TOPOLOGY HW 4

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24.1

(a) Show that no two of the spaces $(0,1)$, $(0,1]$, and $[0,1]$ are homeomorphic.

Proof. Suppose $(0,1)$ and $(0,1]$ are homeomorphic, with the homeomorphism given by f. If $A = (0,1) - \{f^{-1}(1)$, then $f|_A : A \to (0,1)$ is a homeomorphism, by Theorem 18.2(d). However, the interval $(0,1)$ is connected, whereas

$$A = (0, f^{-1}(1)) \cup (f^{-1}(1), 1)$$

is not connected, so A and $(0,1)$ cannot be homeomorphic. From this contradiction, then, we conclude that $(0,1)$ and $(0,1]$ are not homeomorphic.

Similarly, suppose $g : (0,1] \to [0,1]$ is a homeomorphism. Then, if $B = (0,1] - \{g^{-1}(0), g^{-1}(1)\}$, $g|_B : B \to (0,1)$ is a homeomorphism. However, $g^{-1}(0) \neq g^{-1}(1)$, so at most one of these can be 1, meaning one must lie in the interval $(0,1)$. Suppose, without loss of generality, that $g^{-1}(0) \in (0,1)$. Then

$$B = (0, g^{-1}(0)) \cup (g^{-1}(0), 1] - \{g^{-1}(1)$$

is not connected, whereas $(0,1)$ is, so the two cannot be homeomorphic. From this contradiction, then, we conclude that $(0,1]$ and $[0,1]$ are not homeomorphic.

A similar argument easily demonstrates that $(0,1)$ and $[0,1]$ are not homeomorphic, so we see that no two of the spaces $(0,1)$, $(0,1]$, and $[0,1]$ are homeomorphic. □

(b) Suppose that there exist imbeddings $f : X \to Y$ and $g : Y \to X$. Show by means of an example that X and Y need not be homeomorphic.

Example: Let $f : (0,1) \to [0,1]$ be the canonical imbedding and let $g : [0,1] \to (0,1)$ such that

$$g(x) = \frac{x}{3} + \frac{1}{3}.$$

Then $g([0,1]) = [\frac{1}{3}, \frac{2}{3}]$. Obtain g' by restricting the range of g to $g([0,1]) = [\frac{1}{3}, \frac{2}{3}]$. We claim that g' is a homeomorphism. Since multiplication and addition are continuous, as are the inclusion map and compositions of continuous functions, we see that g' is continuous, as is g'^{-1}, where

$$g'^{-1}(x) = 3\left(x - \frac{1}{3}\right).$$

Both of these maps are also bijective, so we see that g' is indeed a homeomorphism, meaning g is an imbedding.

However, as we saw in part (a) above, $(0,1)$ and $[0,1]$ are not homeomorphic, so two spaces need not be homeomorphic for each to be imbedded in the other.

♣

(c) Show $\mathbb{R}^n$ and $\mathbb{R}$ are not homeomorphic if $n > 1$.

Lemma 0.1. *If $f : X \to Y$ is a homeomorphism and X is path-connected, then Y is path-connected.*

Proof. Let $x, y \in Y$. Then there exists a continuous path $g : [a,b] \to X$ such that $g(a) = f^{-1}(x)$ and $g(b) = f^{-1}(b)$. Define

$$h := f \circ g.$$

Then $h : [a,b] \to Y$ is continuous and

$$h(a) = (f \circ g)(a) = f(g(a)) = f(f^{-1}(x)) = x$$

and

$$h(b) = (f \circ g)(b) = f(g(b)) = f(f^{-1}(y)) = y$$

since f is bijective. Hence, h is a path from x to y, so, since our choice of x and y was arbitrary, Y is path connected. □

Proposition 0.2. *$\mathbb{R}^n$ and $\mathbb{R}$ are not homeomorphic if $n > 1$.*

Proof. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a homeomorphism. Then, restricting the domain to $\mathbb{R}^n - \{0\}$ gives a homeomorphism of the punctured euclidean space to $\mathbb{R} - \{f(0)\}$. However, the punctured euclidean space is path-connected (as shown in Example 4), whereas $\mathbb{R} - \{f(0)\}$ is not even connected, let alone path-connected. To see this, we need only note that

$$\mathbb{R} - \{f(0)\} = (-\infty, f(0)) \cup (f(0), \infty)$$

so the open sets $(-\infty, f(0))$ and $(f(0), \infty)$ give a separation of this space. Hence, by the above lemma, the punctured euclidean space and $\mathbb{R} - \{f(0)\}$ are not homeomorphic, a contradiction. Therefore, we conclude that $\mathbb{R}^n$ and $\mathbb{R}$ are not homeomorphic. □

1. 24.8

(a) Is a product of path-connected spaces necessarily path-connected?

Answer: Yes. Suppose X and Y are path-connected. Let $x_1 \times y_1, x_2 \times y_2 \in X \times Y$. Now, we know that $X \times y_1$ is homeomorphic to X and, therefore, is path-connected. Hence, there exists a continuous map $f : [0,1] \to X \times y_1$ such that

$$f(0) = x_1 \times y_1 \; f(1) = x_2 \times y_1.$$

Also, $x_2 \times Y$ is homeomorphic to Y and is, therefore, path connected, so there exists a continuous map $g : [0,1] \to x_2 \times Y$ such that

$$g(0) = x_2 \times y_1 \; g(1) = x_2 \times y_2.$$

Now, define

$$h(x) = \begin{cases} f\left(\frac{x}{2}\right) & 0 \le x \le \frac{1}{2} \\ g\left(\frac{x}{2} + \frac{1}{2}\right) & \frac{1}{2} \le x \le 1. \end{cases}$$

By the pasting lemma, then, h is continuous. Furthermore,

$$h(0) = f\left(\frac{0}{2}\right) = f(0) = x_1 \times y_1$$

and

$$h(1) = g\left(\frac{1}{2} + \frac{1}{2}\right) = g(1) = x_2 \times y_2.$$

Hence, h is a path from $x_1 \times y_1$ to $x_2 \times y_2$. Since our choice of $x_1 \times y_1$ and $x_2 \times y_2$ was arbitrary, we see that $X \times Y$ is path-connected.

♣

(b) If $A \subset X$ and A is path-connected, is $\overline{A}$ necessarily path connected?

Answer: No. In Example 7, we saw that, if $A = \{x \times \sin(1/x) | 0 < x \le 1\}$, then $\overline{A}$, the topologist's sine curve, is not path connected. To see that A is path-connected, let $s, y \in A$. Then $x = a \times \sin(1/a)$ for some $a \in (0, 1]$ and $y = b \times \sin(1/b)$ for some $b \in (0, 1]$. Define the map $f : [a, b] \to A$ by

$$f(z) = z \times \sin(1/z).$$

Then f is continuous since its coordinate functions are continuous and $f(a) = a \times \sin(1/a) = x$ and $f(b) = b \times \sin(1/b) = y$, so f is a path from x to y. Since our choice of x and y was arbitrary, we see that A is path-connected.

♣

(c) If $f : X \to Y$ is continuous and X is path-connected, is $f(X)$ necessarily path connected?

Answer: Yes. Let f be continuous and let $y_1, y_2 \in f(X)$. Let $x_1 \in f^{-1}(y_1)$ and $x_2 \in f^{-1}(y_2)$. Then, since X is path-connected, there exists a continuous map $g : [a, b] \to X$ such that $g(a) = x_1$ and $g(b) = x_2$. Define $h := f \circ g$. Then

$$h(a) = (f \circ g)(a) = f(g(a)) = f(x_1) = y_1$$

and

$$h(b) = (f \circ g)(b) = f(g(b)) = f(x_2) = y_2.$$

Furthermore, h is continuous, since f and g are, so h is a path from y_1 o y_2. Since our choice of y_1 and y_2 was arbitrary, we conclude that $f(X)$ is path-connected.

♣

(d) If $\{A_\alpha\}$ is a collection of path-connected subspaces of X and if $\bigcap A_\alpha \neq \emptyset$, is $\bigcup A_\alpha$ necessarily path-connected?

Answer: Yes. Let $x, y \in \bigcup X_\alpha$ and let $z \in \bigcap A_\alpha$. Then $x \in A_\beta$ and $y \in A_\gamma$ for some β and γ. Furthermore, $z \in A_\beta$, $z \in A_\gamma$. Since A_β is path connected, there exists a path f from x to z. Since A_γ is path connected, there exists a path g from z to y. Using the pasting lemma, we can glue these two paths together to make a path h from x to y (much as we did in part (a) above).

♣

25.1

What are the components and path components of $\mathbb{R}_\ell$? What are the continuous maps $f : \mathbb{R} \to \mathbb{R}_\ell$?

Answer: Each component and path component of $\mathbb{R}_\ell$ consists of a single point. To see this, suppose not. Then there exists a component containing two points, x and y. This means there exists a connected subspace $A \subset \mathbb{R}_\ell$ containing x and y. However, $x \in A \cap (-\infty, y)$ and $y \in A \cap [y, \infty)$, both of these subsets are open in A, and A is equal to their union, so A can be separated by $A \cap (-\infty, y)$ and $A \cap [y, \infty)$, a contradiction. Hence, we conclude that each component (and, therefore, path component) of $\mathbb{R}_\ell$ is a point.

Furthermore, since the continuous image of a connected space is connected, we can conclude that the continuous maps $f : \mathbb{R} \to \mathbb{R}_\ell$ are just the constant maps. This is because $\mathbb{R}$ is connected, so it's continuous image in $\mathbb{R}_\ell$ must be connected. The only connected subspaces of $\mathbb{R}_\ell$ are single points, so such a continuous map must map all of $\mathbb{R}$ to a single point.

♣

26.1

(a) Let τ and τ' be two topologies on the set X; suppose that $\tau' \supset \tau$. What does compactness of X under one of these topologies imply about compactness under the other?

Answer: If X is compact under τ', then it must be compact under τ. To see this, suppose $\mathcal{A}$ is an open cover of X in τ. Then $\mathcal{A}$ is also an open cover of X in τ', since every open set in τ is open in τ'. Since X is compact under τ', $\mathcal{A}$ contains a finite subcover of X. Since our choice of $\mathcal{A}$ was arbitrary, we conclude that X is compact under τ.

On the other hand, if $X = \mathbb{R}$, τ is the trivial topology and τ' is the discrete topology, then $\tau' \supset \tau$ and X is compact under τ, but not under τ'. To see this last, we merely construct the open cover $\{\{x\} | x \in \mathbb{R}\}$, which certainly contains no finite subcover.

♣

(b) Show that if X is compact Hausdorff under both τ and τ', then either τ and τ' are equal or they are not comparable.

Proof. Suppose $\tau \neq \tau'$ and that τ and τ' are comparable. Suppose, without loss of generality, that $\tau \subsetneq \tau'$. Then there exists $U \in \tau'$ such that $U \notin \tau$. Therefore, $X - U$ is not closed in X under τ. Since X is compact under τ, the contrapositive of Theorem 26.3 implies that $X - U$ is not compact.

Let $\{U_\alpha\}$ be an open cover of $X - U$ in τ. Since $\tau \subset \tau'$, $U_\alpha \in \tau'$ for all α. Furthermore, since $U \in \tau'$, $X - U$ is closed in X under τ'. Therefore, by Theorem 26.2, it is compact, so the open cover $\{U_\alpha\}$ contains a finite subcover $\{U_{\alpha_i}\}_{i=1}^n$. Each $U_\alpha \in \tau$, so it is certainly true that $U_{\alpha_i} \in \tau$ for all $i = 1, \ldots, n$. That is to say,

$$\{U_{\alpha_i}\}_{i=1}^n \subseteq \{U_\alpha\}_{\alpha \in J}$$

is an open cover of $X - U$ in τ. In other words, the open cover $\{U_\alpha\}$ contains a finite subcover. Since our choice of open cover was arbitrary, we conclude that, in fact, $X - U$ is compact under τ, a contradiction. From this contradiction, we conclude that either $\tau = \tau'$ or τ and τ' are not comparable. □

26.8

Let $f : X \to Y$; let Y be compact Hausdorff. Then f is continuous if and only if the *graph* of f,

$$G_f = \{x \times f(x) | x \in X\}$$

is closed in $X \times Y$.

Proof. $(\Rightarrow)$ Suppose f is continuous. To show G_f is closed, it suffices to show that $X \times Y - G_f$ is open in $X \times Y$. Therefore, let $x_0 \times y \in (X \times Y) - G_f$. Clearly, $y \neq f(x_0)$ so, since Y is Hausdorff, there exist open neighborhoods V_y and $V_{f(x_0)}$ of y and $f(x_0)$, respectively, such that

$$V_y \cap V_{f(x_0)} = \emptyset.$$

Since f is continuous, there exists an open neighborhood U of x_0 such that

$$f(U) \subseteq V_{f(x_0)}.$$

Note that this implies that $f(U) \cap V_y = \emptyset$. Also, note that $x_0 \times y \in U \times V_y$ and that $U \times V_y$ is open in $X \times Y$.

Now, we want to show that $U \times V_y \subseteq (X \times Y) - G_f$. Let $z \times w \in U \times V_y$. Then, since $f(U)$ and V_y are disjoint, $w \notin f(U)$. However, since $z \in U$, $f(z) \in f(U)$. Therefore, $w \neq f(z)$, so

$$z \times w \notin G_f.$$

In other words,

$$z \times w \in (X \times Y) - G_f.$$

Therefore, we can conclude that, indeed, $(X \times Y) - G_f$ is open, meaning that G_f is closed in $X \times Y$.

$(\Leftarrow)$ On the other hand, suppose that G_f is closed in $X \times Y$. Let $x_0 \in X$ and let V be an open neighborhood of $f(x_0)$ in Y. Since V is open in Y, $Y - V$ is closed and so, since X closed in X, $x \times (Y - V)$ is closed in $X \times Y$. Since G_f closed in $X \times Y$,

$$G_f \cap (X \times (Y - V))$$

is closed in $X \times Y$. Since the projection $\pi_1 : X \times Y \to X$ is closed,

$$U = \pi_1 \left(G_f \cap (X \times (Y - V)) \right) = \{x \in X | f(x) \notin V\}$$

is closed in X. Note that $U = X - f^{-1}(V)$, so we see that $f^{-1}(V)$ is open in X. Also, $x \in f^{-1}(V)$ and $f(f^{-1}(V)) \subseteq V$. Hence, f is continuous at x. Since our choice of x was arbitrary, we see that f is continuous at every point in X, which is to say that f is continuous.

Having demonstrated both directions, we conclude that f is continuous if and only if it's graph is closed in $X \times Y$. □

27.6

(a) Show that the Cantor set C is totally disconnected.

Proof. Suppose not. Then there is an interval $[a, b] \subseteq C$. Let $N \in \mathbb{N}$ such that

$$N > \log_3 \left(\frac{1}{b - a} \right).$$

Then, for $n > N$, $\frac{1}{3^n} < b - a$. However, since $C = \cap A_j$, C, if it contains intervals at all, must contain intervals of length less than $\frac{1}{3^n}$. Hence, C contains no intervals, so C is totally disconnected. □

(b) Show that C is compact.

Proof. We show, by induction, that each A_n is closed in $[0, 1]$. Clearly, $A_0 = [0, 1]$ is closed in $[0, 1]$. Now, suppose A_k is closed. Then $A_k = [0, 1]-U$, where U is open. Define

$$V_k = \bigcup_{i=0}^{\infty} \left(\frac{1 + 3i}{3^{k+1}}, \frac{2 + 3i}{3^{k+1}} \right).$$

Then

$$A_{k+1} = A_k - V_k = ([0, 1] - U) - V_k = [0, 1] - (U \cup V_k).$$

Since U and V_k are open, so is $U \cup V_k$, so A_{k+1} is closed. Hence, by induction, A_n is closed for all $n \in \mathbb{N}$. Therefore, since C is an intersection of closed sets, C is closed. C is also clearly bounded, so, by Theorem 27.3, C is compact. □

(c) Show that each set A_n is a union of finitely many disjoint closed intervals of length $1/3^n$; and show that the end points of these intervals lie in C.

Proof. By induction. Clearly, $A_0 = [0, 1]$ is the union of a single closed interval of length $1 = 1/3^0$. $0 \in C$, since $0 < \frac{1+3k}{3^n}$ for all $n, k \in \mathbb{N} \cup \{0\}$.

Also, $3^{n-1} \in \mathbb{N}$ for all n, so $3^{n-1} - \frac{2}{3} \notin \mathbb{N}$. Hence, there exists $k \in \mathbb{N}$ such that

$$\begin{aligned} k &< 3^{n-1} - \tfrac{2}{3} && < k + \tfrac{2}{3} \\ k &< \tfrac{3^n - 2}{3} && < k + \tfrac{2}{3} \\ 3k &< 3^n - 2 < 3k + 2 \\ 2 + 3k < 3^n &< 4 + 3k \\ \tfrac{2+3^k}{3^n} < 1 &< \tfrac{1+3(k+1)}{3^n} \end{aligned}$$

for all $n \in \mathbb{N}$. Hence, the endpoint $1 \in C$.

Now, suppose A_{k-1} is a union of finitely many disjoint closed intervals of length $1/3^{k-1}$ and that the endpoints of these intervals lie in C. Denote A_{k-1} by

$$A_{k-1} = \bigcup_1^n [a_j, b_j].$$

Note that, if $(c, d) \subset [a, b]$,

$$[a, b] - (c, d) = [a, c] \cup [d, b].$$

Now,

$$\begin{aligned} A_k &= A_{k-1} - \bigcup_{j=0}^{\infty} \left(\tfrac{1+3j}{3^k}, \tfrac{2+3j}{3^k} \right) \\ &= \bigcup_1^n [a_i, b_i] - \bigcup_{j=0}^{\infty} \left(\tfrac{1+3j}{3^k}, \tfrac{2+3j}{3^k} \right) \\ &= \bigcap_{j=0}^{\infty} \left(\bigcup_1^n [a_i, b_i] - \left(\tfrac{1+3j}{3^k}, \tfrac{2+3j}{3^k} \right) \right) \end{aligned}$$

which is simply an intersection of unions of closed intervals, which is itself a union of closed intervals. Furthermore, these closed intervals have length $1/3^k$ since we are deleting the middle third from intervals of length $1/3^{k-1}$. Finally, an argument similar to the one used to show that $1 \in C$ shows that each of these endpoints is in C. Therefore, by induction, each set A_n is a union of finitely many disjoint closed intervals of length $1/3^n$ and the end points of these intervals lie in C. □

(d) Show that C has no isolated points.

Proof. Since every basic open set in C is of the form $(a, b) \cap C$ where (a, b) is open in $[0, 1]$, it suffices to show that no set of this form is a singleton set (a simple modification takes care of basis elements of the form $[0, b)$ and $(a, 1]$). Let $(a, b) \in [0, 1]$ such that $(a, b) \cap C \neq \emptyset$. Let $x \in (a, b) \cap C$. Since we showed in part (c) above that each A_n is a finite union of closed intervals of length $1/3^n$ and that the endpoints of these intervals are in C, we can find such an endpoint not equal to x that is in $(a, b) \cap C$.

Let $p = \min\{x - a, b - x\}$. Choose $N \in \mathbb{N}$ such that $N > \log_3 \left(\frac{1}{p} \right)$. Then, for $m > N$, $x \in A_m$, so x lies in an interval $[c, d]$ of length $1/3^m$. Note that

$$\max\{x - c, d - x\} < p,$$

so $[c, d] \subset (a, b)$. From (c), we know that $c, d \in C$, so

$$\{x, c, d\} \in (a, b) \cap C.$$

Even if x is equal to one of these endpoints, we still see that $(a,b)\cap C$ is not a singleton set.

Since our choice of basis element (a,b) was arbitrary, we conclude that no singleton set is open in C, meaning C has no isolated points. □

(e) Show that C is uncountable.

Proof. C is clearly non-empty, since we showed in part (c) that it contains endpoints of closed intervals; specifically, $0 \in C$. We showed in (b) that C is compact and in (d) that C has no isolated points. Furthermore, we know that C is Hausdorff since $[0,1]$ is. Therefore, we can use Theorem 27.7 to conclude that C is uncountable. □

A

Let $\{X_\alpha\}_{\alpha\in J}$ be an arbitrary collection of connected topological spaces. Show that in the product topology $X = \prod_{\alpha\in J} X_\alpha$ is connected.

Proof. Let $(y_\alpha) = y \in \prod X_\alpha$. Let $A_\beta = \{(a_\alpha)_{\alpha\in J} | a_\alpha = y_\alpha \text{ for all } \alpha \neq \beta\}$. Then let

$$A = \bigcup_{\beta\in J} A_\beta.$$

Since $y \in A_\beta$ for all $\beta \in J$ and each A_β is connected, A is connected. Now, we want to show that $\prod_{\alpha\in J} X_\alpha = \overline{A}$. Let $x \in \prod X_\alpha$ and let

$$U_x = \prod_{\alpha\in J} U_\alpha$$

be a basis element of the product topology containing x. Then $U_\alpha = X_\alpha$ for all but finitely many α. Let $\alpha_1, \ldots \alpha_n$ be the indices for which this equality does not hold. Let

$$a = (a_\alpha)_{\alpha\in J}$$

where $a_{\alpha_i} = x_{\alpha_i}$ and $a_\gamma = y_\gamma$ for all $\gamma \notin \{\alpha_1, \ldots, \alpha_n\}$. Then, clearly,

$$a \in \bigcup_1^n A_{\alpha_j} \subset A$$

and, since $y_\gamma \in U_\gamma = X_\gamma$ for all $\gamma \notin \{\alpha_1, \ldots, \alpha_n\}$,

$$a \in U_x.$$

Hence, we can conclude that $\bigcup_{\alpha\in J} X_\alpha$ is connected, since it is the closure of a connected set. □

B

Suppose $f : (0,1) \to (0,1)$ is a continuous map. Does f have a fixed point?

Answer: No. As a counter-example, consider $f(x) = x^2$. For $c \in (0,1)$,

$$c - f(c) = c - c^2 = c(1-c) < c$$

since $0 < 1 - c < 1$. Hence, $f(c) \neq c$ for all $c \in (0,1)$, so f has no fixed points.

♣

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