AN ASYMPTOTIC WAVE INTERPRETATION OF SONAR IMAGES

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ABSTRACT

This paper studies the inverse problem of determining the shape and medium parameters of a smooth seafloor from sonar data. The first section of the paper reviews beamforming techniques and the Kirchhoff (high-frequency) approximation. The second section carries out a stationary phase calculation for various measurement configurations associated with side-scan sonar and pencil-beam sonar. These results provide an interpretation of the information content of sonar images.

1. INTRODUCTION

This work is motivated by the problem of identifying explosive mines lying on the seafloor. Presently this is done with side-scan sonar, which uses a linear array of acoustic transducers, often positioned on a "towfish" towed behind a boat, to produce images by the following method. The array of transducers is activated in a manner (see section 2b below) so as to to form a beam that is restricted to a plane perpendicular to the array. A time-domain pulse is sent out with this spatial form, and the reflections are recorded along one line as grey-scale intensities as a function of time. Then the transducer array is translated along its axis, so that the planar beam illuminates (or insonifies) a parallel plane, and the corresponding reflections are recorded as grey-scale intensities along a parallel line. After many translations of the array, the recording process builds up an image of the seafloor. [E]

The purpose of this paper is to understand what these side-scan sonar images are telling us about the seafloor. In particular, what does the recorded sonar amplitude and phase tell us about the seafloor's shape and material properties?

To investigate this question, we will use techniques developed to study geophysical problems. These techniques [BCS] apply to problems in which one knows the wave speed in the medium above an unknown smooth reflecting surface. In this paper, we use the Kirchhoff (high-frequency) approximation to linearize the relation between the seafloor and the response from a sonar beam. We then examine this relation in the high-frequency asymptotic limit to determine its information content.

2. BACKGROUND

This section reviews the physical parameters of the mine detection problem, beamforming techniques, and the Kirchhoff approximation.

2a. THE PHYSICAL SETUP

The mines of interest here are those that lie in water of depth less than 100 meters. The mines themselves are often housed in torpedo casings, which are roughly cylindrical in shape, about a foot in diameter and 6 feet long. The sonar systems operate at frequencies up to several hundred kilohertz, so that, in water of speed 1500 m/sec, the sonar wavelength is on the order of centimeters.

We assume that the propagation of sound due to a single-frequency source at x_s is governed by the acoustic equation

$$\rho(x)\nabla\cdot\big(\frac{1}{\rho(x)}\nabla u(\omega,x_s,x)\big)+\frac{\omega^2}{c^2(x)}u(\omega,x_s,x)=-\delta(x-x_s), \qquad (2a.1)$$

together with a radiation condition at infinity. Here u denotes the excess pressure, ω denotes the angular frequency of the sound, ρ denotes the density of the acoustic medium, and c the local speed of sound.

In the water, which we assume fills all of space above the seafloor B, we assume for simplicity that the density and sound speed are constants ρ_0 and c_0, respectively. We choose units so that $1/c_0 = 1$. We use the notation $k=\omega/c_0$.

2b. BEAMFORMING

In a three-dimensional homogeneous medium, the wave at a point $x=(x_1,x_2,x_3)$ emanating from a point source at position $y=(y_1,y_2,y_3)$ has the form

$$g(x,y)=\frac{e^{ik|x-y|}}{4\pi|x-y|}. \qquad (2b.1)$$

In the far field ($|x|\to\infty$), this wave has the form

$$g(x,y)=\frac{e^{ik|x|}}{4\pi|x|}\left(e^{ik\hat{x}\cdot y}+O(|y|/|x|)\right), \qquad (2b.2)$$

where $\hat{x}=x/|x|$.

To create a beam of the type used in side-scan sonar, we position sources at locations $\{s\hat{e}_2, -a<s<a\}$, where $\hat{e}_2$ denotes the unit vector $(0,1,0)$. In the far field, this results in

$$\int_{-a}^{a} g(x,s\hat{e}_2)ds \sim \frac{e^{ik|x|}}{4\pi|x|}\int_{-a}^{a} e^{iks\hat{x}\cdot\hat{e}_2}ds=\frac{e^{ik|x|}}{4\pi|x|}\frac{\sin(ka\hat{x}\cdot\hat{e}_2)}{k\hat{x}\cdot\hat{e}_2}. \qquad (2b.3)$$

If we multiply (2b.3) by k/a and take the large-k limit, we see that the far-field wave due to a linear distribution of sources is concentrated in the plane perpendicular to the source axis.

At finite k, the sinc function on the right side of (2b.3) has "sidelobes" that can be reduced by weighting the sources with, say, a Hamming window function [OS].

The sonar beam can be steered by activating the sources so that the phase varies along the array. In particular, if we multiply the source at position y by $\exp(-ik\hat{\mu}\cdot y)$, then we have

$$\int_{-a}^{a} e^{-ik\hat{\mu}\cdot s\hat{e}_2}g(x,s\hat{e}_2)ds \sim \frac{e^{ik|x|}}{4\pi|x|}\frac{\sin(ka(\hat{x}-\hat{\mu})\cdot\hat{e}_2)}{k(\hat{x}-\hat{\mu})\cdot\hat{e}_2}. \qquad (2b.4)$$

Thus activating the sources in this manner produces a beam concentrated in the plane $\hat{x} \cdot \hat{e}_2 = \hat{\mu} \cdot \hat{e}_2$.

Similarly, by positioning sources over the 2-dimensional square array

$$\{s_1 \hat{e}_1 + s_2 \hat{e}_2 : -a < s_1 < a, -a < s_2 < a\},$$

we can create a pencil beam:

$$\int_{-a}^{a} \int_{-a}^{a} e^{-ik\hat{\mu}\cdot(s_1\hat{e}_1+s_2\hat{e}_2)} g(x, s_1\hat{e}_1+s_2\hat{e}_2) ds_1 ds_2 \sim \frac{e^{ik|x|}}{4\pi|x|} \frac{\sin(ka(\hat{x}-\hat{\mu})\cdot\hat{e}_1)}{k(\hat{x}-\hat{\mu})\cdot\hat{e}_1} \frac{\sin(ka(\hat{x}-\hat{\mu})\cdot\hat{e}_2)}{k(\hat{x}-\hat{\mu})\cdot\hat{e}_2}. \tag{2b.5}$$

In the high-frequency limit, this gives us a beam directed in the direction $\hat{x} = \hat{\mu}$.

The same sort of "beamforming" can be done with an array of receivers. This allows the direction of a received signal to be determined.

2c. THE KIRCHHOFF EXPRESSION FOR THE SCATTERED FIELD

We split u into an incident field g plus a scattered field u_{sc}, so that above the surface B, u_{sc} satisfies

$$(\nabla^2 + k^2) u_{sc}(k, x_s, x) = 0. \tag{2c.1}$$

The free-space Green's function (2b.1) satisifes

$$(\nabla^2 + k^2) g(k, x, y) = -\delta(x - y). \tag{2c.2}$$

We multiply (2c.1) by $g(k, x, y)$, mutiply (2c.2) by $u_{sc}(k, x_s, x)$, subtract the resulting equations, and integrate over all space above B. After using Green's theorem to reduce the integral to the boundary, and using radiation conditions to show that the integrals over surfaces at infinity vanish, we are left with the expression

$$u_{sc}(k, x_s, y) = \int_B \left(u_{sc}(k, x_s, x) \frac{\partial g(k, x, y)}{\partial \hat{\nu}} - g(k, x, y) \frac{\partial u_{sc}(k, x_s, x)}{\partial \hat{\nu}} \right) dS_x, \tag{2c.3}$$

where $(\partial/\partial\hat{\nu})$ denotes differentiation with respect to the *upward* (inward) unit normal vector.

For the scattered field on the seafloor, we use the geometrical optics or Kirchhoff approximation. This approximation is good [O] when a) the wavelength of the incident field is shorter than the local radius of curvature of the reflecting surface and b) there is no multiple scattering. In the geometrical optics approximation, one writes the scattered field as

$$u_{sc}(k, x_s, x) = A(x_s, x) e^{iks(x_s, x)} + O(1/k). \tag{2c.4}$$

We note that the incident field (2b.1) is already in the form (2c.4), where the incident amplitude is $A_{inc} = (4\pi|x - x_s|)^{-1}$ and the incident phase is $s_{inc} = |x - x_s|$.

To determine the amplitude A and phase s, we represent the transmitted field in a form similar to (2c.5), and then we enforce boundary conditions on the surface B. These conditions are continuity of the total field u and of the normal derivative $(1/\rho)(\partial u/\partial\hat{\nu})$.

The boundary conditions imply [KL] that on B, the phases themselves are related by $s(x_s, x) = s_{inc}(x_s, x) = |x - x_s|$, and their normal derivatives are related by $(\partial s(x_s, x)/\partial \hat{\nu}) = -(\partial s_{inc}(x_s, x)/\partial \hat{\nu}) = -(\partial |x - x_s|/\partial \hat{\nu})$. These relations are simply Snell's Law. The amplitude A is found to be $A = RA_{inc}$, where R is the local reflection coefficient at the point x. If the incident ray, which is along the vector $x - x_s$, makes an angle θ with the normal, then the reflection coefficient is

$$R(x_s, x) = \frac{\rho(x)\cos\theta - \rho_0\sqrt{c^{-2}(x) - \sin^2\theta}}{\rho(x)\cos\theta + \rho_0\sqrt{c^{-2}(x) - \sin^2\theta}}. \tag{2c.5}$$

We use the geometrical optics expression for u_{sc} in (2c.3). In the process of taking the normal derivatives, we retain only the leading order term in k:

$$\frac{\partial(Ae^{iks})}{\partial \hat{\nu}} = ikA\frac{\partial s}{\partial \hat{\nu}}e^{iks} + O(1). \tag{2c.6}$$

Using (2c.4) and (2c.6) in (2c.3) results in

$$u_{sc}(x_s, x_r) \sim \frac{ik}{(4\pi)^2}\int_B \frac{e^{ik\phi(x,x_s,x_r)}}{|x - x_r||x - x_s|}R(x_s, x)\frac{\partial\phi(x, x_s, x_r)}{\partial\hat{\nu}}d^2S_x, \tag{2c.7}$$

where

$$\phi(x, x_s, x_r) = |x - x_r| + |x - x_s|. \tag{2c.8}$$

Equation (2c.7) is the Kirchhoff approximation for the scattered field. It is the basic equation used by the Kirchhoff geophysical inversion methods. In these methods, one considers (2c.7) as a Fourier integral operator operating on $R\delta_B$, where δ_B denotes a delta function supported on the surface B (the "singular function of the surface"). One can then use a Generalized Radon Transform [BB] to obtain an approximate inverse of the Fourier integral operator (2c.7), thus obtaining, to leading order in k, an approximation to $R\delta_B$.

The paper takes a different approach to (2c.7). We apply sonar beamforming techniques, and carry out a high-frequency asymptotic analysis to determine the information content of the received signals.

3. ANALYSIS OF SONAR SIGNALS

We assume that the water is deep relative to the size of the sonar array ($|x_r|, |x_s| << |x|$), so that a far-field expansion can be used in (2c.7). This results in the expression

$$u_{sc}(x_s, x_r) \sim \frac{ik}{(4\pi)^2}\int_B \frac{e^{ik\varphi(x,x_s,x_r)}}{|x|^2}R(x_s, x)\frac{\partial\phi(x, x_s, x_r)}{\partial\hat{\nu}}d^2S_x, \tag{3.1}$$

where

$$\varphi(x, x_r, x_s) = 2|x| - \hat{x}\cdot(x_r + x_s). \tag{3.2}$$

We assume that the surface B is parametrized as

$$x = x(\alpha_1, \alpha_2). \tag{3.3}$$

In (3.1), we hold x_r and x_s constant, and apply the two-dimensional stationary phase formula [BH]

$$\int g(z)e^{ik\varphi(z)}d^2z \sim (\frac{2\pi}{k})\sum_j \frac{g(z^j)}{\sqrt{|\det H|}}e^{ik\varphi(z^j)}e^{(i\pi/4)\mathrm{sig}H}, \tag{3.4}$$

where H denotes the Hessian of φ evaluated at the stationary point z^j, and sig$H$ denotes the signature of H. Eq. (3.4) gives the first term in an expansion in decreasing powers of $k|\det H|$.

The critical points $x = z^j$ are points on the seafloor where

$$\frac{\partial\varphi(x(\alpha), x_r, x_s)}{\partial\alpha_i} = 2\left(\hat{x}\cdot\frac{\partial x}{\partial\alpha_i} - (x_r + x_s)\hat{e}_2\cdot\frac{1}{|x|}(\frac{\partial x}{\partial\alpha_i} - (\hat{x}\cdot\frac{\partial x}{\partial\alpha_i})\hat{x})\right) = 0. \tag{3.5}$$

To leading order in $1/|x|$, this is simply

$$2\hat{x}\cdot\frac{\partial x}{\partial\alpha_i} = 0, \tag{3.6}$$

which is Snell's Law.

Thus (3.4) becomes

$$u_{sc}(x_r, x_s) \sim \frac{i}{8\pi}\sum_j \frac{R(z^j)e^{ik(2|z^j| - \hat{z}^j\cdot(x_r + x_s))}e^{(i\pi/4)\mathrm{sig}H}}{|z^j|^2\sqrt{(\kappa_1^j - (1/|z^j|)(\kappa_2^j - (1/|z^j|)}} \tag{3.7}$$

where κ_1^j and κ_2^j denote the principal curvatures at the criticial point z^j,

$$\mathrm{sig}H = \frac{(\kappa_1^j - (1/|z^j|)}{|(\kappa_1^j - (1/|z^j|)|} + \frac{(\kappa_2^j - (1/|z^j|)}{|(\kappa_2^j - (1/|z^j|)|}, \tag{3.8}$$

and

$$R(z^j) = \frac{\rho(z^j) - \rho_0 c^{-1}(z^j)}{\rho(z^j) + \rho_0 c^{-1}(z^j)} \tag{3.8}$$

is the normal-incidence reflection coefficient at the critical point z^j. Details of the stationary phase computation are given in the appendix.

We see that the backscattering specular reflection involves the reflection coefficient multiplied by curvature terms. These curvature terms depend on the difference between the principal curvatures of the bottom surface and the curvature $(1/|z^j|$ of the isochrone or constant-time sphere of the incident wave.

The asymptotic form (3.7) depends on a number of geometrical assumptions. First, this formula holds only in the case in which the stationary points are isolated. It does not hold, for example, in regions where the curvature of the surface is the same as the curvature of the isochrone. Second, it holds only under conditions when the Kirchhoff approximation is valid: no multiple scattering, and $k >> \kappa_1, \kappa_2$.

Next we consider specific sonar source-receiver configurations.

fan-beam (side-scan)

For side-scan sonar, we assume that the sources and receivers are distributed over a line segment, so that $x_s = s\hat{e}_2$ and $x_r = r\hat{e}_2$. Beamforming the result (3.7) gives us

$$\int_{-a}^{a}\int_{-a}^{a} u_{sc}(s\hat{e}_2, r\hat{e}_2)dsdr \sim \frac{i}{8\pi}\sum_j \frac{R(z^j)e^{2ik|z^j|}e^{(i\pi/4)\mathrm{sig}H}}{|z^j|^2\sqrt{(\kappa_1^j - (1/|z^j|)(\kappa_2^j - (1/|z^j|)}}(\mathrm{sinc}(kaz^j \cdot \hat{e}_2))^2 \tag{3.9}$$

We note that (3.9) is effectively a sum over critical points that lie in the main lobe of the sonar beam. A side-scan sonar image is formed by plotting the received signal as a function of time. The phase $2ik|z^j|$ corresponds to the two-way travel time from the center of the array to the critical point z^j. Equation (3.9) shows that in the high-frequency limit, the image of a smooth surface would be composed of isolated spots and streaks corresponding to points where Snell's law is satisfied. The intensity at these spots is due to a combination of the local acoustic impedance and curvature. Moreover, these reflections appear in the image convolved by the Fourier transform of the sinc function factor.

pencil-beam

A similar analysis can be carried out for the pencil beam case. Thus we write $x_s = s_1\hat{e}_1 + s_2\hat{e}_2$ and $x_r = r_1\hat{e}_1 + r_2\hat{e}_2$. Here we consider the integral corresponding to a beam steered to point in direction $\hat{\mu}$:

$$I_{PB} = \int_{-a}^{a}\int_{-a}^{a}\int_{-a}^{a}\int_{-a}^{a} e^{ik\hat{\mu}\cdot(x_s+x_r)}u_{sc}(x_s, x_r)d^2sd^2r \tag{3.10}$$

Applying this beamforming to (3.7) gives

$$I_{PB} \sim \frac{i}{8\pi}\frac{R(z^0)e^{2ik|z^0|}e^{(i\pi/4)\mathrm{sig}H}}{|z^0|^2\sqrt{(\kappa_1^0 - (1/|z^0|)(\kappa_2^0 - (1/|z^0|)}}(\mathrm{sinc}(kaz^0 \cdot \hat{e}_1))^2(\mathrm{sinc}(kaz^0 \cdot \hat{e}_2))^2 \tag{3.11}$$

Here the point z^0 is the point on the seafloor to which the beam has been steered, namely $z^0 = |z^0|\hat{\mu}$. If the surface at that point is perpendicular to the vector μ, then we obtain the specular reflection given by (3.11); otherwise the response is of lower order.

As before, this result depends on certain geometrical assumptions about the seafloor at the stationary point.

4. DISCUSSION AND FUTURE WORK

Side-scan sonar images are formed by plotting the received intensity as a function of time. Although the resulting images give the appearance of being maps of the seafloor, in reality they are traveltime maps. Thus, for example, if there are several different reflectors that are the same distance from the sonar array and are all in the same plane probed by the sonar beam, they will all appear at the same spot on the image. It is not clear whether the shape of the seafloor can be reconstructed from this traveltime information, even if images from several different directions are combined.

If the seafloor shape information is lacking, then the acoustic impedance of the seafloor cannot be determined from the side-scan data, because of the geometrical factors multiplying R in (3.7).

Pencil-beam sonar, however, contains more information. In particular, one can first focus a pencil-beam sonar at a particular point on the seafloor. If Snell's law is satisfied at this point, then (3.11) gives the round-trip travel time, and hence the distance, to this point. In theory, the geometry of the bottom could be mapped by an exhaustive survey (tow the sonar array over the entire ocean surface, and from each location scan the entire seafloor). Once the geometry of the seafloor is known, the curvatures could be obtained, and disentangled from the reflection coefficient to provide the acoustic impedance at each point. However, in practice the focus of the beam is unlikely to be fine enough to obtain such detailed curvature information.

At points on the seafloor where Snell's law is not satisfied, these calculations predict that the leading-order high-frequency term should vanish. However, side-scan sonar images certainly appear to show regions of the bottom at which Snell's law is not satisfied. These reflections are most likely from points at which the seafloor is rough and at which the Kirchhoff theory used here does not apply. However, the theory developed here should be valid for mines with smooth surfaces.

In order to obtain the density and sound speed themselves, rather than their product, it is necessary to measure the reflection coefficient at non-normal incidence. Doing this would require different data collection geometries, such as towed arrays, that are not considered in this paper. However, the Generalized Radon Transform inversion methods [BCS, BB], when applied to appropriate data, do provide methods of obtaining this non-normal reflection coefficient. The Generalized Radon Transform method has the additional advantage that it can be extended to the case in which the sound speed of the water varies [BCS, L].

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APPENDIX. DETAILS OF THE STATIONARY PHASE CALCULATIONS

The following quantities need to be evaluated at the stationary point: the phase, its Hessian, the signature of the Hessian, and the integrand. We use the far-field approximation throughout.

The phase.

At the criticial point z^j, the phase itself is $\varphi(z^j) = 2|z^j|$, which is simply the two-way travel time.

The Hessian.

The Hessian is more complicated. The second derivatives of the phase are

$$\frac{\partial^2 \varphi}{\partial \alpha_i \partial \alpha_j} \sim 2\big(\hat{z} \cdot \frac{\partial^2 z}{\partial \alpha_i \partial \alpha_j} - \frac{1}{|z|}\frac{\partial z}{\partial \alpha_i} \cdot \frac{\partial z}{\partial \alpha_j} + \frac{1}{|z|}\hat{z} \cdot \frac{\partial z}{\partial \alpha_i}\hat{z} \cdot \frac{\partial z}{\partial \alpha_j}\big) \tag{A.1}$$

The last term is zero at a critical point by (3.6). We choose the parametrization of the surface in the neighborhood of the critical point to be the arc lengths along the directions of principle curvatures. Then we have

$$\frac{\partial z}{\partial \alpha_i} \cdot \frac{\partial z}{\partial \alpha_j} = \delta_{ij}, \tag{A.2}$$

and

$$\frac{\partial^2 z}{\partial \alpha_i \partial \alpha_j} = \kappa_i \hat{\nu} \delta_{ij}, \tag{A.3}$$

where δ_{ij} denotes the Kroeneker delta that is one when $i = j$ and zero otherwise, and the κs denote the prinicpal curvatures.

The Hessian is thus a diagonal matrix whose diagonal elements are $2(\kappa_1 - 1/|z|)$ and $2(\kappa_2 - 1/|z|)$. These correspond to the difference in the curvature between the isochrone and the surface.

We note that if the curvature $1/|x|$ of the isochrone is equal to one of the curvatures κ_1 or κ_2 of the surface, then the Hessian is zero, and the validity of the stationary phase approximation breaks down. Such spherical regions of the surface would cause a stronger sonar return than predicted by (3a.7).

The signature of the Hessian.

The signature of the Hessian is the sum of the signs of the factors $2(\kappa_1 - 1/|z|)$ and $2(\kappa_2 - 1/|z|)$. In the case of a flat surface, when the principal curvatures of the surface are both zero, the signature is -2.

The integrand.

The integrand to be evaluated at a stationary point $x = z^j$ is

$$|x|^{-2} R(x_s, x) \frac{\partial(|x - x_s| + |x - x_r|)}{\partial \hat{\nu}} \left| \frac{\partial x}{\partial \alpha_1} \times \frac{\partial x}{\partial \alpha_2} \right|. \tag{A.4}$$

The expression $\partial(|x - x_s| + |x - x_r|)/\partial \hat{\nu}$ is equal to the normal ν dotted with the sum of the unit vectors joining the point x^j to the source and receiver. This quantity is precisely $2 \cos \theta$, where θ is the angle between the normal and the vector from x^j to the source (or receiver). When $|x|$ is much larger than the array size, this angle θ is close to 0, so the factor $\partial(|x - x_s| + |x - x_r|)/\partial \hat{\nu}$ is close to 2.

The Jacobian $|(\partial x/\partial \alpha_1) \times (\partial x/\partial \alpha_2)|$, in the case of the coordinatization chosen above to study the Hessian, is simply 1.

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