TOPOLOGY HW 1

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18.1

Prove that for functions $f : \mathbb{R} \to \mathbb{R}$, the ϵ-δ definition of continuity implies the open set definition.

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous under the ϵ-δ definition of continuity. Then we want to show that f is continuous under the open set definition. Let (a, b) be a basis element of the standard topology of $\mathbb{R}$. Let $x \in f^{-1}(a, b)$. Then $f(x) \in (a, b)$. Let $\epsilon = \min\{f(x) - a, b - f(x)\}$. Then, since f is continuous under the ϵ-δ definition, there exists $\delta > 0$ such that, if $|x-y| < \delta$,

$$|f(x) - f(y)| < \epsilon$$

which is to say

$$f(x - \delta, x + \delta) \subseteq (f(x) - \epsilon, f(x) + \epsilon) \subseteq (a, b).$$

In turn, this means that

$$(x - \delta, x + \delta) \subseteq f^{-1}(a, b).$$

$(x - \delta, x + \delta)$ is open and contains x, so $f^{-1}(a, b)$ is open, meaning f is continuous under the open set definition. Since our choice of f was arbitrary, we can conclude that the ϵ-δ defintion of continuity implies the open set defintion.

□

18.6

Find a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at precisely one point.

Example: Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Since we've just shown that the ϵ-δ definition of continuity is equivalent to the open set definition, we use an ϵ-δ argument to show that f is continuous at 0 and nowhere else. Let $\epsilon > 0$. Define $\delta < \sqrt{\epsilon}$. Then, if $|x - 0| < \delta$,

$$|f(x) - f(0)| = |f(x)| \leq |x^2| = |x|^2 < |\delta|^2 = \epsilon.$$

Hence, f is continuous at 0. Suppose f is continuous at some point $c \neq 0$. Then, for $\epsilon < c^2/2$, there exists $\delta > 0$ such that, if $|x - c| < \delta$,

$$|f(x) - f(c)| < \epsilon.$$

If c is rational, then there exists an irrational number $b \in (c-\delta, c+\delta)$. Then

$$|f(b) - f(c)| = |0 - c^2| = c^2 > \epsilon.$$

On the other hand, suppose c is irrational. Since $f(x) = x^2$ is continuous, there exists $\delta_0 > 0$ such that

$$|x - c| < \delta_0 \Rightarrow |f(x) - f(c)| < \epsilon.$$

Let $\delta_1 = \min\{\delta, \delta_0\}$. Then there exists rational $a \in (c - \delta_1, c + \delta_1$, so

$$|f(a) - f(c)| = |a^2 - 0| = a^2$$

where

$$|c^2 - a^2| < \epsilon$$

or $a^2 > \epsilon$. In either case, we see that f is not continuous at any point other than 0.

♣

18.8

Let Y be an ordered set in the order topology. Let $f, g : X \to Y$ be continuous.

(a) Show that the set $\{x | f(x) \leq g(x)\}$ is closed in X.

Lemma 0.1. *If Z is a topological space under the order topology, then Z is Hausdorff.*

Proof. Let Z be a topological space under the order topology. Let $x, y \in Y$ such that $x < y$. If there exists $z \in Z$ such that $x < z < y$, then we can construct disjoint open sets containing x and y, respectively, in the following way:

If there exist $a, b \in Z$ such that $a < x$ and $y < b$, then (a, z) and (z, b) are open in Z, are disjoint, and contain x and y, respectively. If x is the minimal element in Z, then, if $b > y$, $[x, z), (z, b)$ are disjoint, open in Z and contain x and y, respectively. If y is the maximal element of Z, then, if there exists $a < x$, $(a, z), (z, y]$ are disjoint, open in Z and contain x and y, respectively. Finally, if x is the minimal element and y is the maximal element in Z, then $[x, z), (z, y]$ are disjoint, open in Z and contain x and y, respectively. Hence, whenever there exists $z \in Z$ such that $x < z < y$, we can construct disjoint open sets containing x and y respectively.

On the other hand, if there exists no $z \in Z$ such that $x < z < y$, then, if $a < x$ and $b > y$, $(a, y), (x, b)$ are disjoint open sets in Z that contain x and y, respectively. Again, if x is the minimal element and/or y is the maximal element in Z, we can still construct two disjoint open sets, on containing x and one containing y.

Hence, we can conclude that Z is Hausdorff. □

Proposition 0.2. *$\{x | f(x) \leq g(x)\}$ is closed in X.*

Proof. To show that $\{x|f(x) \leq g(x)\}$ is closed, it is sufficient to demonstrate that $X \backslash \{x|f(x) \leq g(x)\} = \{x|f(x) > g(x)\} = A$ is open in X. Let $x_0 \in A$. Then $g(x_0) < f(x_0)$. Since Y is Hausdorff by the above lemma, there exist disjoint open sets U and V contained in Y such that $f(x_0) \in U$, $g(x_0) \in V$. Then, since f, g are continuous, $f^{-1}(U)$ and $g^{-1}(V)$ are open in X, so their intersection

$$f^{-1}(U) \cap g^{-1}(V)$$

is open in X. Furthermore, $x_0 \in f^{-1}(U) \cap g^{-1}(V)$, so there exists open $W \subseteq f^{-1}(U) \cap g^{-1}(V)$ containing x_0. Since U and V are disjoint,

$$f(W) \cap g(W) = \emptyset$$

meaning, for all $w \in W$,

$$g(w) < f(w).$$

In other words, $W \subseteq A$. Therefore, we can conclude that A is open, meaning that its complement, $\{x|f(x) \leq g(x)\}$ is closed. □

(b) Let $h : X \to Y$ be the function

$$h(x) \min\{f(x), g(x)\}.$$

Show that h is continuous.

Proof. By a similar argument to that made in (a) above, we can show that $B = \{x|g(x) \leq f(x)\}$ is closed. Also, since f and g are continuous on X, it is true that f is continuous on $A = \{x|f(x) \leq g(x)\}$ and g is continuous on B. Furthermore, $A \cup B = X$ and $A \cap B = \{x|f(x) = g(x)\}$, which is to say that $f(x) = g(x)$ for all $x \in A \cap B$. By the Pasting Lemma, then, we can construct $h' : X \to Y$ such that $h'(x) = f(x)$ when $x \in A$ and $h'(x) = g(x)$ when $x \in B$. Finally, we need only note that $h \equiv h'$ to demonstrate that h is continuous. □

18.9

Let $\{A_\alpha\}$ be a collection of subsets of X; let $X = \cup_\alpha A_\alpha$. Let $f : X \to Y$; suppose that $f|A_\alpha$ is continuous for each α.

(a) Show that if the collection $\{A_\alpha\}$ is finite and each set A_α is closed, then f is continuous.

Proof. If the collection $\{A_\alpha\}$ is finite, then $X = \cup_{i=1}^n A_i$ for some integer n. If $n = 2$, then $X = A_1 \cup A_2$, A_1 and A_2 are closed in X, $f|A_1 : A_1 \to Y$ and $f|A_2 : A_2 \to Y$ are continuous and $f|A_1(x) = f|A_2(x)$ for every $x \in A_1 \cap A_2$. Hence, by the pasting lemma, we can construct continuous $f' : X \to Y$ such that $f'(x) = f|A_1(x)$ if $x \in A_1$ and $f'(x) = f|A_2(x)$ if $x \in A_2$. It is clear that $f \equiv f'$, so f is continuous.

Now, suppose that every map f fulfilling the above hypotheses is continuous on any $X = \bigcup_{i=1}^{n} A_i$. Let $X = \bigcup_{i=1}^{n+1} A_i$. Then,

$$\left(\bigcup_{i=1}^{n} A_i\right) \cup A_{n+1}.$$

Since $\cup_{i=1}^{n} A_i$ is a finite union of closed sets, it is closed, as is A_{n+1}. Furthermore, by the inductive hypothesis, $f| \cup_1^n A_i$ is continuous, as is $f|A_{n+1}$. Hence, by the pasting lemma, we can construct continuous $f' : X \to Y$ such that $f'(x) = f| \cup_1^n A_i(x)$ if $x \in \cup_1^n A_i$ and $f'(x) = f|A_{n+1}(x)$ if $x \in A_{n+1}$. Again, $f \equiv f'$, so f is continuous.

Therefore, by induction, if the collection $\{A_\alpha\}$ is finite and each set A_α is closed, then f is continuous. □

(b) Find an example where the collection $\{A_\alpha\}$ is countable and each A_α is closed, but f is not continuous.

Example: Let $f : \mathbb{R}_{std} \to \mathbb{R}_K$ be the identity map. Then f is not continuous, as $(-1, 1)-K$ is open in $\mathbb{R}_K$ but not in $\mathbb{R}$, since no neighborhood of $\{0\}$ is contained in $(-1, 1) - K$. Let

$$A_n = (-\infty, -\frac{1}{n}] \cup \{0\} \cup [\frac{1}{n}, \infty).$$

Then $\{A_n\}$ is countable and $\mathbb{R} = \cup_1^\infty A_n$.

We want to show, then, that $f|A_n : A_n \to \mathbb{R}_K$ is continuous. Let $U \subseteq \mathbb{R}_K$ be open. Then

$$\begin{aligned} f^{-1}|A_n(U) &= f^{-1}|_{(-\infty, \frac{1}{n}]}(U) \cup f^{-1}|_{\{0\}}(U) \cup f^{-1}|_{[\frac{1}{n}, \infty)}(U) \\ &= (U \cap (-\infty, -\tfrac{1}{n}]) \bigcup (U \cap \{0\}) \bigcup (U \cap [\tfrac{1}{n}, \infty)) \end{aligned}$$

Since f is the identity map. Each of the three terms in this union are open in A_n, the outer two by definition of the subspace topology and $U \cap \{0\}$ because

$$U \cap \{0\} = \begin{cases} \emptyset, \text{ open by definition} \\ \{0\} = A_n \cap (-\frac{1}{n+1}, \frac{1}{n+1}) \end{cases}$$

Which is open in A_n. Hence, $f^{-1}|_{A_n}(U)$ is the union of three open sets in A_n, so $f^{-1}|_{A_n}(U)$ is open. In turn, this implies that $f|_{A_n}$ is continuous for all n.

♣

(c) An indexed family of sets $\{A_\alpha\}$ is said to be *locally finite* if each point x of X has a neighborhood that intersects A_α for only finitely many values of α. Show that if the family $\{A_\alpha\}$ is locally finite and each A_α is closed, then f is continuous.

Proof. Let $\{A_\alpha\}$ be locally finite. Let $x \in X$. Then there exists a neighborhood U_x such that $U_x \cap A_\alpha \neq \emptyset$ for only finitely many α. Index the α for which this intersection is nonempty by $\{\alpha_i\}_{i=1,\ldots n}$. Then

$$f|U_x(U_x) = \bigcup_{i=1}^{n} f|A_{\alpha_i}(U_x).$$

Since each $f|A_{\alpha_i}$ is continuous, we can use an analog of the argument used in (a) above to show that the pasting lemma implies that $f|U_x$ is continuous. Since

$$X = \bigcup_{x \in X} U_x$$

then, by Theorem 18.2(f) (the local formulation of continuity), f is continuous. $\square$

19.1

Prove Theorem 19.2

Proof. Let each X_α be a space given by a basis $\mathcal{B}_\alpha$. Then we want to show that the collection of sets of the form

$$\mathcal{P} = \prod_{\alpha \in J} B_\alpha$$

is a basis for the box topology on $\prod_{\alpha \in J} X_\alpha$. Let

$$U = \prod_{\alpha \in J} U_\alpha$$

be an open set in the box topology and let $x \in U$. Then

$$x = (x_\alpha)_{\alpha \in J}$$

where each $x_\alpha \in X_\alpha$. For each x_β, there exists an element $B_\beta \subseteq U_\beta$ of the basis for the topology on X_β that contains x_β. Since

$$\prod_{\alpha \in J} B_\alpha \in \mathcal{P},$$

we can conclude that, if $\mathcal{T}'$ is a topology generated by the collection $\mathcal{P}$, then $\mathcal{T}'$ will be finer than the box topology. Obviously, the box topology is finer than $\mathcal{T}'$, if it is a topology, as every basis element of $\mathcal{T}'$ (again, assuming it is a topology) is contained in the standard basis for the box topology. Hence the equality of the topologies will be clear once we demonstrate that $\mathcal{P}$ is, in fact, the basis of a topology. The fact that any element $x \in \prod X_\alpha$ is contained in an element of $\mathcal{P}$ is merely a special case of the result proved above, so we turn our attention to the second prerequisite for a basis. If x is contained in the intersection of two elements

$$x \in (\prod_{\alpha_a \in J} B_{\alpha_a}) \cap (\prod_{\alpha_b \in J} B_{\alpha_b}$$

then, for each $B_{\beta_a}, B_{\beta_b} \subseteq X_\beta$, $x_\beta \in B_{\beta_a}$ and $x \in B_{\beta_b}$, which is to say that

$$x \in B_{\beta_a} \cap B_{\beta_b}.$$

Since X_β is a subspace, there exists a basis element

$$B_{\beta_c} \subseteq B_{\beta_a} \cap B_{\beta_b}$$

that contains x_α. Then

$$\prod_{\alpha\in J} B_{\alpha_c} \subseteq \left(\prod_{\alpha\in J} B_{\alpha_a}\right) \cap \left(\prod_{\alpha\in J} B_{\alpha_b}\right)$$

is a basis element and contains x. Hence, $\mathcal{B}_\alpha$ is a basis for a topology on $\prod_{\alpha\in J} X_\alpha$. Thus, our collection of $\prod B_\alpha$ is a basis for the box topology.

Now, we turn our attention to the collection of all sets of the same form, where $B_\alpha \in \mathcal{B}_\alpha$ for finitely many α and $B_\alpha = X_\alpha$ for all the remaining indices. We want to show that this collection $\mathcal{Q}$ serves as a basis for the product topology on $\prod_{\alpha\in J} X_\alpha$. Let

$$U = \prod_{\alpha\in J} U_\alpha$$

be a basis element of the product topology and suppose $x \in U$. Let $x = (x_\alpha) \in U$. Then, for each α for which $U_\alpha \neq X_\alpha$ we assign an index U_{α_i}. Then, since $\mathcal{B}_{\alpha_j}$ is a basis for X_{α_j}, there exists a basis element B_{α_i} containing x_α and contained in U_{α_j}. Then the product

$$\prod_{\alpha\in J} B_\alpha$$

where $B_\alpha = B_{\alpha_i}$ for each of the finite number of properly contained B_{α_i} and $B_\alpha = X_\alpha$ for all the remaining indices is an element of our collection. Thus, the topology $\mathcal{T}$ generated by this basis (assuming it is a basis) is finer than the product topology on $\prod X_\alpha$. Furthermore, by arguments virtually identical to those above concerning the box topology, $\mathcal{Q}$ is a basis for a topology on $\prod X_\alpha$. Finally, we note that any element of $\mathcal{Q}$ is an element of the standard basis for the product topology, so $\mathcal{Q}$ generates the product topology. □

19.2

Prove Theorem 19.3

Proof. Let

$$U = \prod_{\alpha\in J} U_\alpha$$

be a basis element as described in problem 19.1 above for either the box or product topology. Then, for each α,

$$U_\alpha \cap A_\alpha$$

is an element of the basis for the subspace topology on A_α. Then it is clear that

$$B = \prod_{\alpha\in J} (U_\alpha \cap A_\alpha)$$

is an element of the basis of the box or product topology on $\prod A_\alpha$. Let $x = (x_\alpha) \in B$. Then each $x_\alpha \in U_\alpha \cap A_\alpha$. Specifically, $x \in U$ and $x \in \prod A_\alpha$, so

$$x \in U \cap \prod_{\alpha \in J} A_\alpha$$

a basis element for the subspace topology on $\prod A_\alpha$. Hence the subspace topology on $\prod A_\alpha$ is finer than the box or product topology on $\prod A_\alpha$.

On the other hand, if

$$x \in U \cap \prod_{\alpha \in J} A_\alpha,$$

an element of the basis for the subspace topology, then $x_\alpha \in U_\alpha$ and $x_\alpha \in A_\alpha$. In other words,

$$x_\alpha \in U_\alpha \cap A_\alpha$$

for all α, so

$$x \in \prod_{\alpha \in J} (U_\alpha \cap A_\alpha),$$

a basis element of the box or product topology on $\prod A_\alpha$. Therefore, the box or product topology on $\prod A_\alpha$ is finer than the subspace topology.

Since each is finer than the other, we conclude that $\prod A_\alpha$ is a subspace of $\prod X_\alpha$ if both products are given either the box or the product topology. □

19.9

Show that the choice axiom is equivalent to the statement that for any indexed family $\{A_\alpha\}_{\alpha \in J}$ of nonempty sets, with $J \neq 0$, the cartesian product

$$\prod_{\alpha \in J} A_\alpha$$

is not empty.

Proof. According to the choice axiom, there exists a choice function

$$c : \{A_\alpha\}_{\alpha \in J} \to \bigcup_{\alpha \in J} A_\alpha$$

such that $c(A_\alpha) = a_\alpha \in A_\alpha$. Then

$$(c(A_\alpha))_{\alpha \in J} = (a_\alpha)_{\alpha \in J} \in \prod_{\alpha \in J} A_\alpha.$$

□

19.10

Let A be a set; let $\{X_\alpha\}_{\alpha\in J}$ be an indexed family of spaces; and let $\{f_\alpha\}_{\alpha\in J}$ be an indexed family of functions $f_\alpha : A \to X_\alpha$.

(a) Show there is a unique coarsest topology $\mathcal{T}$ on A relative to which each of the functions f_α is continuous.

Proof. In order for f_α to be continuous in $\mathcal{T}$, then, for each open $U_\alpha \in X_\alpha$, $f_\alpha^{-1}(U_\alpha)$ must be open in A. Furthermore, for any α,

$$f_\alpha^{-1}(X_\alpha) = A$$

so, if $x \in A$, $x \in f_\alpha^{-1}(X_\alpha)$. So we let $\mathcal{T}$ consist of all countable unions of finite intersections of such $f_\alpha^{-1}(U_\alpha)$, for all open U_α in each X_α. Suppose there exists a topology $\mathcal{T}_1$ in which each f_α is continuous which is coarser than $\mathcal{T}$. Then there exists an element U of $\mathcal{T}$ which is not an element of $\mathcal{T}_1$. By construction,

$$U = \bigcup_{\alpha\in J}\left(\bigcap_{i=1}^{n} f_{\alpha_i}^{-1}(U_{\alpha_i}\right)$$

where J is a countable set and each U_{α_i} is an open set in some X_{α_i}. Then, for each α_i, $i = 1, \ldots, n$, $f_{\alpha_i}^{-1}(U_{\alpha_i})$ must be open in $\mathcal{T}_1$. Hence, since U is a countable union of a finite intersection of open sets, $U \in \mathcal{T}_1$. Hence, $\mathcal{T}_1 \subseteq \mathcal{T}$, a contradiction. Hence, $\mathcal{T}$ is the unique coarsest topology of A under which each of the f_α is open. □

(b)Let

$$\mathcal{S}_\beta = \{f_\beta^{-1}(U_\beta) | U_\beta \text{ is open in } X_\beta\},$$

and let $\mathcal{S} = \cup\mathcal{S}_\beta$. Show that $\mathcal{S}$ is a subbasis for $\mathcal{T}$.

Proof. As we've constructed it, $\mathcal{T}$ is the topology generated by the subbasis $\mathcal{S}$. □

(c) Show that a map $g : Y \to A$ is continuous relative to $\mathcal{T}$ if and only if each map $f_\alpha \circ g$ is continuous.

Proof. Suppose, first of all, that g is continuous. Then, since each f_α is continuous, $f_\alpha \circ g$ is continuous, since the composition of continuous functions is continuous.

On the other hand, suppose that each $f_\alpha \circ g$ is continuous. To show g is continuous, it suffices to show that, for each subbasis element $\mathcal{S}_\beta$ of $\mathcal{T}$,

$$g^{-1}(\mathcal{S}_\beta)$$

is open in Y. However, each $\mathcal{S}_\beta = f_\beta^{-1}(U_\beta)$ for open $U_\beta \subseteq X_\beta$. Furthermore, since each $f_\alpha \circ g$ is continuous,

$$g^{-1}(\mathcal{S}_\beta) = g^{-1}(f_\beta^{-1}(U_\beta)) = (f_\beta \circ g)^{-1}(U_\beta)$$

is open in Y. Thus, we can conclude that g is continuous. □

(d) Let $f : A \to \prod X_\alpha$ be defined by the equation

$$f(a) = (f_\alpha(a))_{\alpha \in J};$$

let Z denote the subspace $f(A)$ of the product space $\prod X_\alpha$. Show that the image under f of each element of $\mathcal{T}$ is an open set of Z.

1

Let X be a topological space. $X \subseteq X$ is said to be *dense* in X if $\overline{S} = X$.

(a) Let X be an infinite set with the finite complement topology. Show that any infinite subset S of X is dense in X.

Proof. Since X is an infinite set, by definition we know that the closed subsets of X are the finite subsets of X and X itself. Specifically, the only infinite closed subset of X is X itself. Let S be an infinite subset of X. Then

$$S \subseteq \overline{S}.$$

Hence, the closed set $\overline{S}$ must be infinite; the only infinite closed subset is X, so $\overline{S} = X$. Hence, S is dense in X. □

(b) Now suppose $f, g : X \to Y$ are continuous functions between two topological spaces, for which there is a dense subset $S \subseteq X$ such that $f(x) = g(x)$ for any $x \in S$. Show that if Y is Hausdorff, then $f(x) = g(x)$ for all $x \in X$.

Proof. Suppose not. Let $x \in X \backslash S$ such that $f(x) \neq g(x)$. Since Y is Hausdorff, this means that there exist disjoint open neighborhoods U and V of $f(x)$ and $g(x)$, respectively. Hence, since f and g are continuous, there exist open neighborhoods U_x and V_x of x such that $U_x \subseteq f^{-1}(U)$ and $V_x \subseteq g^{-1}(V)$. Clearly,

$$f(U_x) \cap g(V_x) = \emptyset$$

so $f(a) \neq g(a)$ for $a \in U_x \cap V_x$. However, $x \in U_x \cap V_x$, so

$$U_x \cap V_x \neq \emptyset.$$

Since S is dense in X, x is a limit point of S, so

$$S \cap (U_x \cap V_x) \neq \emptyset.$$

Let $s \in S \cap (U_x \cap V_x)$. Then $f(s) = g(s)$, a contradiction. From this contradiction, we can conclude that, in fact, $f(x) = g(x)$ for all $x \in X$. □

2

Let A and B be homeomorphic subsets of $\mathbb{R}^n$. If A is closed in $\mathbb{R}^n$ does it follow that B is also closed in $\mathbb{R}^n$? For each dimension in which this is true please provide a proof and in each dimension for which it is false please provide a counterexample.

No, this does not hold true for any dimension. In general, let $A = \mathbb{R}^n$ and

$$B = (-\pi/2, \pi/2) \times \mathbb{R}^{n-1}.$$

Clearly, A is closed in $\mathbb{R}^n$ and B is open in $\mathbb{R}^n$. However, the map

$$f : A \to B$$

defined by

$$f(x) = f(x_1, \dots, f_n) = (\tan^{-1} x_1, x_2, \dots, x_n)$$

is a homeomorphism from A to B. To see that f is continuous, we need only note that it's coordinate functions (namely $\tan^{-1}$ and the identity map) are continuous. Furthermore, tan is continuous on $(-\pi/2, \pi/2)$, so

$$f^{-1}(x) = f^{-1}(x_1, \dots, x_n) = (\tan x_1, x_2, \dots, x_n)$$

is continuous, since its coordinate functions are.

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