SYNTHETIC APERTURE INVERSION FOR NON-FLAT TOPOGRAPHY

C. J. Nolan [*], *M. Cheney* [**]

[*] Department of Mathematics and Statistics
University of Limerick, Limerick, Ireland
Tel: +353 61 202700, Fax: +353 61 334927, E-mail: clifford.nolan@ul.ie

[**] Department of Mathematical Sciences
Rensselaer Polytechnic Institute, Troy, NY 12180, USA
Tel: 1 518 276 2646, Fax: 1 518 276 4824, E-mail: cheney@rpi.edu

ABSTRACT

This paper considers Synthetic Aperture Radar and other synthetic aperture imaging systems in which a backscattered wave is measured from positions along a single flight track. We assume that the ground topography is known but not necessarily flat.

We consider two cases, corresponding to the degree of directionality of the antenna. For the high-directivity case, we propose an imaging algorithm involving backprojection and a spatially varying filter that corrects for the antenna beam pattern, source waveform, and other geometrical factors. We give conditions on the relationship between the flight track and the topography to avoid artifacts. We show that the algorithm correctly reproduces certain features of the scene.

For the case of an antenna with poor directionality, the image produced by the above algorithm contains artifacts. For this case, we analyze the strength of the artifacts relative to the strength of the true image. The analysis of this paper shows that the artifacts can be somewhat suppressed by increasing the curvature of the flight track and by keeping the desired target in view for as long as possible.

1 INTRODUCTION

We consider Synthetic Aperture Radar imaging in the case of antennas with poor directivity, where the antenna footprint is large and standard Fourier-based imaging methods are not useful. This is typically the case for foliage-penetrating radar [9] [11], whose low frequencies do not allow for much beam focusing.

In the case of a non-directional antenna on a straight-line flight track above a flat earth, it is not possible to determine from the data whether a given reflection originated from the left or the right side of the flight track. This gives rise to an image artifact which we call an *ambiguity artifact*. Similar ambiguity artifacts arise in the case of curved flight paths and non-flat earth topography. One goal of this paper is to give conditions on the relationship between the antenna footprint, the flight path, and the topography for which these ambiguity artifacts do not arise.

We consider imaging algorithms based on backprojection. Such algorithms produce an image I via

$$I(z) = \int w(z,s,t)d(s,t)dsdt \tag{1}$$

where d denotes the data, which depends on time t and a flight path parameter s, and w denotes a weighting function that will be explained in the text. This weighting function w depends on factors such as the flight path and on the topography.

In the case in which ambiguity artifacts can be avoided, the weighting function can also compensate for the antenna beam pattern, the source waveform, and other geometrical factors. In this case, we show that the image I is related to the desired ground reflectivity function V by

$$I(z) = \int K_I(z,x)V(x)d^2x, \tag{2}$$

where K_I is approximately a delta function. From analysis of K_I, we find that the image I has the property that certain features such as edges and boundaries beween different materials are positioned correctly and have the correct amplitudes.

In the case when ambiguity artifacts are unavoidable, we cannot arrange for w to compensate for all geometrical factors. We can still form a backprojected image, but this image contains ambiguity artifacts. We analyze the strength of these artifacts relative to the strength of the true image. Our analysis shows that the artifacts can be suppressed to some degree by increasing the the curvature of the flight track, orienting the flight path in a favorable way, keeping the target in view for as long as possible.

2 MODEL FOR RECEIVED SIGNAL

We assume the earth's surface is located at the position given by $X = \psi(x)$, where $\psi : \mathbf{R}^2 \to \mathbf{R}^3$ is known. We assume that the scattering is due to perturbations in the permittivity of the form $V(x)\delta(X - \psi(x))$, where V, the *ground reflectivity function*, is the quantity we wish to image. We denote the map from scene V to data d by F, where

$$(FV)(s,t) = \int e^{-i\omega(t-2|\psi(x)-\gamma(s)|/c_0)} \cdot A(x,s,t,\omega)V(x)\, d\omega d^2x, \tag{3}$$

where γ is the flight path, and $A = \omega^2 p(\omega)a(x,s,t,\omega)$, where p is the frequency-domain waveform of the transmitted signal, and a contains factors such as the antenna beam pattern, geometrical spreading factors, and taper functions to avoid abrupt transitions at the ends of the data collection region. We assume that a is approximately independent of ω, at least to the extent that A satisfies the estimate

$$\sup_{(s,t,x)\in K} |\, \partial_\omega^\alpha \partial_s^\beta \partial_t^\delta \partial_{x_1}^{\rho_1} \partial_{x_2}^{\rho_2} A(x,s,t,\omega)\| \leq$$

$$C\,(1+\omega^2)^{(2-|\alpha|)/2} \tag{4}$$

where K is any compact subset of $\mathbf{R}_s \times \mathbf{R}_t \times \mathbf{R}_x^2$, and the constant C depends on $K, \alpha, \beta, \delta, \rho$. This assumption is needed in order to make various stationary phase calculations hold; in fact this assumption makes the "forward" operator F a Fourier Integral Operator [6] [17].

3 IMAGE FORMATION

We form the image by means of a *filtered backprojection* operator:

$$I(z) := \int Q(z,s,t,\omega) e^{i\omega(t-2|\psi(z)-\gamma(s)|/c_0)} d(s,t)\, d\omega ds dt, \tag{5}$$

where Q is determined below.

To determine Q, we investigate the degree to which the image I faithfully reproduces features of the ground reflectivity function V. We will show that under favorable circumstances, singular features such as edges appear in the correct locations.

Using $d = FV$ in (5) results in an equation of the form

$$I(z) = \int K(z,x) V(x) d^2x, \tag{6}$$

where

$$K(z,x) = \int Q(z,s,t,\omega) e^{i\omega(t-2|\psi(z)-\gamma(s)|/c_0)} \cdot e^{-i\tilde{\omega}(t-2|\psi(x)-\gamma(s)|/c_0)} A(x,s,t,\tilde{\omega}) d\tilde{\omega}\, d\omega\, ds\, dt. \tag{7}$$

The kernel K is the imaging *point-spread function*, which, when considered as a function of the variable z, is the reconstructed (backprojected) image due to a delta point source located at x. If we had $K_I(z,x) = \delta(z-x)$, then the image I would be perfect; we want to determine Q so that K comes as close as possible to being a delta function.

In (7) we perform a large-ω stationary phase calculation in the variables $\tilde{\omega}$ and t. This results in

$$I(z) = 2\pi \int K(z,x) V(x)\, dx + E_1(z) \tag{8}$$

where E_1 denotes a function smoother than the first term on the right side of (8) and

$$\begin{aligned} K(z,x) \quad = \quad & \int e^{i2\omega(|\psi(x)-\gamma(s)|-|\psi(z)-\gamma(s)|)/c_0} \\ & \cdot Q(z,s,2|\psi(x)-\gamma(s)|/c_0,\omega) \\ & \cdot A(x,s,2|\psi(x)-\gamma(s)|/c_0,\omega)\, d\omega ds. \end{aligned} \tag{9}$$

The main contributions to K come from those critical points of its phase at which the amplitude A is nonzero; the criticality conditions are

$$\begin{aligned} |\psi(z)-\gamma(s)| \quad &= \quad |\psi(x)-\gamma(s)| \\ \widehat{(\psi(z)-\gamma(s))} \cdot \dot{\gamma}(s) \quad &= \quad \widehat{(\psi(x)-\gamma(s))} \cdot \dot{\gamma}(s). \end{aligned} \tag{10}$$

The first condition of (10) says that x should be at the same range as z. The second says that the direction $\widehat{(\psi(x)-\gamma(s))}$ should have the same projection onto the flight velocity vector as the direction $\widehat{(\psi(z)-\gamma(s))}$. We will call (s,x,z) a *contributing critical point* if it satisfies both conditions of (10) and if $A(x,s,2|\psi(x)-\gamma(s)|/c_0,\omega)$ is nonzero for some ω (and hence, by assumption (4), for a large interval of ω).

For a high-fidelity image, we would like K to be as close as possible to the delta function $\delta(z-x) \propto \int \exp(i(z-x)\cdot\xi) d\xi$. In particular, we should have contributing critical points only when $z = x$. In other words, if (s,x,z) satisfies (10) when $z \neq x$, the amplitude A should be zero there. Flight paths for which this is the case can be found when the the antenna beam pattern is sufficiently focused to one side of the flight heading.

For example, in the case of flat topography, there are points (s,x,z) satisfying (10) in two cases, one when $z = x$ and the other when z is at a "mirror" point x^*, which is the reflection of x across the horizontal projection of the line tangent to the flight path at $\gamma(s)$. This "mirror" critical point contributes to the image, and hence gives rise to an ambiguity artifact, unless the amplitude A is zero there. To make the amplitude A zero at all such "mirror" points, the antenna beam should be negligible to one side of the flight direction [8].

We show below [15] that the conditions for avoiding ambiguity artifacts are the following.

1. The only contributing critical points are those for which $z = x$.

2. At no point x on the earth's surface should the surface be perpendicular to the plane formed by the the range vector $\psi(x)-\gamma(s)$ and the flight velocity vector $\dot{\gamma}(s)$. This should hold for every position $\gamma(s)$ along the flight path.

If both these conditions are satisfied, then the image formed by (5) is free of ambiguity artifacts.

3.1 THE CASE OF NO AMBIGUITY ARTIFACTS

Under the above conditions for avoiding ambiguity artifacs, we make a certain change of variables that makes the phase of $K(z,x)$ the same as that of the delta function $\delta(z-x) \propto \int \exp(i(z-x)\cdot\xi) d\xi$.

To determine the change of variables that makes the phase of K into the phase of a delta function, we first use the integral form of the remainder for Taylor's theorem to write the imaginary part of the exponent of (9) as

$$2\omega(|\psi(x)-\gamma(s)|-|\psi(z)-\gamma(s)|)/c_0 = (z-x)\cdot\Xi(z,x,s,\omega); \tag{11}$$

explicitly, Ξ is given by

$$\begin{aligned} \Xi(z,x,s,\omega) &= -\frac{2\omega}{c_0} \int_0^1 \nabla |\psi(y)-\gamma(s)| \Big|_{y=x+\lambda(z-x)} d\lambda \\ &= \frac{2\omega}{c_0} \int_0^1 \widehat{(\gamma(s)-\psi(y))} \cdot D\psi(y) \Big|_{y=x+\lambda(z-x)} d\lambda, \end{aligned} \tag{12}$$

where the differentiations on the right side are with respect to $y \in \mathbf{R^2}$. When $z = x$, (12) is simply

$$\Xi(z,z,s,\omega) = (2\omega/c_0) \widehat{(\gamma(s)-\psi(z))} \cdot D\psi(z). \tag{13}$$

In a neighborhood of $z = x$, we make the change of variables

$$(s,\omega) \to \xi = \Xi(z,x,s,\omega). \tag{14}$$

This transforms the integral (8) into

$$\begin{aligned} I(z) &= 2\pi \int e^{i(z-x)\cdot\xi} Q(z,s,2|\psi(x)-\gamma(s)|/c_0,\omega) \\ &\quad \cdot A(x,s,2|\psi(x)-\gamma(s)|/c_0,\omega) \\ &\quad \cdot \left|\frac{\partial(s,\omega)}{\partial\xi}\right| (z,x,s,\omega) V(x)\, d^2\xi\, d^2x + E_1(z). \end{aligned} \tag{15}$$

where s and ω are understood to refer to $s(\xi)$ and $\omega(\xi)$, respectively. This exhibits the operator with kernel K as a pseudodifferential operator. Pseudodifferential operators have the *pseudolocal* property [17], i.e., they do not move singularities or change their orientation.

The Jacobian determinant $|\partial(s,\omega)/\partial\xi|$ is also called the *Beylkin determinant* [2] [4]. When $x = z$, its reciprocal is given by

$$\left|\frac{\partial\xi}{\partial(s,\omega)}\right| (z,z,s,\omega) = \frac{4\omega}{c_0^2} \left| \begin{array}{cc} (\widehat{\gamma(s)-\psi(z)}) \cdot \frac{\partial\psi(z)}{\partial z_1} & P_\perp\dot\gamma(s) \cdot \frac{\partial\psi(z)}{\partial z_1} \\ (\widehat{\gamma(s)-\psi(z)}) \cdot \frac{\partial\psi(z)}{\partial z_2} & P_\perp\dot\gamma(s) \cdot \frac{\partial\psi(z)}{\partial z_2} \end{array} \right| \tag{16}$$

where $P_\perp\dot\gamma(s)$ denotes the scaled projection of $\dot\gamma(s)$ onto the plane perpendicular to $\widehat{\gamma(s)-\psi(z)}$:

$$P_\perp\dot\gamma(s) = \frac{\dot\gamma(s) - (\widehat{\gamma(s)-\psi(z)})\left((\widehat{\gamma(s)-\psi(z)})\cdot\dot\gamma(s)\right)}{|\gamma(s)-\psi(z)|}. \tag{17}$$

We note that $P_\perp\dot\gamma(s)$ remains in the plane $T_{s,z}$ defined by

$$T_{s,z} := \mathrm{Span}\,\{\gamma(s)-\psi(z), \dot\gamma(s)\}. \tag{18}$$

Conditions under which the change of variables (14) can be made (locally) are those under which the right side of (16) is nonzero; this gives us the second condition on the relation between the flight track and the ground topgraphy. This condition can be understood by noting that the vectors $\widehat{\gamma-\psi}$ and $P_\perp\dot\gamma$ are orthogonal and thus determine a coordinate system in the plane $T_{s,z}$; the rows of (16) are the coordinates of the tangent vectors $X^1 = \partial\psi/\partial z_1$ and $X^2 = \partial\psi/\partial z_2$ in this coordinate system. Thus the right side of (16) is nonzero provided X^1 and X^2 project to two linearly independent vectors in the plane $T_{s,z}$. In [15] we show that this condition is equivalent to the condition that the earth's surface at z not be orthogonal to the plane $T_{s,z}$.

Equation (15) shows how we should choose Q to make K an approximate delta function. In particular, we should choose

$$Q(z,s,t\omega) = (2\pi)^{-3}\left(b(z,z,\xi(s,\omega))\right)^{-1}\chi(z,z,\xi(s,\omega)) \tag{19}$$

where χ is a smooth cutoff function that prevents us from dividing by zero, and where

$$b(z,x,\xi) = 2\pi A\bigg(x, s(\xi), 2|\psi(x)-\gamma(s(\xi))|/c_0, \omega(\xi)\bigg) \cdot \left|\frac{\partial(s,\omega)}{\partial\xi}\right| (z,x,s(\xi),\omega(\xi)) \tag{20}$$

3.2 ANALYSIS OF AMBIGUITY ARTIFACTS

In this section, we discuss the effect of an antenna with poor directivity, where (10) has more than one solution in the support of A.

In this case, the change of variables (14) can be made only in the neighborhood of $z = x$, so we cannot use the above method for determining Q. We therefore consider the case $Q = 1$; we denote the corresponding kernel (9) by K_1.

In the flat-earth case, we see that there are two points z on the earth for which (s,z,x) satisfies (10): one at $z = x$, which gives rise to the correct image, and one at a "mirror" point, which can give rise to an artifact. For non-flat topograpy, it is possible to have a curve of points z on the earth for which (s,z,x) satisfies (10). Such a curve is composed of points at the same range $|\psi(z)-\gamma(s)|$ whose directions $(\widehat{\psi(z)-\gamma(s)})$ have the same projection onto the flight velocity vector $\dot\gamma(s)$.

We refer to contributing critical points for which $z \neq x$ as "extraneous" critical points; it is these critical points that give rise to the ambiguity artifacts. We analyze the relative contributions to the image from these points by investigating the size of K_1 there.

At an extraneous critical point (s,z,x), we carry out a stationary phase analysis in the s variable. This is possible only when the second derivative of the phase is nonzero. We write $H(z,x,s) = \phi''(z,s,x,\lambda)/\omega$, where ϕ denotes the phase in (9). We obtain to leading order of approximation

$$\begin{aligned} K_1(z,x) &\approx \textstyle\int A(x,s(z,x),2|\psi(x)-\gamma(s(z,x))|/c_0,\omega) \\ &\quad \cdot|\omega|^{-1/2}\exp(i\pi/4)\,|H|^{-1/2}(z,x,s(z,x))\,d\omega \\ &\approx a(x,s(z,x),2|\psi(x)-\gamma(s(z,x))|/c_0)\exp(i\pi/4) \\ &\quad \cdot|H|^{-1/2}(z,x,s(z,x))\,d\omega \textstyle\int |\omega|^{3/2}p(\omega)\,d\omega. \end{aligned} \tag{21}$$

On the other hand, if we consider the reconstruction at x we find

$$K_1(x,x) \approx \int a(x,s,2|\psi(x)-\gamma(s)|/c_0)\,ds \cdot \int |\omega|^{3/2}p(\omega)\,d\omega. \tag{22}$$

Equations (21) and (22) tell us the degree to which a point scatterer at x creates an artifact at the point z. The ratio of artifact to image is

$$\left|\frac{K_1(z,x)}{K_1(x,x)}\right| \approx \frac{a(x,s(z,x),2|\psi(x)-\gamma(s(z,x))|/c_0)}{\int a(x,s,2|\psi(x)-\gamma(s)|/c_0)\,ds} \cdot|H|^{-1/2}(z,x,s(z,x)) \tag{23}$$

This ratio can be made smaller (i.e., the strength of the artifact at z can be decreased) by increasing the denominator of (23) and by increasing H. The former can be done by increasing the the stretch of γ for which x is illuminated. The Hessian H can be increased by increasing the curvature of the flight track while keeping its direction unchanged, or by changing the direction of the flight track so as to make the acceleration more closely aligned with the difference of unit vectors $\Big(\,(\widehat{\psi(z)-\gamma(s)}) - (\widehat{\psi(x)-\gamma(s)})\,\Big)$.

4 CONCLUSION

We have exhibited a filtered backprojection algorithm (5), including a variable filter Q given by (19) and (20), that corrects for the topography, antenna beam pattern, transmitted waveform, and geometrical spreading. If the antenna beam pattern is negligible to one side of the flight direction and the flight path satisfies conditions 1) and 2), then this backprojection algorithm results in an image free of ambiguity artifacts, which moreover has the property that edges appear in the correct location, with the correct orientation and the correct strength. For an omnidirectional antenna pattern, ambiguity artifacts are present; their strengths are affected by the curvature and orientation of the flight path and the aperture over which the target is in view.

5 ACKNOWLEDGMENTS

We thank Lars Ulander and his group for discussions and copies of their papers. This work was partially supported by National Science Foundation through the Engineering Research Centers Program (award number EEC-9986821) and the Focused Research Groups in the Mathematical Sciences Program.

6 REFERENCES

[1] G. Beylkin and R. Burridge, "Linearized inverse scattering problems in acoustics and elasticity", Wave Motion 12 (1990) 15–52.

[2] G. Beylkin, "Imaging of discontinuities in the inverse scattering problem by inversion of a causal generalized Radon transform", J. Math. Phys., 26 (1985) 99–108.

[3] N. Bleistein and R.A. Handelsman, Asymptotic Expansions of Integrals, Dover, New York, 1986.

[4] N. Bleistein, J. K. Cohen, and J.W. Stockwell, The Mathematics of Multidimensional Seismic Inversion, Springer, New York, 2000.

[5] M. Cheney, "A mathematical tutorial on Synthetic Aperture Radar", SIAM Review 43 (2001) 301–312.

[6] J.J. Duistermaat, Fourier Integral Operators, Birkhauser, Boston, 1996.

[7] G. Franceschetti and R. Lanari, Synthetic Aperture Radar Processing, CRC Press, New York, 1999.

[8] S. E. Fisher, D. S. Weile, and E. Michielssen, "Pareto genetic algorithm design of log-periodic monopole arrays mounted on realistic platforms," Journal of Electromagnetic Waves and Applications, vol. 13, no. 5, pp. 571-598, 1999.

[9] A. Gustavsson, P.-O. Frölind, H. Hellsten, T. Jonsson, B. Larsson, G. Stenström, and L.M.H. Ulander, "Development and operation of the FOA CARABAS HF/VHF-SAR System", Proc. Fourth International Workshop on Radar Polarimetry, July 13-17, 1998, Nantes, France.

[10] H. Hellsten and L.E. Andersson, "An inverse method for the processing of synthetic aperture radar data", Inverse Problems 3 (1987), 111–124.

[11] H. Hellsten, L.M.H. Ulander, A. Gustavsson, and B. Larsson, "Development of VHF CARABAS II SAR", SPIE Conference Proceedings Vol. 2747 (SPIE, Bellingham, 1996), 48–60.

[12] A. Louis and E.T. Quinto, "Local Tomographic Methods in SONAR", in Surveys on Solution Methods for Inverse Problems, Ed. D. Colton, H. W. Engl, A.K. Louis, J.R. McLaughlin and W. Rundell, Springer Verlag, New York, 2000.

[13] S. Nilsson, "Application of fast backprojection techniques for some inverse problems of integral geometry", Linköping Studies in Science and Technology, Dissertation No. 499 (1997).

[14] C. J. Nolan and M. Cheney, "Synthetic Aperture Inversion", Inverse Problems 18 (2002) 221–236.

[15] C.J. Nolan and M. Cheney, "Synthetic Aperture Inversion with Non-Flat Topography", preprint.

[16] C. J. Nolan and W. W. Symes, "Global solution of a linearized inverse problem for the acoustic wave equation", Comm. in P.D.E., Vol. 22, Nos 5-6, 1997.

[17] F. Treves, Introduction to Pseudodifferential and Fourier Integral Operators, volumes I and II, Plenum Press, New York, 1980.