TOPOLOGY TAKE-HOME

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1. The Discrete Topology

Let $Y = \{0, 1\}$ have the discrete topology. Show that for any topological space X the following are equivalent.

(a) X has the discrete topology.

(b) Any function $f : X \to Y$ is continuous.

(c) Any function $g : X \to Z$, where Z is some topological space, is continuous.

Proof. (a ⇒ c) Suppose X has the discrete topology and that Z is a topological space. Let $f : X \to Z$ be any function and let $U \subseteq Z$ be open. Then $f^{-1}(U) \subseteq X$ is open, since X has the discrete topology. Since our choice of U was arbitrary, we see that f is continuous. Since our choice of f was arbitrary, we see that any function $f : X \to Z$ is continuous.

(c ⇒ b) Suppose any function $g : X \to Z$, where Z is a topological space, is continuous. Then, since Y is a topological space, we see that any function $f : X \to Y$ is continuous.

(b ⇒ a) Suppose any function $f : X \to Y$ is continuous. Let $p \in X$. Then certainly the function $f_p : X \to Y$ defined by

$$f_p(x) = \begin{cases} 1 & x = p \\ 0 & x \neq p \end{cases}$$

is continuous. Since $\{1\}$ is open in Y, this means that $f^{-1}(\{1\}) = \{p\}$ is open in X. Now, since our choice of p was arbitrary, we see that $\{x\}$ is open in X for all $x \in X$. Since every singleton is open in X, that means that every subset of X is open in X (since each subset of X can be constructed from a union of singletons), so X has the discrete topology.

Since we've shown that $a \Rightarrow c \Rightarrow b \Rightarrow a$, we see that (a), (b) and (c) are equivalent. □

2. Compactness

Prove or disprove: If K_1 and K_2 are disjoint compact subsets of a Hausdorff space X, then there exist disjoint open sets U and V such that $K_1 \subseteq U$ and $K_2 \subseteq V$.

Proof. Let $y \in K_2$. Then, since X is Hausdorff, for all $x \in K_1$ there exist disjoint neighborhoods U_x and V_x of x and y, respectively. Then $\{U_x\}_{x \in K_1}$

is an open cover of K_1. Since K_1 is compact, there exist $x_1, \ldots, x_n \in K_1$ such that

$$K_1 \subseteq \bigcup_{j=1}^{n} U_{x_j} = U_y.$$

Now, let $V_y = \bigcap_{j=1}^{n} V_{x_j}$. This is a neighborhood of y and $V_y \cap U_y = \emptyset$. We can perform the same procedure to yield an open neighborhood V_y of each $y \in K_2$ that is disjoint from an open set U_y containing K_1. Now, $\{V_y\}_{y \in K_2}$ forms an open cover of K_2. Since K_2 is compact, there exist $y_1, \ldots, y_m$ such that

$$K_2 \subseteq \bigcup_{k=1}^{m} V_{y_k} = V.$$

Now, if we let

$$U = \bigcap_{k=1}^{m} U_{y_k},$$

then U is open and $K_1 \subseteq U$. Furthermore, $U \cap V = \emptyset$, since no point in V is in every U_{y_k}. □

3. Dense Sets

(a) Suppose $O_1, O_2 \subseteq X$ are open and dense subsets of X. Show that $O_1 \cap O_2$ is dense in X.

First, we prove the following lemma:

Lemma 3.1. *If $A, B \subseteq X$ such that A is open, then $\overline{A} \cap \overline{B} \subseteq \overline{A \cap B}$*

Proof. Let $x \in \overline{A} \cap \overline{B}$ and let U be a neighborhood of x. Then, since $x \in \overline{A}$,

$$U \cap A \neq \emptyset.$$

Furthermore, since A is open, $U \cap A$ is a neighborhood of x. Therefore, since $x \in \overline{B}$,

$$\emptyset \neq (U \cap A) \cap B = U \cap (A \cap B),$$

which, since our choice of the neighborhood U was arbitrary, is to say that $x \in \overline{A \cap B}$. Since our choice of x was arbitrary, this means that

$$\overline{A} \cap \overline{B} \subseteq \overline{A \cap B}.$$

□

Now, we are ready to show that $O_1 \cap O_2$ is dense in X. To do so, we merely note that, since O_1 and O_2 are dense in X, $\overline{O_1} = X$ and $\overline{O_2} = X$. Now, applying Lemma 3.1,

$$X = X \cap X = \overline{O_1} \cap \overline{O_2} \subseteq \overline{O_1 \cap O_2} \subseteq X,$$

we see that, in fact, $\overline{O_1 \cap O_2} = X$, so $O_1 \cap O_2$ is dense in X.

♣

(b) Suppose $A, B \subseteq X$ are nowhere dense subsets. Show that $A \cup B$ is nowhere dense in X.

To show this, first we prove the following three lemmas:

Lemma 3.2. *If $A \subseteq X$, then $x \in IntA \Leftrightarrow$ there is a neighborhood U of x such that $U \subseteq A$.*

Proof. If $x \in \text{Int}A$, then clearly there is a neighborhood U of x such that

$$x \in U \subseteq \text{Int}A \subseteq A.$$

On the other hand, if $x \in A$ and there exists a neighborhood U of x contained in A, then $U \subseteq \text{Int}A$, since $\text{Int}A$ is the union of all open sets contained in A. □

Lemma 3.3. *If $A, B \subseteq X$, then*

$$Int(A \cup B) \subseteq (IntA) \cup (IntB).$$

Proof. Let $x \in \text{Int}(A \cup B)$. Then, by Lemma 3.2, there is a neighborhood U of x such that $U \subseteq A \cup B$. Now, either $x \in A$ or $x \in B$. Suppose, without loss of generality, that $x \in A$. Then

$$x \in A \cap U \subseteq A,$$

so $x \in \text{Int}A$. Hence, $x \in (\text{Int}A) \cup (\text{Int}B)$. Since our choice of x was arbitrary, we conclude that $\text{Int}(A \cup B) \subseteq (\text{Int}A) \cup (\text{Int}B)$. □

Lemma 3.4. *If $A, B \subseteq X$,*

$$\overline{A \cup B} = \overline{A} \cup \overline{B}.$$

Proof. Let $x \in \overline{A \cup B}$ and let U be a neighborhood of x. Since U is non-empty, either $A \cap U \neq \emptyset$ or $B \cap U \neq \emptyset$. Suppose, without loss of generality, that $A \cap U \neq \emptyset$. Then, since our choice of neighborhood was arbitrary, $x \in \overline{A} \subseteq \overline{A} \cup \overline{B}$. Hence, since our choice of x was arbitrary,

$$\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}.$$

On the other hand, let $y \in \overline{A} \cup \overline{B}$. Then either $y \in \overline{A}$ or $y \in \overline{B}$. Suppose, without loss of generality, that $y \in \overline{B}$. Then every neighborhood of y intersects B, which means that every neighborhood of y intersects $A \cup B$, so $y \in \overline{A \cup B}$. Since our choice of y was arbitrary, we see that

$$\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}.$$

Having proved containment in both directions, we conclude that

$$\overline{A \cup B} = \overline{A} \cup \overline{B}.$$

□

Now, we are ready to prove the desired result. Let $A, B \subseteq X$ be nowhere dense subsets. That is to say,

$$\text{Int}\overline{A} = \emptyset = \text{Int}\overline{B}.$$

Now, by Lemmas 3.3 and 3.4,

$$\mathrm{Int}\overline{(A \cup B)} = \mathrm{Int}(\overline{A} \cup \overline{B}) \subseteq (\mathrm{Int}\overline{A}) \cup (\mathrm{Int}\overline{B}) = \emptyset \cup \emptyset = \emptyset.$$

In other words, $A \cup B$ is nowhere dense.

♣

4. Maps from the Circle to $\mathbb{R}$

Show that for any continuous map $f : S^1 \to \mathbb{R}$ there exists $x_0 \in S^1$ such that $f(x_0) = f(-x_0)$. So, in particular, this shows that if temperature is a continuous funcion , then at any given moment there are two antipodal points on the equator with exactly the same temperature.

Proof. Let $f : S^1 \to \mathbb{R}$ be a continuous map. Define $h : S^1 \to \mathbb{R}$ by

$$h(x) = f(x) - f(-x).$$

Then h is continuous, as it is merely the difference of two continuous functions. Since S^1 is connected (if this isn't immediately obvious, we need only note that S^1 is the continuous image of $\mathbb{R}^2\backslash\{(0,0)\}$ as we see in problem 6; since $\mathbb{R}^2\backslash\{(0,0)\}$ is connected, so is S^1) and $\mathbb{R}$ is an ordered set in the order topology, we can apply the Intermediate Value Theorem to h. Note that

$$h(-x) = f(-x) - f(-(-x)) = f(-x) - f(x) = -(f(x) - f(-x)) = -h(x)$$

for all $x \in S^1$. Hence, either $h \equiv 0$, in which case $f(x) = f(-x)$ for all $x \in S^1$, or there is some point $z \in S^1$ such that $h(-z) < 0 < h(z)$. In this second case, we see, by the Intermediate Value Theorem, that there exists some point $x_0 \in S^1$ such that

$$0 = h(x_0) = f(x_0) - f(-x_0) \Longrightarrow f(x_0) = f(-x_0).$$

In either case, there exists $x_0 \in S^1$ such that $f(x_0) = f(-x_0)$. □

5. Manifolds

An m-dimensional manifold is a topological space M such that

(a) M is Hausdorff

(b) M has a countable basis for its topology. That is, M is second countable.

(c) For each $p \in M$ there exists a neighborhood U of p and a homeomorphism $\phi : U \to V \subseteq \mathbb{R}^m$, where V is an open subset of $\mathbb{R}^m$. The ordered pair (U, ϕ) is often referred to as a *coordinate chart*.

Now, recall Brouwer's theorem on the invariance of domain.

Theorem 5.1. *Let $A, B \subseteq \mathbb{R}^n$ be homeomorphic subspaces. If A is open in $\mathbb{R}^n$, then B is also open in $\mathbb{R}^n$.*

Let M and N be m and n-dimensional topological manifolds, respectively. Use Brouwer's theorem to show that if $m \neq n$ then M and N cannot be homeomorphic.

Proof. By contradiction. Suppose $m \neq n$ and $f : M \to N$ is a homeomorphism. Without loss of generality, assume $m < n$. Let $p \in N$. Since f is a homeomorphism, $f^{-1}(p)$ is a single point. Furthermore, since M is an m-manifold, there exists a neighborhood U of $f^{-1}(p)$ such that $\phi : U \to V$ is a homeomorphism, where $V \subseteq \mathbb{R}^m$ is open. Also, U is homeomorphic to $f(U)$, which is a neighborhood of p. Since f and ϕ are homeomorphisms,

$$h = \phi \circ f^{-1} : f(U) \to V$$

is a homeomorphism. Furthermore, since N is an n-manifold, there exists a neighborhood A of p such that

$$g : A \to B \subseteq \mathbb{R}^n$$

is a homeomorphism, where B is open. Let $C = f(U) \cap A$. Then C is homeomorphic to both $h(C)$ and $g(C)$. The inclusion map

$$i : \mathbb{R}^m \to \mathbb{R}^n$$

is a homeomorphism, so $h(C)$ is naturally homeomorphic to some $D := i(h(C)) \subseteq \mathbb{R}^n$. Following the chain of homeomorphisms, we see that D is homeomorphic to $g(C)$, which is open since g^{-1} is continuous and $f(U) \cap A$ is open in A. Therefore, by Brouwer's Theorem, D is open in $\mathbb{R}^n$. However,

$$D = h(C) \times \underbrace{(\{0\} \times \ldots \times \{0\})}_{n-m}.$$

Since $h(C)$ is non-empty (at the very least it contains $h(p)$), there is a point

$$d = (d_1, d_2, \ldots, d_m, 0, 0, \ldots 0) \in D.$$

Let

$$W = \prod_{i=1}^{n} (a_i, b_i)$$

be a basic open set containing d. Then

$$d' = \left(d_1, d_2, \ldots, d_m, \frac{b_{m+1}}{2}, \ldots, \frac{b_n}{2} \right) \in W,$$

but $d' \notin D$. Therefore, since our choice of basic neighborhood W was arbitrary, we conclude that D is not open, contradicting the result from Brouwer's Theorem. From this contradiction, then, we conclude that if $m \neq n$, then M and N cannot be homeomorphic. □

6. Quotient Spaces

Let $X = \mathbb{R}^2 \backslash \{(0,0)\}$. On X define the equivalence relation $\sim$ by saying that $x \sim y$ if and only if there exists a $0 \neq t \in \mathbb{R}$ such that $x = ty$. Let X^* be the collection of equivalence classes in the quotient topology. What familiar space is X^* homeomorphic to? Prove it

We claim that X^* is homeomorphic to P, the projective plane; in order to prove this, we need the following two lemmas, originally proved in Homework 3 (this is problem 2 from Section 22):

Lemma 6.1. *Let $p : X \to Y$ be a continuous map. If there is a continuous map $f : Y \to X$ such that $p \circ f$ equals the identity map of Y, then p is a quotient map.*

Proof. If there exists a continuous map $f : Y \to X$ such that $p \circ f \equiv id_Y$, then we want to show that p is a quotient map. p is clearly surjective since, if it were not, $p \circ f$ could not be equal to the identity map. Now, let $U \subset Y$. If $p^{-1}(U)$ is open in X, then

$$U = (p \circ f)^{-1}(U) = f^{-1}(p^{-1}(U))$$

is open in Y since f is continuous. Hence, p is a surjective, continuous open map, so it is necessarily a quotient map. □

Lemma 6.2. *If $A \subset X$, a retraction of X onto A is a continuous map $r : X \to A$ such that $r(a) = a$ for each $a \in A$. A retraction is a quotient map.*

Proof. Let $i : A \to X$ be the inclusion map. Then i is continuous and, for all $a \in A$,

$$(r \circ i)(a) = r(i(a)) = r(a) = a$$

so $r \circ i \equiv id_A$. Hence, since r is continuous, we can conclude by Lemma 6.1 that r is a quotient map. □

With these two lemmas in hand, we are now ready to prove the following proposition:

Proposition 6.3. *X^* is homeomorphic to P, the projective plane.*

Proof. Define the map $g : X \to S^1$ by

$$g(z) = \frac{z}{||z||}$$

(here we are thinking of X as $\mathbb{C}\backslash\{0\}$). Since $||z|| > 0$, this is a continuous map. Furthermore, this map is constant on S^1 (since, for all $x \in S^1$, $||x|| = 1$), so g is surjective and g is a retraction. By Lemma 6.2, then, we see that g is a quotient map.

Now, consider the space $P = \{\{-z, z\} | z \in S^1\}$, the projective space. Note that P is a partition of S^1 into disjoint subsets whose union is S^1, so we let $p : S^1 \to P$ be the surjective map which carries each point of S^1 to the element of P containing it. Then, by definition, p is a quotient map. If $U \subseteq P$ is open, then

$$(p \circ g)^{-1}(U) = g^{-1}(p^{-1}(U))$$

which is open in $\mathbb{C}\backslash\{0\}$, since $p^{-1}(U)$ is open and g is continuous. Similarly, if $(p \circ g)^{-1}(U) \subseteq \mathbb{C}\backslash\{0\}$ is open, then

$$U = ((p \circ g) \circ (p \circ g)^{-1})(U) = p(g((p \circ g)^{-1}(U)))$$

is open in P, since g is a quotient map (meaning $g((p \circ g)^{-1}(U))$ is open in S^1) and p is a quotient map. Therefore, $p \circ g$ is a quotient map.

Let $z_1, z_2 \in \mathbb{C}\backslash\{0\}$. Note that if $z_1 \sim z_2$, then $z_1 = tz_2$ for some $t \neq 0$, which means that

$$\begin{aligned}(p \circ g)(z_1) = p(g(z_1)) = p\left(\frac{z_1}{||z_1||}\right) &= \left\{\frac{z_1}{||z_1||}, \frac{-z_1}{||z_1||}\right\} \\ &= \left\{\frac{t}{|t|}\frac{z_1}{||z_1||}, \frac{-t}{|t|}\frac{z_1}{||z_1||}\right\} \\ &= p\left(\frac{tz_1}{||tz_1||}\right) \\ &= p(g(tz_1)) \\ &= (p \circ g)(tz_1) \\ &= (p \circ g)(z_2).\end{aligned}$$

Also, if $(p \circ g)(z_1) = (p \circ g)(z_2)$, then

$$z_1 = \pm\frac{||z_1||}{||z_2||}z_2,$$

so $z_1 \sim z_2$. Hence, $X^* = \{(p \circ g)^{-1}(z) | z \in P\}$. Therefore, since $p \circ g$ is a quotient map, we can use Corollary 22.3 to conclude that X^* is homeomorphic to P. □

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