OPTIMAL MEASUREMENTS, TIME-REVERSAL, AND AUTOMATIC FREQUENCY TUNING

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1 ABSTRACT

In problems in which one wants to use electromagnetic waves to make an image of a small, weak scatterer, the question arises of which incident field will produce the scattered field with greatest energy. This problem can be solved by means of a certain adaptive algorithm that involves an iterative time-reversal process. The resulting best incident field is a time-harmonic (CW) wave, whose frequency is automatically chosen by the interative algorithm.

2 INTRODUCTION AND PROBLEM FORMULATION

This work concerns the ultimate limits of electromagnetic imaging systems. What measurements should we make to detect the weakest possible scatterers? What measurements should we make to determine that an unknown scatterer is different from a guessed one? To answer these questions, we must first formulate them more precisely. What sort of measurements do we allow? What do we mean by "detectable"? How do we know what measurements are "best"?

We assume that the electromagnetic fields are generated and measured in the upper half-space, $z > 0$, which consists of free space away from the sources of the field. We assume that the sources are separated from the boundary $z = 0$, so that there is a layer of free space between the sources and the boundary, In this region, we write Maxwell's equations as

$$\nabla \times \boldsymbol{\mathcal{E}} = -\mu_0 \frac{\partial \boldsymbol{\mathcal{H}}}{\partial t} \qquad \nabla \times \boldsymbol{\mathcal{H}} = \epsilon_0 \frac{\partial \boldsymbol{\mathcal{E}}}{\partial t}. \tag{1}$$

We will make use of the Fourier transform relating the time and frequency domains: $\boldsymbol{\mathcal{E}}(\boldsymbol{r},t) = \int \boldsymbol{E}(\boldsymbol{r},k) e^{-ic_0kt} c_0 dk$, where $c_0 = 1/\sqrt{\epsilon_0\mu_0}$ is the vacuum speed of light, and $k = \omega/c_0$ is the wave number in vacuum.

In the upper half-space, we split the waves [2] into upgoing and downgoing parts as $\boldsymbol{\mathcal{E}}(\boldsymbol{r},t) = \boldsymbol{\mathcal{E}}^{\downarrow}(\boldsymbol{r},t) + \boldsymbol{\mathcal{E}}^{\uparrow}(\boldsymbol{r},t)$, where we have written $\boldsymbol{r} = (x,y,z)$ and where the downgoing component is

$$\boldsymbol{\mathcal{E}}^{\downarrow}(\boldsymbol{r},t) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \iint_{\mathbb{R}^2} \mathsf{E}^{\downarrow}(\boldsymbol{e}_{xy},k) e^{ik\boldsymbol{e}^{-}\cdot\boldsymbol{r}} e^{-ic_0kt}\, k^2 de_x de_y c_0 dk, \tag{2}$$

and the upgoing component is

$$\boldsymbol{\mathcal{E}}^{\uparrow}(\boldsymbol{r},t) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \iint_{\mathbb{R}^2} \mathsf{E}^{\uparrow}(\boldsymbol{e}_{xy},k) e^{ik\boldsymbol{e}^{+}\cdot\boldsymbol{r}} e^{-ic_0kt}\, k^2 de_x de_y c_0 dk, \tag{3}$$

where $\boldsymbol{e}_{xy} = (e_x, e_y)$ and

$$e_z(k) = \begin{cases} \sqrt{1-e_{xy}^2} & \text{for } e_{xy} < 1 \\ i\,\mathrm{sgn}(k)\sqrt{e_{xy}^2 - 1} & \text{for } e_{xy} > 1, \end{cases} \tag{4}$$

and $\boldsymbol{e}^{\pm} = \boldsymbol{e}_{xy} \pm \hat{\boldsymbol{z}} e_z$, which satisfy $\boldsymbol{e}^{\pm}\cdot\boldsymbol{e}^{\pm} = 1$.

Our criterion for the best incident field is the field that gives rise to the upgoing wave with the most total energy. The total energy flow into the lower half-space $z < 0$ is

$$W(\boldsymbol{\mathcal{E}}) = W(\boldsymbol{E}) = \int_{-\infty}^{\infty} \iint_{z=0} -\hat{\boldsymbol{z}} \cdot (\boldsymbol{\mathcal{E}}(\boldsymbol{r}_{xy}, 0, t) \times \boldsymbol{\mathcal{H}}(\boldsymbol{r}_{xy}, 0, t))\, dxdy\, dt \tag{5}$$

In this paper we assume that the sources (in the upper half-space) of the downgoing field are far from the scatterers in the lower half-space, so that upgoing and downgoing evanescent waves are not both present at the plane $z = 0$. Under these conditions, $W(\mathsf{E}^{\downarrow} + \mathsf{E}^{\uparrow}) = W(\mathsf{E}^{\downarrow}) + W(\mathsf{E}^{\uparrow})$; moreover, evanescent waves do not contribute to either $W(\mathsf{E}^{\downarrow})$ or $W(\mathsf{E}^{\uparrow})$ [2].

On the space of upgoing waves and on the space of downgoing waves, we introduce an inner product that corresponds to the flux W:

$$(\mathsf{U}, \mathsf{V})_W = \frac{1}{\eta_0 (2\pi)^3} \int_{-\infty}^{\infty} \iint_{\{e_{xy}<1\}} (\mathsf{U}(\boldsymbol{e}_{xy}, k) \cdot \mathsf{V}^*(\boldsymbol{e}_{xy}, k))\, e_z\, k^2 de_x de_y\, c_0 dk. \tag{6}$$

2.1 Criterion for detectability and distinguishability

The downgoing wave that is best for detecting a scatterer is obtained from the optimization problem

$$\sup_{\boldsymbol{\mathcal{E}}^{\downarrow}} \frac{W(\boldsymbol{\mathcal{E}}^{\uparrow})}{W(\boldsymbol{\mathcal{E}}^{\downarrow})} = \sup_{\mathsf{E}^{\downarrow}} \frac{W(\mathsf{E}^{\uparrow})}{W(\mathsf{E}^{\downarrow})} = \sup_{\mathsf{E}^{\downarrow}} \frac{(\mathsf{E}^{\uparrow}, \mathsf{E}^{\uparrow})_W}{(\mathsf{E}^{\downarrow}, \mathsf{E}^{\downarrow})_W}. \tag{7}$$

For a scatterer to be detectable, (7) should be greater than the measurement precision of our instruments (expressed in units so that the denominator of (7) is one).

The upgoing and downgoing fields of (7) are related by the reflection operator; thus carrying out the optimization of (7) requires an analysis of this operator.

2.1.1 The reflection operator

The reflection operator $\boldsymbol{\mathcal{R}}$ for the electric field is the linear operator that maps the downgoing electric field to the upgoing one at $z = 0$:

$$\boldsymbol{\mathcal{E}}^{\uparrow} = \boldsymbol{\mathcal{R}} \cdot \boldsymbol{\mathcal{E}}^{\downarrow} \tag{8}$$

The reflection operator is a 3×3 dyadic-valued operator; the dot in (8), as thoughout this paper, denotes dyadic contraction. When the dyadics are expressed in a basis, this contraction is simply multiplication of the corresponding matrices. In the transform domain, we write the reflection operator as R; it satisfies

$$\mathsf{E}^{\uparrow}(\boldsymbol{e}_{xy}, z = 0, k) = \int \mathsf{R}(\boldsymbol{e}_{xy}, \boldsymbol{e}'_{xy}, k) \cdot \mathsf{E}^{\downarrow}(\boldsymbol{e}'_{xy}, z = 0, k) d\boldsymbol{e}'_{xy} \tag{9}$$

2.1.2 Detectability

In terms of the reflection operator, the criterion (7) for choosing the downgoing wave that is best for detecting the presence of a scatterer is obtained from

$$\sup_{\boldsymbol{\mathcal{E}}^{\downarrow}} \frac{W(\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{E}}^{\downarrow})}{W(\boldsymbol{\mathcal{E}}^{\downarrow})} = \sup_{\mathsf{E}^{\downarrow}} \frac{W(\mathsf{R}\mathsf{E}^{\downarrow})}{W(\mathsf{E}^{\downarrow})} = \sup_{\mathsf{E}^{\downarrow}} \frac{(\mathsf{R}\mathsf{E}^{\downarrow}, \mathsf{R}\mathsf{E}^{\downarrow})_W}{(\mathsf{E}^{\downarrow}, \mathsf{E}^{\downarrow})_W} = \sup_{\mathsf{E}^{\downarrow}} \frac{(\mathsf{E}^{\downarrow}, \mathsf{R}^{\#}\mathsf{R}\mathsf{E}^{\downarrow})_W}{(\mathsf{E}^{\downarrow}, \mathsf{E}^{\downarrow})_W} \tag{10}$$

where $\mathsf{R}^{\#}$ denotes the adjoint of R with respect to the W inner product.

2.1.3 *Criterion for distinguishability*

Suppose we have a guess as to the scatterer. For our guessed scatterer, we denote the reflection operator by $\boldsymbol{\mathcal{R}}_0$. Then the unknown scatterer can be distinguished from our guess if

$$\begin{aligned}\sup_{\boldsymbol{\mathcal{E}}^{\downarrow}} \frac{W((\boldsymbol{\mathcal{R}}-\boldsymbol{\mathcal{R}}_0)\boldsymbol{\mathcal{E}}^{\downarrow})}{W(\boldsymbol{\mathcal{E}}^{\downarrow})} &= \sup_{\mathsf{E}^{\downarrow}} \frac{((\mathsf{R}-\mathsf{R}_0)\mathsf{E}^{\downarrow},(\mathsf{R}-\mathsf{R}_0)\mathsf{E}^{\downarrow})_W}{(\mathsf{E}^{\downarrow},\mathsf{E}^{\downarrow})_W} \\ &= \sup_{\mathsf{E}^{\downarrow}} \frac{(\mathsf{E}^{\downarrow},(\mathsf{R}-\mathsf{R}_0)^{\#}(\mathsf{R}-\mathsf{R}_0)\mathsf{E}^{\downarrow})_W}{(\mathsf{E}^{\downarrow},\mathsf{E}^{\downarrow})_W}\end{aligned} \tag{11}$$

corresponds to a value greater than the measurement precision of our instruments. Here the $\#$ denotes the adjoint in the energy flux inner product.

2.2 The adjoint of the reflection operator

Provided the scattering medium is linear and reciprocal, the kernel $\mathsf{R}^{*T}(-\boldsymbol{e}'_{xy},-\boldsymbol{e}_{xy},k)$ corresponds to the time-domain kernel $\boldsymbol{\mathcal{R}}(\boldsymbol{r},\boldsymbol{r}',\tau-t)$. This latter is [2] the kernel of the operator $\boldsymbol{\mathcal{T}}\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{T}}$, where $\boldsymbol{\mathcal{T}}$ denotes the operator of time-reversal: $\boldsymbol{\mathcal{T}}f(t)=f(-t)$.

3 ALGORITHM FOR PRODUCING THE OPTIMAL FIELD

Expressions (10) and (11) give rise to algorithms for producing the optimal field. In the case of (10), if $\mathsf{R}^{\#}\mathsf{R}$ were a finite-dimensional matrix $\mathbf{A}$, then we could use the power method [3] to construct the eigenfunction E_∞ of $\mathbf{A}$ corresponding to the largest eigenvalue. In particular, the power method constructs E_∞ as the limit of the sequence $\mathbf{A}^n\mathsf{E}_0/c_n$, where c_n is a certain normalizing factor.

It was shown in [1] that the same method applies to the case of an operator such as $\mathsf{R}^{\#}\mathsf{R}$, an analytic compact-operator-valued function of k. In the case when the adjoint can be interpreted in terms of time reversal so that $\boldsymbol{\mathcal{R}}^{\#}\boldsymbol{\mathcal{R}}=\boldsymbol{\mathcal{T}}\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{T}}\boldsymbol{\mathcal{R}}$, where $\boldsymbol{\mathcal{T}}$ denotes the operator of time-reversal, we have the following algorithm for producing the optimal field.

1. Set $j=0$. Apply any downgoing field $\boldsymbol{\mathcal{E}}_j^{\downarrow}(\boldsymbol{r},t)$.
2. Measure the resulting scattered field $\boldsymbol{\mathcal{E}}_j^{\uparrow}=\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{E}}_j^{\downarrow}$.
3. Time-reverse the measured field, and apply this as a downgoing field: $\boldsymbol{\mathcal{E}}_{j+1}^{\downarrow}(\boldsymbol{r},t)=\boldsymbol{\mathcal{E}}_j^{\uparrow}(\boldsymbol{r},-t)$.
4. Add one to j. If this new j is even, so that $j=2n$, divide by c_n (defined below). Go to step 2.

Two iterations, that is, a scattering experiment followed by a time-reversal and another scattering experiment, constitute one application of the operator $\boldsymbol{\mathcal{R}}^{\#}\boldsymbol{\mathcal{R}}=\boldsymbol{\mathcal{T}}\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{T}}\boldsymbol{\mathcal{R}}$ and is thus one step in the power method algorithm for constructing the optimal field.

The normalization c_n is obtained by first choosing an arbitrary test function $\boldsymbol{\psi}$ in a certain space X_B. This space X_B consists of smooth functions of space and time whose temporal Fourier transforms are (uniformly) supported in the frequency band B. The frequency band B should be chosen to lie within the frequency band of the experimental equipment. The normalization factor c_n is then chosen as $c_n=(\boldsymbol{\mathcal{E}}_j^{\downarrow},\boldsymbol{\psi})_W$.

To determine what the above algorithm eventually converges to, we can apply the theorem of [1]. It depends on the fact that the frequency-domain operator $\boldsymbol{R}^{\#}\boldsymbol{R}$ is compact at every k and therefore has a spectral decomposition of the form $\boldsymbol{R}^{\#}\boldsymbol{R}=\sum_l\lambda_l(k)\boldsymbol{P}_l(k)$, where the λ_l are the eigenvalues of the self-adjoint operator $\boldsymbol{R}^{\#}\boldsymbol{R}$ and the $\boldsymbol{P}_l$ are projections onto the corresponding eigenspaces.

The algorithm converges to fixed-frequency waves whose frequency is determined by the values of k at which the eigenvalues λ_l attain a maximum. If the same maximum M is attained at several k, then the limiting field is a sum of fixed-

frequency waves. Exactly which terms contribute to the sum depends on the detailed behavior of the λ_l in the neighborhood of the maximum. In particular, if a maximum of λ_l occurs at k_j, and in the neighborhood of k_j, λ_l has a Taylor expansion of the form $M - b_j(k - k_j)^{p_j}$, then we call p_j the *order* of the maximum at k_j.

Theorem 3.1 *Let c_n be chosen as above. Then in the space X_B, $(\boldsymbol{\mathcal{R}}^{\#}\boldsymbol{\mathcal{R}})^n \boldsymbol{\mathcal{E}}_0/c_n$ converges to*

$$\frac{1}{2\pi \sum_{l,i} \beta_l (\boldsymbol{P}_l \boldsymbol{E}_0, \boldsymbol{\psi})(k_i)} \sum_{l,j} \beta_j \boldsymbol{P}_j \boldsymbol{E}_0 e^{i k_j c_0 t}$$

where the sums are over those indices j and l for which the eigenvalue $\lambda_l(k_j)$ attains its maximum M and has maximal order. The β_j are certain constants depending on b_j, p_j, and k_j (see [1]).

This theorem applies to the case of band-limited signals. It says that in the generic case, the power method converges to a fixed-frequency wave, whose frequency is determined by the frequency k_∞ at which the largest eigenvalue λ_0 attains its maximum in the relevant frequency band. In other words, the algorithm "tunes" automatically to the best available frequency. The spatial shape of the wave is determined by the spectral projector $\boldsymbol{P}_0$ at that frequency. In other words, in the generic case, the spatial shape is given by the eigenfunction corresponding to the eigenvalue $\lambda_0(k_\infty)$. The "generic case" here is the case in which the initial field $\boldsymbol{E}_0$ has a nonzero projection $\boldsymbol{P}_0\boldsymbol{E}_0$ at k_∞ and λ_0 has only one maximum in the relevant frequency band. If λ_0 happens to have several identical maxima, for example at k_0 and k_1, then the algorithm converges to a field that is a sum of the fields described above. The relative weighting of the different contributions is determined by the projections $\boldsymbol{P}_0(k_0)\boldsymbol{E}_0(k_1)$ and $\boldsymbol{P}_0(k_1)\boldsymbol{E}_0(k_1)$, which also give the spatial shape of the wave.

4 An experimental prediction

The algorithm and theorem of the previous section constitute an experimental prediction. In particular, this prediction concerns electromagnetic time-reversal experiments analogous to those done by Fink *et al.* [4, 5] for acoustics. The theorem predicts that an interative time-reversal process will "tune" itself to a fixed-frequency wave whose frequency gives the maximum scattering. For example, the iterative time-reversal process should "tune" to the resonant frequency of a Lorentz medium.

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