + (kg+G)(v')^2) \\ &= \tfrac{1}{h_r(p)}(I_p(w) - h_r(p)II_p(w)) \\ &= \tfrac{1}{h_r(p)} - II_p(w). \end{aligned}$$

Now, we need only recall that, since w is a unit vector, $II_p(w) = k_n$. Plugging this into the above equation, we see that

$$H_p h_r(w) = \frac{1}{h_r(p)} - k_n.$$

Since $h_r(p)$ is fixed in this context, we see that the value of the quadratic form $H_p h_r$ depends solely on the value of k_n. Now, k_1 and k_2 are the maximum and minimum values, respectively, of $k_n = II_p(w)$. With these values are associated the unit vectors e_1 and e_2, respectively, which are mutually orthogonal. Now, it is clear that e_1 minimizes $H_p h_r$ and e_2 maximizes $H_p h_r$. Hence, e_1 and e_2 are the eigenvectors of the associated self-adjoint linear map $A_p h_r$. Since $\{e_1, e_2\}$ is an orthonormal set, these vectors diagonalize the map $A_p h_r$.

Finally, p is a degenerate critical point of h_r if and only if $A_p h_r$ is singular; $A_p h_r$ is singular if and only if $H_p h_r(e_1) = 0$ or $H_p h_r(e_2) = 0$. Now, solving for $h_r(p)$, we see that

$$0 = H_p h_r(e_1) = \frac{1}{h_r(p)} - k_1 \quad \Leftrightarrow \quad h_r(p) = \frac{1}{k_1}$$

and

$$0 = H_p h_r(e_2) = \frac{1}{h_r(p)} - k_2 \quad \Leftrightarrow \quad h_r(p) = \frac{1}{k_2}.$$

Therefore, p is a degenerate critical point if and only if $h_r(p) = \frac{1}{k_1}$ or $h_r(p) = \frac{1}{k_2}$. □

(c) Show that the set

$$B = \{r \in \mathbb{R}^3 : h_r \text{ is a Morse function}\}$$

is an open and dense set in $\mathbb{R}^3$; here dense in $\mathbb{R}^3$ means that in each neighborhood of a given point of $\mathbb{R}^3$ there exists a point of B.

Proof. A point $a \in \mathbb{R}^3$ is associated with a non-Morse function h_a if and only if $h_a(p) = \frac{1}{k_1}$ or $h_a(p) = \frac{1}{k_2}$ for some $p \in S$ such that the line from p to a is parallel to N_p. Hence, if we attach line segments of length $\frac{1}{k_1}$ and $\frac{1}{k_2}$ parallel to N_p and $-N_p$ to each $p \in S$, then the endpoints of these line segments will comprise the entirety of the set $\mathbb{R}^3\backslash B$. We want to show that, for any point $q \in \mathbb{R}$, and any ϵ-ball $B_\epsilon(q)$ centered at q, there is a point $r \in B_\epsilon(q)$ such that $r \in B$; that is, r is not an endpoint of one of these line segments.

Now, suppose B is not dense in $\mathbb{R}^3$. Then there exists a $q \in \mathbb{R}^3$ and an $\epsilon > 0$ such that the open ϵ-ball $B_\epsilon(q)$ centered at q is contained in the complement of B. That is to say, every point in $B_\epsilon(q)$ is the endpoint of one of the line-segments constructed above. Specifically, q is the endpoint of such a line-segment l; suppose, without loss of generality, that the length of l is $\frac{1}{k_1}$. In this situation, there are two possibilities:

(1) If we translate $B_\epsilon(q)$ by a distance $\frac{1}{k_1}$ such that the image of q is in S, then the image of the ball is contained in S. However, this is clearly impossible, as S is locally diffeomorphic to $\mathbb{R}^2$ and, as such, certainly cannot contain such a ball.

(2) If, when we perform the above, there are points in the ball whose image is not contained in S. Denote by $B'_\epsilon(q)$ the ball with all points whose image under this $\frac{1}{k_1}$-translation is contained in S deleted. Then, if we translate $B'_\epsilon(q)$ by a distance of $\frac{1}{k_2}$, then its image will be contained in S. However, again this is impossible, since $B'_\epsilon(q)$ is not locally diffeomorphic to $\mathbb{R}^2$, either.

From this contradiction, then, we conclude that, in fact, B is dense in $\mathbb{R}^3$. □

3.4.1

Prove that the differentiability of a vector field does not depend on the choice of a coordinate system.

Proof. Suppose w is a vector field on a neighborhood $U \subseteq S$ with a coordinate chart $\phi(u, v)$ such that $a(u, v)$ and $b(u, v)$ are differentiable, where

$$w(p) = a(u, v)\Phi_1 + b(u, v)\Phi_2.$$

Now, suppose there is another coordinate chart $\psi(u, v)$ on U such that

$$w(p) = c(u', v')\Psi_1 + d(u', v')\Psi_2.$$

Then it suffices to show that c and d are differentiable.

Recall that

$$\Phi_1 = \frac{\partial u'}{\partial u}\Psi_1 + \frac{\partial v'}{\partial u}\Psi_2$$

and

$$\Phi_2 = \frac{\partial u'}{\partial v}\Psi_1 + \frac{\partial v'}{\partial v}\Psi_2.$$

Hence,

$$\begin{aligned} w(p) &= a(u,v)\Phi_1 + b(u,v)\Phi_2 \\ &= a(u,v)\left(\tfrac{\partial u'}{\partial u}\Psi_1 + \tfrac{\partial v'}{\partial u}\Psi_2\right) + b(u,v)\left(\tfrac{\partial u'}{\partial v}\Psi_1 + \tfrac{\partial v'}{\partial v}\Psi_2\right) \\ &= \left(a(u,v)\tfrac{\partial u'}{\partial u} + b(u,v)\tfrac{\partial u'}{\partial v}\right)\Psi_1 + \left(a(u,v)\tfrac{\partial v'}{\partial u} + b(u,v)\tfrac{\partial v'}{\partial v}\right)\Psi_2 \\ &= c(u',v')\Psi_1 + d(u',v')\Psi_2. \end{aligned}$$

Hence, since we see that c and d are merely sums of products of differentiable functions, c and d are differentiable. Therefore, w is a differentiable vector field with respect to ψ as well. Since our choice of ψ was arbitrary, we see that differentiability of the vector field w is independent of the choice of coordinate system. □

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