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Microlocal Analysis of Synthetic Aperture Radar Imaging

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ABSTRACT. *We consider Synthetic Aperture Radar (SAR) in which backscattered waves are measured from locations along a single flight path of an aircraft. Emphasis is on the case where it is not possible to form a beam with the radar. The paper uses a scalar linearized mathematical model of scattering, based on the wave equation. This leads to a forward (scattering) operator, which maps singularities in the coefficient of the wave equation (viewed as a singular perturbation about a constant coefficient) to singularities in the scattered wave field. The goal of SAR is to recover a picture of the singular support of the coefficient, i.e., an* image *of the underlying terrain.*

Traditionally, images are produced by "backprojecting the data". This is done by applying the adjoint of the scattering operator to the data. This backprojected image is equivalent to that obtained by applying to the perturbed coefficient the composition of the scattering operator followed by its adjoint. We analyze this composite operator, and show that it is a paired Lagrangian operator *(also called a* Fourier integral operator with singular symbol*) [21]. The well-understood properties of such operators explains the origin of certain artifacts in the backprojected image.*

1. Introduction

In Synthetic Aperture Radar (SAR) imaging, a plane or satellite carrying an antenna moves along a flight path. The antenna emits pulses of electromagnetic radiation, which scatter off the terrain, and the scattered waves are detected with the same antenna. The received signals are then used to produce an image of the terrain. (See Figure 1)

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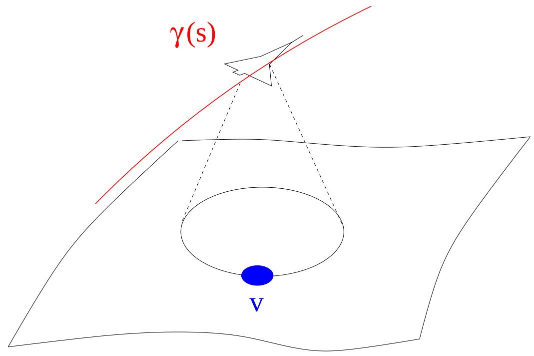

FIGURE 1 Acquisition geometry for SAR

A similar procedure is used for Synthetic Aperture Sonar, using an array of transducers instead of an antenna; here the goal is to map the seafloor. Synthetic aperture focusing techniques are also used in non-destructive evaluation [22] and geophysics [5] [4] [6] [28] [31]. In this paper we discuss explicitly the radar case, but the analysis applies equally well to sonar, ultrasound, and seismic imaging.

Most present SAR systems operate at microwave frequencies, where antennas of a manageable size can form an illuminating beam and rapid Fourier-based imaging methods can be used [14]. Microwave frequencies, however, do not penetrate well through foliage, and there is much interest in developing lower-frequency SAR systems [23] [36] able to image regions under a forest canopy. At these lower frequencies, antennas usually have poor directivity, and the standard Fourier-based imaging methods are not useful.

If the antenna is modeled as an isotropically-radiating point source, the imaging problem reduces to the inversion of a spherical Radon transform. For a flat earth, this becomes the problem of reconstructing a function from its integrals over circles. For this case, an inversion formula for reconstructing symmetric functions from straight flight tracks was given in [13] and [1]; the case of reconstructing a function from its integrals over circular arcs was analysed in [32]. A uniqueness theorem for general flight paths having non-vanishing curvature was proved in [2]. The problem of multiple flight passes and three-dimensional imaging was considered in [25]; for this case an approximate reconstruction formula for a non-ideal antenna was given in [29].

We consider the case in which a single pass is made over the scene, so that backscattered data is known for positions that sample a curve of sensor positions. The data depend on two variables, namely the (fast) time variable and the position on the flight track (changing on slower time-scale). Here we are making what is known as the *start-stop* approximation, in which we assume that the radar is stationary while the radio wave is transmitted and received. This is a very accurate approximation [14] for typical SAR systems.

Because the data is a function of two variables, we expect to be able to reconstruct a two-dimensional image of the scene.

We model the propagation of electromagnetic waves by the wave equation, which involves a local speed of wave propagation. In in free space, this speed is c_0, the speed of light in vacuum. We model scatterers as position-dependent perturbations of this wave speed $c(X)$, where $X \in \mathbf{R}^3$. Boundaries of scatterers (interfaces between objects with different material properties) correspond, for example, to jumps in $c(X)$. To image different objects we therefore want to estimate the singular components of $c(X)$.

For simplicity we use a weak-scattering approximation; this makes the forward scattering operator (the operator that maps the unknown perturbation to the measured data) a linear one. In particular, this operator maps singular components of the wave speed to singularities in the measured data. We show that this operator is a Fourier Integral Operator (FIO) [10] [18] [35]; such operators map singular distributions to other singular distributions (on a different domain in general). The relationship between the input and output singularities forms what is called a canonical relation. Canonical relations associated to FIOs are Lagrangian manifolds [10] and have a rich geometric structure. We study this in detail for the case of radar.

The conventional method for obtaining an image (a picture of the singular support of the wave speed) is to "backproject" the data; this amounts to application of the adjoint of the forward scattering operator to the data. We analyze this procedure for reconstructing the singularities in the scene. We show that in general, this procedure produces artifacts. Moreover, we are able to show that the strength of the artifacts are the same of as those of true singularities; e.g., for every Heaviside jump discontinuity that appears in the image in the correct position, one will also appear in the wrong position.

2. The Mathematical model

For SAR, the correct model is of course Maxwell's equations, but the simpler scalar wave equation is commonly used:

$$\left(\nabla^2 - \frac{1}{c^2(X)}\partial_t^2\right) u(t, X) = 0, \tag{2.1}$$

where c is the wave propagation speed. Each component of the electric and magnetic fields in free space satisfies (2.1); thus it is a good model for the propagation of electromagnetic waves in dry air.

For sonar and ultrasound, (2.1) is a good model. For geophysics it is sometimes used but the equations of linear elasticity are more appropriate.

We assume

Assumption 1 Locally, the earth's surface $\mathcal{X} = \{X = (x, 0) : x \in \mathbf{R}^2\})$ is flat; the flight path is level and is well separated from the surface; and in the intervening region, $c(X) = c_0$.

For radar applications, c_0 is the speed of light in vacuum.

Because electromagnetic waves are rapidly attenuated in the earth, we assume that the scattering takes place in a thin region near the surface . In particular, we assume

Assumption 2 *The perturbation in wave speed c is of the form $c_0^{-2} - c^{-2}(X) = V(x)\delta(x_3)$.*

Here V, the *ground reflectivity function*, is the distribution we wish to reconstruct and δ is the Dirac distribution.

We denote the time-domain waveform sent to the antenna by $P(t)$. We write P in terms of its Fourier transform p:

$$P(t) = \int e^{-i\omega t} p(\omega) d\omega, \tag{2.2}$$

where ω denotes the angular frequency. In practice, the waveform P is such that only a certain union of intervals $[-\omega_{\max}, -\omega_{\min}] \cup [\omega_{\min}, \omega_{\max}]$ contributes significantly to (2.2); we call this set the *effective support* of p. The difference $(\omega_{\max} - \omega_{\min})$ is the (angular-frequency) *bandwidth*. The fact that P is bandlimited means that ultimately we reconstruct band-limited approximations to singular components of the coefficient c.

We show in Appendix 1 that the received wave field at sensor location Y and time t can be approximated by the expression

$$\begin{aligned} S(Y, t) = \int_{\mathbf{R} \times \mathcal{X}} & e^{-i2\omega(t - 2|(x,0) - Y|/c_0)}\ p(\omega) \\ & W(x, Y, t, \omega) V(x)\ \omega^2 d\omega dx, \end{aligned} \tag{2.3}$$

where W can be obtained from (1.15). W contains geometrical factors such as the antenna beam pattern and the attenuation from geometrical spreading of the wave.

In particular, we have assumed

Assumption 3 The data S is *linearly* related to the wave speed perturbation V, i.e., we use a single-scattering (Born) approximation.

The idealized inverse problem is to determine V from knowledge of S for $t \in (T_1, T_2)$ and for Y on a curve. This curve we parametrize by

$$\gamma := \{\ \gamma(s) = (\gamma_1(s), \gamma_2(s), h) :\ s^{\min} < s < s^{\max}\ \} \tag{2.4}$$

where h is the (constant) altitude at which the aircraft flies. We denote the (s,t) parameter space by $\mathcal{Y}$:

$$\mathcal{Y} = (s^{\min}, s^{\max}) \times (T_1, T_2). \tag{2.5}$$

A number of technical difficulties arise if we attempt to image points directly underneath the antenna. In particular, we will see that the technique of backprojection does not apply to such data coming from locations directly underneath the current location of the the antenna. We therefore assume

Assumption 4 The height h of the flight track and the time T_1 at which data recording begins are related by $T_1 > T_0 > 2h/c_0$ for some T_0.

The abrupt ends of the flight track and the recording time interval cause artifacts in the image; consequently it is useful to multiply the data by a smooth taper or *mute* function $m(s,t)$ which is zero outside $\mathcal{Y}$.

We denote the map from scene V to data $d = m \cdot S$ by F, where

$$FV(s,t) = \int_{\mathbf{R}\times\mathcal{X}} e^{-i\omega(t-2|R(x,s)|/c_0)} A(x,s,t,\omega)V(x)\, d\omega dx, \tag{2.6}$$

where $R(x,s) = (x,0) - \gamma(s)$,

$$\begin{aligned} A(x,s,t,\omega) &= m(s,t)\omega^2 p(\omega)\, W(x,\gamma(s),t,\omega) \\ &:= \omega^2 p(\omega)\alpha(x,s,t,\omega). \end{aligned} \tag{2.7}$$

For a broadband antenna, α is approximately independent of ω. More specifically, we make the following assumption. This assumption is needed in order to make various stationary phase approximations we make hold; in fact, as we will see, this assumption makes the "forward" operator F a FIO.

Assumption 5 The amplitude A of (2.6) satisfies

$$\sup_{(s,t,x)\in K} \mid \partial_\omega^\alpha \partial_s^\beta \partial_t^\delta \partial_x^\rho A(x,s,t,\omega) \mid \leq C\ (1+\omega^2)^{(2-|\alpha|)/2} \tag{2.8}$$

where K is any compact subset of $\mathcal{Y} \times \mathcal{X}$, $\rho = (\rho_1, \rho_2)$ is a multi-index and the constant C depends on $K, \alpha, \beta, \delta, \rho$.

This assumption is valid for example when the source waveform P is approximately a delta function and the antenna is sufficiently broadband.

Finally, we remark that the case when the scattering surface is nonplanar can be treated similarly [30].

3. Analysis of the scattering operator

The "forward" operator F (2.6) is an example of a Fourier Integral Operator (FIO) [10], [35], [18]. Standard theorems from FIO theory therefore give us information about how F maps singularities in the scene to those in the data. How F maps singularities is determined [10] by its wavefront relation $WF'(F)$, which is a subset of its (twisted) canonical relation Λ', which in turn is a subset of $T^*(\mathcal{Y} \times \mathcal{X}) \setminus \{0\}$, i.e., of the cotangent bundle $\mathcal{Y} \times \mathcal{X}$ with its zero section removed.

Definition 1 If the Fourier Integral Operator $T : \mathcal{E}'(\mathcal{X}) \to \mathcal{E}'(\mathcal{Y})$ is given by the oscillatory integral

$$Tf(y) = \int e^{i\phi(y,x,\omega)} a(y, x, \omega)\ f(x) d\omega dx \tag{3.1}$$

then its (twisted) *canonical relation* is the set

$$\begin{gathered}\Lambda'_T = \{((y, \eta), (x, \xi)) : D_\omega \phi(y, x, \omega) = 0, \\ \eta = D_y \phi(y, x, \omega), \xi = -D_x \phi(y, x, \omega),\ \xi, \eta \neq 0\},\end{gathered} \tag{3.2}$$

where D denotes the gradient. The *wavefront relation* $WF'(T)$ is the subset of points in Λ'_T that are also in the essential support of the amplitude a, i.e., in the region outside of which a and all its derivatives decrease faster than any negative power of ω, as $|\omega| \to \infty$.

The canonical relation $\Lambda'_F = \Lambda'$ for F is computed from the phase of (2.6):

$$\begin{aligned}\Lambda' \quad = \quad & \{\ ((s, t, \sigma, \tau), (x, \xi)) \mid t = 2|R(x, s)|/c_0, \\ & \sigma = -2\tau \widehat{R}(x, s) \cdot \dot{\gamma}(s)/c_0\ , \xi = 2\tau \widehat{R}(x, s)_H/c_0\ \}\end{aligned} \tag{3.3}$$

where the subscript H denotes the horizontal component of a vector. This horizontal component arises because we are identifying the scattering surface $\mathcal{X}$ with $\mathbf{R}^2$.

The wavefront relation $WF'(F)$ is the set of points in Λ' for which the antenna beam pattern is nonzero and for which $\tau = \omega$ is in the effective support of p (i.e., points $((s, t, \sigma, \tau), (x, \xi)) \in \Lambda'$ such that the associated (x, s, t, τ) belong to the essential support of A).

Some more terminology [16], [17] we need is the following:

Definition 2 Suppose a mapping $f : M \to N$ between two manifolds has full rank everywhere except on a submanifold $\Sigma \subset M$ where it drops rank by one, and suppose the determinant of the Jacobian of f vanishes to exactly first order on Σ. We say that f has a *simple fold* singularity or *drops rank simply* along Σ if

$$T_l\Sigma \cap \ker(df(l)) = \{0\}, \;\; \forall l \in \Sigma, \tag{3.4}$$

where df denotes the exterior derivative of f, i.e., a one-form. If, on the other hand, for every l in Σ we have

$$Ker(df(l)) \subset T_l\Sigma, \tag{3.5}$$

then we say that f has a *blow-down* singularity, and that Σ gets *blown down* by f. If the condition (3.4) holds in a punctured neighborhood of a point l_0, and at l_0 we have $ker(df)$ is simply tangent to Σ, we say the map has a *cusp* singularity at l_0.

Definition 3 If $\Lambda' \subset T^*\mathcal{X} \times T^*\mathcal{Y}$ is a wavefront relation such that the only singularities of the natural projections $\pi_L : \Lambda' \to T^*\mathcal{Y}$ and $\pi_R : \Lambda' \to T^*\mathcal{X}$ are simple folds along a common codimension-one submanifold, called the 'fold' of Λ', then we say [26] that Λ' is a *folding canonical relation.*

Definition 4 If the assumptions of definition (3) hold for the maps π_L, π_R when restricted to the wavefront relation $WF'(F) \subset \Lambda'$ of a Fourier integral operator F, we say that F has a *folding wavefront relation.*

Lemma 1 *The natural projection $\Lambda' \to T^*\mathcal{Y}$ has a simple fold along the submanifold Σ given by (3.8).*

Proof. One may easily verify that $\{(s, x, \tau) \in \mathbf{R^4}\}$ constitutes a global coordinate system for Λ'. We use this coordinate system to express (local representatives of) the natural projection $\pi_L \; : \Lambda' \to T^*\mathcal{Y}$:

$$\begin{aligned}\pi_L(s, x, \tau) &= (s, t, \sigma, \tau) \\ &= (s, \frac{2}{c_0}\,|R(x, s)|\,, \frac{2\tau}{c_0}(\widehat{R}(x, s)) \cdot \dot{\gamma}(s)\;, \tau),\end{aligned}$$

where $R(x, s)$ is defined below (2.6).

Let π denote the operator of horizontal projection (projection onto first two components) of a vector. The action of π'_L (the Jacobian of π_L) is given by

$$\begin{pmatrix} \delta s \\ \delta t \\ \delta\sigma \\ \delta\tau \end{pmatrix} = \pi'_L \begin{pmatrix} \delta s \\ \delta x \\ \delta\tau \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \\ L_1 & \frac{2}{c_0}\widehat{R}(x, s)^T_H & 0 \\ \\ L_2 & -\frac{2\tau}{c_0}\dot{\gamma} \cdot P^R_H & L_3 \\ \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \delta s \\ \delta x \\ \delta\tau \end{pmatrix} \tag{3.6}$$

where the superscript T denotes transpose, where L_1, L_2, L_3 are linear operators whose specific form is immaterial, and where for arbitrary 3-vectors X, Y,

$$\begin{aligned} X_H &= \pi X = (x_1, x_2, 0) = (x, 0) \\ P^Y X &= \frac{X - (\hat{Y} \cdot X)\hat{Y}}{|Y|} \\ P^Y_H X &= \pi P^Y X \end{aligned} \tag{3.7}$$

We note that P^Y projects the vector X onto the plane perpendicular to Y.

First, we show that π'_L has full rank everywhere except on the critical submanifold

$$\Sigma := \{\ ((s,t,\sigma,\tau),(x,\xi)) \in \Lambda' \mid \dot{\gamma}(s) \text{ is co-linear with } R_H(x,s)\ \} \tag{3.8}$$

of Λ' where its rank drops by one. To see this, we note that the determinant of the matrix in (3.6) is equal to a non-zero multiple of the determinant J of the matrix formed by the two vectors $\hat{R}_H$ and $P^R_H \cdot \dot{\gamma}$, which can be written

$$J = \begin{vmatrix} \hat{R}_1 & (P^R\dot{\gamma})_1 & 0 \\ \hat{R}_2 & (P^R\dot{\gamma})_2 & 0 \\ \hat{R}_3 & (P^R\dot{\gamma})_3 & 1 \end{vmatrix} = \hat{R} \times (P^R\dot{\gamma}) \cdot \hat{e}_3 \tag{3.9}$$

The vectors $\hat{R}$ and $P^R\dot{\gamma}$ are orthogonal. Since, $R \neq 0$, the scalar-triple product of (3.9) is zero only when either (i) $P^R\gamma = 0$, or (ii) when the plane formed by $\hat{R}$ and $P^R\dot{\gamma}$ is vertical. Condition (i) or (ii) holds precisely when $\hat{R}_H(x,s)$ is co-linear with $\dot{\gamma}(s)$, i.e., when $((s,t,\sigma,\tau),(x,\xi)) \in \Sigma$. Therefore, the rank of π'_L drops (by one) only at points on Σ.

Next we show that the determinant of π'_L vanishes to precisely first order on Σ. For this we need only exhibit a direction in which the derivative of J is non-zero. We choose a direction $\delta X = (\delta x, 0) = (\delta x_1, \delta x_2, 0)$ such that $\dot{\gamma} \cdot \delta X = 0$, which, on Σ, is the same condition as $\hat{R} \cdot \delta X = 0$. In this case, we have

$$D_x \hat{R} \delta X = P^R_H \delta X = \delta X / |R| \tag{3.10}$$

Under these conditions, we have that the action of the x-directional derivative of $P^R\dot{\gamma}$ is given by

$$\begin{aligned} D_x(P^R\dot{\gamma})\delta X &= -\left[\frac{D_x(\hat{R}\cdot\dot{\gamma})\hat{R}}{|R|} + \frac{(\hat{R}\cdot\dot{\gamma})D_x\hat{R}}{|R|} - \frac{(\hat{R}\cdot\dot{\gamma})\hat{R}D_x|R|}{|R|^2}\right]\cdot\delta X \\ &= -\left(\frac{P^R\dot{\gamma}}{|R|}\hat{R}\cdot\delta X + \frac{\hat{R}\cdot\dot{\gamma}(P^R\delta X)}{|R|} - \frac{\hat{R}(P^R\delta X)\cdot\dot{\gamma}}{|R|}\right) \end{aligned}$$

$$= -\left(0 + \frac{\delta X}{|R|}\frac{\hat{R}\cdot\dot{\gamma}}{|R|} + 0\right), \tag{3.11}$$

where in the second line we used $D_x|R| \cdot \delta X = \hat{R} \cdot \delta X$. We thus find the x-directional derivative of J is given by

$$\begin{aligned} D_x J \cdot \delta X &= \frac{\delta X}{|R|} \times (P^R \dot{\gamma}) \cdot \hat{e}_3 - \hat{R} \times \frac{\delta X}{|R|}\frac{\hat{R}\cdot\dot{\gamma}}{|R|} \cdot \hat{e}_3 \\ &= \frac{\delta X}{|R|} \times \frac{\dot{\gamma} - \hat{R}\hat{R}\cdot\dot{\gamma}}{|R|} \cdot \hat{e}_3 \; - \hat{R} \times \frac{\delta X}{|R|}\frac{\hat{R}\cdot\dot{\gamma}}{|R|} \cdot \hat{e}_3 \\ &= \frac{\delta X \times \dot{\gamma} \cdot \hat{e}_3}{|R|^2} - \frac{\delta X}{|R|} \times \hat{R}\frac{\hat{R}\cdot\dot{\gamma}}{|R|} \cdot \hat{e}_3 - \hat{R} \times \frac{\delta X}{|R|}\frac{\hat{R}\cdot\dot{\gamma}}{|R|} \cdot \hat{e}_3 \\ &= \frac{\delta X \times \dot{\gamma} \cdot \hat{e}_3}{|R|^2}, \end{aligned} \tag{3.12}$$

which is nonzero because $\{\delta X, \dot{\gamma}, e_3\}$ is an orthogonal triad.

To check condition (3.4), we need to calculate the tangent space of Σ at a typical point l_0. In terms of the coordinates on Λ', points in Σ can be written in the form

$$\sigma_0 = (s_0, x_0 = \gamma(s_0) + \beta_0\dot{\gamma}(s_0), \tau_0). \tag{3.13}$$

In terms of the coordinates on Λ', the tangent space $T_{l_0}\Sigma$ is spanned by the following three vectors:

$$\begin{aligned} (0, 0, \tau'(0)) &= \left.\frac{d}{d\alpha}\right|_{\alpha=0} (s_0, x_0, \tau(\alpha)) \\ (s'(0), [\dot{\gamma}(s_0) + \beta_0\ddot{\gamma}(s_0)]\, s'(0), 0) &= \left.\frac{d}{d\alpha}\right|_{\alpha=0} (s(\alpha), \gamma(s(\alpha)) + \beta_0\dot{\gamma}(s(\alpha)), \tau_0) \\ (0, \dot{\gamma}(s_0)\beta'(0), 0) &= \left.\frac{d}{d\alpha}\right|_{\alpha=0} (s_0, \gamma(s_0) + \beta(\alpha)\dot{\gamma}(s_0), \tau_0) \end{aligned} \tag{3.14}$$

Here $\tau(\cdot), s(\cdot)$, and $\beta(\cdot)$ are smooth functions whose values at 0 are τ_0, s_0, and β_0, respectively. The three vectors of (3.14) can be regarded as tangent vectors of the ambient space $T_{l_0}\Lambda'$ and can be written in the form: $(0, 0, \delta\tau)$, $(\delta s, \delta x = [\dot{\gamma}(s_0) + \beta_0\ddot{\gamma}(s_0)]\,\delta s, 0)$, and $(0, \delta x = \dot{\gamma}(s_0)\delta\beta, 0)$ respectively, where $\delta\beta$ is an arbitrary scalar.

To check the transversality condition (3.4), we compute the kernel of $d\pi_L|_\Sigma$. We see from (3.6) that in order to have $(\delta s, \delta x, \delta\tau) \in ker(d\pi_L(s_0, x_0, \tau_0))$, we need the following two sets of conditions to hold:

$$\delta\tau = \delta s = 0, \;\; \widehat{R}_H \cdot \delta x = 0 \tag{3.15}$$

$$\dot{\gamma}(s_0) \cdot P^R_H \delta x = 0 \tag{3.16}$$

To determine whether such a vector of $ker(d\pi_L(l_0))$ can belong to $T_{l_0}\Sigma$, we attempt to write the tangent vectors satisfying (3.15, 3.16) as linear combinations of the vectors spanning $T_{l_0}\Sigma$:

$$(0, \delta x, 0) = a(0, 0, \delta\tau) + b(\delta s, Z\delta s, 0) + c(0, \dot{\gamma}(s_0)_H \delta\beta, 0), \tag{3.17}$$

where $Z = \dot{\gamma}(s_0) + \beta_0\ddot{\gamma}(s_0)$. We see immediately that the first two summands must vanish, otherwise either the first or third component of the right side would be non-zero, which is incompatible with the left hand side. In the case that $\beta_0 \neq 0$, c must be zero because, on Σ, a δx proportional to $\dot{\gamma}(s_0)$ cannot be orthogonal to $R_H(x_0, s_0)$ as required by (3.15). In the case where $\beta_0 = 0$, then $\hat{R}_H = 0$, so (3.15) is vacuous and the only condition on δx is (3.16). Similar reasoning using (3.10) then leads to $c = 0$. We have therefore confirmed the transversality condition (3.4). QED

Lemma 2 *If $\ddot{\gamma}(s) \neq 0$, $\forall s \in (s^{\min}, s^{\max})$, then the natural projection $\pi_R : WF'(F) \to T^*\mathcal{X}$ has a simple fold singularity. At points s where $\ddot{\gamma}(s) = 0$, (3.5) holds; thus if $\ddot{\gamma} = 0$ for all s, π_R has a blow-down singularity; if $\ddot{\gamma}(s)$ has only simple zeros (e.g., a 'snaking' flight track, with simple inflection points), then π_R has only cusp singularities.*

Remark Notice that the domain of π_R in the lemma is the wavefront relation $WF'(F) \subset \Lambda'$, rather than the canonical relation Λ'. This is because the transversality condition (3.4) breaks down for points on the ground located directly underneath the antenna. This can be observed in the proof below, where $ker(d\pi_R)$ and $T\Sigma$ have non-trivial tangent vectors in common; namely $(\delta s = 0, \delta x = 0, \delta\tau \neq 0)$.

Proof. In the coordinates (s, x, τ) of Λ', $\pi_R : WF'(F) \to T^*\mathcal{X}$ can be written

$$\pi_R(s, x, \tau) = (x, \xi) = (x, \frac{2\tau}{c_0}\widehat{R}_H(x, s)). \tag{3.18}$$

The action of its Jacobian π'_R is given by

$$\begin{pmatrix} \delta x \\ \delta\xi \end{pmatrix} = \begin{pmatrix} 0 & I & 0 \\ & & \\ -\frac{2\tau}{c_0} \cdot P_H^R\dot{\gamma} & L_4 & \frac{2}{c_0}\hat{R}_H(x, s) \end{pmatrix} \begin{pmatrix} \delta s \\ \delta x \\ \delta\tau \end{pmatrix}$$

where L_4 is a 2×2 matrix whose particular form is unimportant.

First, we note that π'_R has full rank everywhere except on the critical submanifold (3.8) of Λ' where its rank drops by one. This is because the determinant of π'_R happens to be a non-zero multiple of $|\pi'_L|$, so the arguments of Lemma 1 apply. Moreover, the arguments of Lemma 1 show $|\pi'_R|$ vanishes to order one on Σ.

To check the transversality condition (3.4), we calculate the kernel of $d\pi_R$. The condition for $(\delta s, \delta x, \delta\tau)$ to be in $ker(d\pi_R(s_0, x_0, \tau_0))$ amounts to

$$\delta x = 0, \ \left(-\tau_0 P_H^R \dot{\gamma}(s_0)\right) \ \delta s + \left(\hat{R}_H\right) \ \delta\tau = 0. \tag{3.19}$$

To determine whether $T_{l_0}\Sigma$ contains the vector $(\delta s, 0, \delta\tau)$ of $ker(d\pi_R)$, we attempt to write it as a linear combination of the vectors spanning $T_{l_0}\Sigma$:

$$(\delta s, 0, \delta\tau) = a(0, 0, \delta\tau) + b(\delta s, [\dot{\gamma}(s_0) + \beta_0\ddot{\gamma}(s_0)]\, \delta s, 0) + c(0, \dot{\gamma}(s_0)\delta\beta, 0). \tag{3.20}$$

Given that we only consider the restriction of π_R to $WF'(F)$, we must have $\beta_0 \neq 0$. We see immediately that the latter pair of summands must vanish by linear independence of $\dot{\gamma}(s_0)$ and $\ddot{\gamma}(s_0)$). This means that $\delta s = 0$ and hence, since $\hat{R}_H \neq 0$ in (3.19), $\delta\tau = 0$. QED.

Theorem 1 Under Assumptions 1, 4 and 5, F is a FIO of order 3/2. If, in addition, $\ddot{\gamma}_H(s) \neq 0$, $\forall s \in (s^{\min}, s^{\max})$, then F has a folding wavefront relation.

Proof. The map F (2.6) is a FIO because a simple calculation shows that the phase function $\phi(s, t, x, \omega) = \omega(t - 2|R(x, s)|/c_0)$ is homogeneous of degree one and non-degenerate [10], as $\nabla_\omega\phi$ never vanishes. From the definition of the order [10] of a FIO, it now follows that F has order $2 + 1/2 - 4/4 = 3/2$.

Lemmas 1 and 2 show that $WF'(F)$ is a folding wavefront relation. QED

4. Imaging

We form an image by applying the *backprojection* operator F^* to the data:

$$I(z) := \ (F^*d)(z) = \int e^{i\omega(t-2R(z,s)/c_0)}\overline{A(z, s, t, \omega)}d(s, t) \ d\omega ds dt. \tag{4.1}$$

Next we determine the degree to which the image I faithfully reproduces features of the ground reflectivity function V. Using $d = FV$ in (4.1) results in

$$\begin{aligned} I(z) = F^*FV(z) \quad &= \quad \int e^{i\omega(t-2R(z,s)/c_0)}e^{-i\tilde{\omega}(t-2R(x,s)/c_0)} \\ & \qquad \overline{A(z, s, t, \tilde{\omega})}A(x, s, t, \tilde{\omega})V(x) \ d\tilde{\omega} \ d\omega \ d^2x \ ds \ dt \end{aligned} \tag{4.2}$$

We show that F^*F falls into a certain class of operators that has been studied extensively [19], [20],[21]. This class of operators is associated with a pair of cleanly intersecting Lagrangian manifolds [10][35] $\Lambda_1, \Lambda_2 \subset T^*(\mathcal{X} \times \mathcal{X})$.

Definition 5 A pair of manifolds (M_1, M_2) is said to be *cleanly intersecting* if $M_1 \bigcap M_2$ is a manifold and

$$T_m M_1 \bigcap T_m M_2 = T_m(M_1 \cap M_2), \ \forall m \in M_1 \cap M_2.$$

For a pair of cleanly intersecting Lagrangian manifolds Λ_1, Λ_2, we define the class of paired Lagrangian distributions $I^{p,q}(\Lambda_1, \Lambda_2)$:

Definition 6 Let $\mathcal{X}$ be a C^∞-manifold. The distribution u is said to be a *paired Lagrangian distribution* on $\mathcal{X} \times \mathcal{X}$, written $u \in \bigcup_{p,q \in R} I^{p,q}(\Lambda_1, \Lambda_2; \mathcal{X} \times \mathcal{X})$, iff
a) $u \in H^s_{loc}(\mathcal{X} \times \mathcal{X})$ for some s and
b) if $P_1, \ldots, P_M$ are first order pseudodifferential operators on $\mathcal{X} \times \mathcal{X}$, which are all characteristic on $\Lambda_1 \cup \Lambda_2$ (i.e. their principal symbols vanish on the union of Λ_1 and Λ_2), then

$$P_1 \cdots P_N u \in H^s_{loc}(\mathcal{X} \times \mathcal{X}), \ \forall \ N = 1, 2, \ldots \tag{4.3}$$

Definition 7 We use the term *paired Lagrangian operator* for a linear operator $Q : \ \mathcal{E}'(\mathcal{X}) \to \mathcal{E}'(\mathcal{X})$ whose the Schwartz distribution kernel K_Q of Q is a paired Lagrangian distribution on $\mathcal{X} \times \mathcal{X}$.

Other authors call such an operator a *FIO with singular symbol* [20] or a *pseudodifferential operator with singular symbol* [19]; however such operators are neither pseudodifferential operators nor FIOs.

A consequence [20] of the above definitions is: If A and B are pseudodifferential operators on $X \times X$ whose wavefront relations are contained in

$$\{ \ (\ (x,\xi),(x,\xi) \ )| \ (x,\xi) \in \Lambda_0 \setminus \Lambda_1 \ \},$$

$$\{ \ (\ (x,\xi),(x,\xi) \ )| \ (x,\xi) \in \Lambda_1 \setminus \Lambda_0 \ \},$$

respectively, and u satisfies (4.3), then (automatically) Au and Bu can be identified as kernels of FIOs whose canonical relations are contained in $\Lambda_0 \setminus \Lambda_1$ and $\Lambda_1 \setminus \Lambda_0$, respectively. If the orders of Au and Bu are $p+q$ and q, respectively, then we admit u to the class of FIOs with singular symbol and order (p, q), denoted by $I^{p,q}(\Lambda_0, \Lambda_1, \mathcal{X} \times \mathcal{X})$.

Corollary 1 If the curvature of the flight track $\ddot{\gamma}$ is never zero, then $F^*F \in I^{0,3}(\triangle, \mathcal{C}, \mathcal{X} \times \mathcal{X})$, where $\triangle, \mathcal{C}$ are respectively the identity relation and a folding canonical relation in $T^*(\mathcal{X} \times \mathcal{X})$.

Proof. It was shown in Theorem 4.3 of [28] that when F is a FIO with folding wavefront relation and order $(n-1)/2$, then there is a folding wavefront relation $\mathcal{C}$ on $T^*(\mathcal{X} \times \mathcal{X})$ such that F^*F is in the class $I^{0,n-1}(\triangle, \mathcal{C}, \mathcal{X} \times \mathcal{X})$.

5. Implications of the Analysis

¿From the point of view of reconstructing singularities, Corollary (1) is a negative result. The implication of corollary (1) is that when one backprojects the data, singularities will be reconstructed in their correct locations (explained by the relation Δ) along with artifacts (explained by the relation $\mathcal{C}$). Moreover the strength of these singularities will be equal in the sense that F^*F behaves as pseudodifferential operator of order three on $\triangle \setminus \mathcal{C}$ and as a FIO of order three on $\mathcal{C} \setminus \triangle$. Moreover, the relations $\triangle \setminus \mathcal{C}$ and $\mathcal{C} \setminus \triangle$ are local canonical graphs, which means that the action of F^*F on singularities can be calculated in a straightforward way.

To the authors' knowledge, no parametrix (asymptotic inverse) is available for F^*F in such a situation. In fact, operators such as F^*F with the same structure have turned up in other applications, such as acoustical imaging when the background medium produces fold caustics in the incident field [28] and certain problems involving the X-ray transform [21] and Radon transform [19] [20]. However, in practice one can attempt to minimize the relative strength of the artifacts; this is discussed in [30].

1. Appendix: The Mathematical Model

The model we use for wave propagation is (2.1). We also model the field emanating from the antenna and its scattering.

1.1 A model for the field from an antenna

In free space, the field $G_0(t, X)$ at time t and position $X \in \mathbf{R}^3$ due to a delta function point source at the origin at time 0 is given by [34]

$$G_0(t, X) = \frac{\delta(t - |X|/c_0)}{4\pi|X|}, \tag{1.1}$$

which satisfies

$$\left(\nabla^2 - c_0^{-2}\partial_t^2\right) G_0(t, X) = -\delta(t)\delta(X).$$

The antenna (or transducer array), however, is not a point source [37] $\delta(X)$, and the signal sent to the antenna is not a delta function $\delta(t)$. Therefore we replace $\delta(X)$ by $J_s(X)$ and $\delta(t)$ by the waveform $P(t)$. In the radar case, J_s is the time derivative of the current distribution over the antenna, and $P(t)$ is the waveform sent to the antenna.

The field U^{in} emanating from the antenna then satisfies

$$\left(\nabla^2 - c_0^{-2}\partial_t^2\right) U^{in}(t, X) = -P(t)J_s(X) \tag{1.2}$$

so that

$$\begin{aligned} U^{in}(t,X) &= G_0 * (PJ_s) \\ &= \int \frac{P(t-|X-Y|/c_0)}{4\pi|X-Y|} J_s(Y) d^3Y \end{aligned} \tag{1.3}$$

where the star denotes convolution in t and X. The waveform $P(t)$ can be of almost any shape, but commonly a *chirp* of the form $P(t) = \mathrm{rect}(t)\exp(i\alpha t^2)$ is used.

We denote the Fourier transform (2.2) of P by p.

With the notation (2.2), (1.3) becomes

$$U^{in}(t,X) = \int \tfrac{e^{-i\omega(t-|X-Y|/c_0)}}{4\pi|X-Y|} p(\omega) J_s(Y) d\omega d^3Y. \tag{1.4}$$

Next we assume that the antenna is small compared with the distance to the scatterers. We denote the center of the antenna by Y^0; thus a point on the antenna can be written $Y = Y^0 + Q$, where Q is a vector from the center of the antenna to a point on the antenna. In this notation, the assumption that the scattering location X is far from the antenna can be expressed $|Q| << |X - Y^0|$. For such X, we can write

$$|X-Y| = |X-Y^0| - (\widehat{X-Y^0}) \cdot Q + O(|Q|^2/|X-Y^0|), \tag{1.5}$$

where $\hat{y}$ denotes a unit vector in the same direction as y.

We use the expansion (1.5) in (1.4) to obtain

$$\begin{aligned} U^{in}(t,X) &\approx \int \frac{e^{-i\omega(t-|X-Y^0|/c_0)}}{4\pi|X-Y^0|} e^{-i\omega(\widehat{X-Y^0})\cdot Q} p(\omega) J_s(Y^0+Q) d\omega d^3Q \\ &\approx \int \frac{e^{-i\omega(t-|X-Y^0|/c_0)}}{4\pi|X-Y^0|} p(\omega) j_s(\omega(\widehat{X-Y^0}), Y^0) d\omega \end{aligned} \tag{1.6}$$

where we have written

$$j_s(\omega(\widehat{X-Y^0}), Y^0) = \int e^{-i\omega(\widehat{X-Y^0})\cdot Q} J_s(Y^0+Q) d^3Q. \tag{1.7}$$

This Fourier transform of the current density gives the antenna beam pattern in the far-field at each fixed frequency. Generally the antenna beam pattern is nearly independent of frequency over the effective support of $p(\omega)$; indeed, a great deal of work goes into designing antennas for which this is the case. Antennas whose beam patterns are nearly constant over a wide frequency band are called *broadband* antennas.

We see from (1.6) that the field emanating from the antenna is a superposition of fixed-frequency point sources that are each shaped by the antenna beam pattern. The field (1.6) clearly depends on Y^0, the location of the center of the antenna; consequently we write $U^{in}_{Y^0}$ to indicate this dependence.

1.2 A linearized scattering model

Since we are ignoring polarization, the current density J_s is scalar and the model we use for wave propagation, including the source, is

$$\left(\nabla^2 - c^{-2}(X)\partial_t^2\right) U(t,X) = -P(t)J_s(X). \tag{1.8}$$

We write $U = U^{in} + U^{sc}$ in (1.8) and use (1.2) to obtain

$$\left(\nabla^2 - c_0^{-2}\partial_t^2\right) U^{sc}(t,X) = -\tilde{V}(X)\partial_t^2 U(t,X), \tag{1.9}$$

where

$$\tilde{V}(X) = c_0^{-2} - c^{-2}(X). \tag{1.10}$$

We make the assumption that $\tilde{V}(X) = V(x)\delta(x_3)$, where $X = (x, x_3)$. The *reflectivity function* V contains all the information about how the medium differs from free space. It is V, or at least its discontinuities and other singularities, that we want to recover.

We can write (1.9) as an integral equation

$$U^{sc}(t,X) = \int G_0(t-\tau, X-(z,0))V(z)\partial_\tau^2 U(\tau,(z,0))d\tau dz. \tag{1.11}$$

A commonly used approximation [24] [22], often called the *Born approximation* or the *single scattering approximation*, is to replace the full field U on the right side of (1.11) and (1.9) by the incident field U^{in}, which converts (1.11) to

$$\begin{aligned} U^{sc}(t,X) &\approx \int G_0(t-\tau, X-(z,0))V(z)\partial_\tau^2 U^{in}(\tau,(z,0))d\tau dz \\ &= \int \frac{V(z)}{4\pi|X-(z,0)|}\partial_t^2 U^{in}(t-|X-(z,0)|/c_0, z)dz. \end{aligned} \tag{1.12}$$

The value of this approximation is that it removes the nonlinearity in the inverse problem: it replaces the product of two unknowns (V and U) by a single unknown (V) multiplied by the known incident field.

The Born approximation makes the problem simpler, but it is not necessarily a good approximation. Another linearizing approximation that can be used for reflection from smooth surfaces is the *Kirchhoff approximation*, in which the scattered field is replaced by its geometrical optics approximation [6] [24]. Here, however, we consider only the Born approximation.

For the incident field (1.6), (1.12) becomes

$$U^{sc}_{Y^0}(t,X) \approx \int \frac{e^{-i\omega((t-(|X-(z,0)|+|(z,0)-Y^0|)/c_0)}}{(4\pi)^2|X-(z,0)||(z,0)-Y^0|} \omega^2 p(\omega) j_s(\omega(\widehat{(z,0)-Y^0}), Y^0)V(z)\, d\omega dz.$$

(1.13)

At the center $X = Y_0$ of the antenna,

$$U_{Y^0}^{sc}(t, Y^0) \approx \int \frac{e^{-i\omega(t-2|(z,0)-Y^0|/c_0)}}{(4\pi)^2|(z,0)-Y^0|^2}\omega^2 p(\omega) j_s(\omega(\widehat{(z,0)-Y^0}), Y^0)V(z)\ d\omega dz. \tag{1.14}$$

In practice, the field is measured not at the center of the antenna but is integrated over the whole antenna. The resulting signal can be calculated with the help of (1.5) and involves a beam pattern for reception $j_r(\omega(\widehat{(z,0)-Y^0}), Y^0)$. (Normally, when the same antenna is used for transmission and reception, $j_r = j_s$.) Thus an expression for the signal measured at antenna location Y is

$$\begin{aligned} S(t,Y) &= \int_{\text{antenna}} U_Y^{sc}(t,X)dX \\ &\approx \int \frac{e^{-i\omega(t-2|(z,0)-Y|/c_0)}}{(4\pi)^2|(z,0)-Y|^2}\omega^2 p(\omega) j_s(\omega(\widehat{(z,0)-Y}), Y) \\ &\qquad j_r(\omega(\widehat{(z,0)-Y}), Y)V(z)\ d\omega dz. \end{aligned} \tag{1.15}$$

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