OPTIMAL ACOUSTIC MEASUREMENTS

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INTRODUCTION

This paper is motivated by the question "How can we design the best possible system to do acoustic imaging?" If we want to make the best possible images, we must begin with data that contain the most possible information. In particular, since all practical measurements are of limited precision, some scatterers may be undetectable because their scattered fields are below the precision of the measuring instrument: our data will contain no information about them. What incident fields should we apply that will result in the biggest measurements?

There are many ways to formulate this question, depending on the measuring instruments. In this paper we consider a formulation involving wave-splitting in the accessible half-space: what downgoing wave will result in an upgoing wave of greatest energy?

A closely related question arises in the case when we have a guess about the configuration of the inaccessible half-space. What measurements should we make to determine whether our guess is accurate? In this case we compare the scattered field to the field computed from the guessed configuration. Again we look for the incident field that results in the greatest energy difference.

This optimal measurement problem has been studied for fixed-frequency problems in electrical impedance tomography (Isaacson, 1986) and acoustic scattering (Mast, Nachman, and Waag, 1997). For time-domain problems, the issue of optimal time-dependent waveforms in a special $1+1$ – dimensional case was studied in (Cherkaeva and Tripp, 1997), where a time-harmonic waveform was found to be optimal.

In this paper we study the question of optimal time-dependent waveforms in the $3+1$ – dimensional case. In particular, we consider the half-space geometry: we imagine that a plane divides space into accessible and inaccessible regions, and we assume that we can make measurements everywhere on the plane.

We show that the optimal incident field can be found by an iterative process involving time reversal "mirrors". The connection between optimal measurements and an iterative time-reversal process was first pointed out in (Mast, Nachman, and Waag, 1997), (Prada and Fink, 1994), and (Prada, Thomas, and Fink, 1995); in all these papers, the analysis was carried out a a single frequency.

In this work we consider time-dependent fields. We show that for band-limited incident fields and compactly supported scatterers, the iterative time-reversal algorithm converges to a sum of time-harmonic fields. This provides a theoretical foundation for the pulse-broadening observed in the computations of Cherkaeva and Tripp (1997) and in the time-reversal experiments of Prada and Fink (1994), and Prada, Thomas, and Fink (1995).

Section 2 contains a careful formulation of the idealized problem: the wave equation model, the measurements, the notion of "biggest". Section 3 gives the adaptive experimental algorithm that can be used to find the optimal field even if the scatterer is unknown. In general the iterates converge to a time-harmonic field. The paper concludes with a discussion.

Full details can be found in (Cheney, Isaacson, and Lassas, 2000).

PRECISE FORMULATION OF THE PROBLEM

We consider the constant-density acoustic wave equation

$$(\nabla^2 - c^{-2}(x)\partial_t^2)U(t,x) = 0. \tag{1}$$

in the case in which $c = c_0$ everywhere in the upper half-space $x_3 > 0$. Here $x = (x_1, x_2, x_3)$. This model includes neither dispersion nor dissipation.

We consider the half-space geometry, in which the lower half-space is inaccessible and the upper half-space is accessible. The measurements we consider are those in which we send a downgoing wave into the lower half-space and measure, on the plane $x_3 = 0$, the corresponding upgoing wave. "Upgoing" and "downgoing" can be defined by means of Fourier transforms (in time and in the lateral space variables $x' = (x_1, x_2)$); see (Cheney, Isaacson, and Lassas, 2000) for details. We denote by $\mathcal{S}$ the scattering operator that maps the downgoing wave $U^\downarrow$ to the upgoing wave $U^\uparrow$ on the plane $x_3 = 0$. The scattering operator for a reference configuration is denoted by $\mathcal{S}_0$.

In order to find the downgoing field $U^\downarrow$ that maximizes the difference between the scattered waves $\mathcal{S}U^\downarrow$ and $\mathcal{S}_0U^\downarrow$, we want to maximize the quotient

$$\max_{U^\downarrow} \frac{\|\mathcal{S}U^\downarrow - \mathcal{S}_0U^\downarrow\|}{\|U^\downarrow\|}. \tag{2}$$

Here it is necessary to divide by $\|U^\downarrow\|$ to account for the fact that in practice the energy we can apply is always finite.

In (2), we need to define the measured of "size" $\|\cdot\|$. We do this with the help of the energy identity

$$\int_{\partial V} (\partial_t U)\partial_\nu U dS = \frac{\partial_t}{2} \int_V \left(|\nabla U|^2 + \frac{1}{c^2(x)} (\partial_t U)^2 \right) dx, \tag{3}$$

where the right side corresponds to the increase in energy in the volume V and the left side is the energy flux through the surface $x_3 = 0$. We thus use as our measure of "size" the total (time-integrated) energy flux into the region $x_3 < 0$:

$$\|U\|_W = \left| \int_{-\infty}^{\infty} \int_{x_3=0} (\partial_t U) \partial_{x_3} U d^2x' \, dt \right|$$

(In the electromagnetic case, the corresponding quantity would be the integral of the Poynting vector, dotted with the unit normal vector $\hat{e}_3$, integrated over the plane $x_3 = 0$ and integrated over all time.)

Thus the best downgoing field is the one that maximizes

$$\max_{U^\downarrow} \frac{\|\mathcal{S}U^\downarrow - \mathcal{S}_0 U^\downarrow\|_W}{\|U^\downarrow\|_W}. \tag{4}$$

AN ADAPTIVE METHOD FOR PRODUCING THE BEST FIELD

To maximize (4) in the case when the medium is unknown, we can use the following adaptive method.

0) Begin with any downgoing wave $U_0^\downarrow$; let $j = 0$.

1) Send $U_j^\downarrow$ into the lower half-space; measure the resulting upgoing field

$$U_j^\uparrow(t, x) = \mathcal{S}U_j^\downarrow(t, x).$$

2) Calculate the corresponding scattering from the reference configuration $\mathcal{S}_0 U_j^\downarrow(t, x)$. Calculate the difference field

$$U_j^\uparrow(t, x) = U_j^\uparrow(t, x) - \mathcal{S}_0 U_j^\downarrow(t, x).$$

3) The next downgoing wave is the (normalized) *time-reversed* difference

$$U_{j+1}^\downarrow(t, x) = \frac{U_j^\uparrow(-t, x) - (\mathcal{S}_0 U_j^\downarrow)(-t, x)}{c_j};$$

add one to j; go to step 1).

To understand why this iterative process converges to the right answer, imagine the case of a single point scatterer, and take the reference medium to be empty space. Any incident wave will scatter from the point scatterer into an outgoing spherical wave. If this outgoing spherical wave is time-reversed, it becomes a spherical wave that focuses on the point scatterer. Focusing all the energy of the incident wave on the scatterer gives rise to the largest possible scattered wave.

For fixed frequencies, the iterative time-reversal algorithm has been analyzed by Prada, Thomas, and Fink (1995). They considered the case of a finite array of transducers and the corresponding "transfer matrix" in which the (i, j)th element of the matrix is the scattered field measured at the jth transducer resulting from the activation of only the ith transducer. Prada et al. (1995) showed that the iterative time-reversal algorithm converges to the spatial eigenfunction corresponding to the largest eigenvalue of the transfer matrix. Their paper contains diagrams that show what happens when the scatterering region contains a strong scatterer and a weak scatterer. The strong scatterer returns a stronger wave, which, when time-reversed, becomes a stronger wave focusing on the stronger scatterer. After a few iterations, the signals from the weak scatterer disappear because all energy becomes focused on the stronger scatterer.

Here we consider the question of the time-domain waveform of the optimal field. In (Cheney, Isaacson, and Lassas, 2000) we analyze the problem by Fourier transforming to the frequency domain. At each fixed frequency, the scattering operator has eigenvalues, and these eigenvalues depend analytically on the frequency. Suppose the largest eigenvalue is biggest at the frequency ω_0. Then in general, the iterative time-reversal algorithm converges to a *time-harmonic* (fixed-frequency) wave with the frequency ω_0. The spatial shape of the waveform is given by the eigenfunction corresponding to this largest eigenvalue.

This prediction is consistent with the experiments of Prada, Thomas, and Fink (1995), in which pulse broadening and frequency shifts are observed as the iterative algorithm proceeds. This prediction is also consistent with the one-dimensional work of (Cherkaeva and Tripp, 1997), in which the optimal time-domain waveform was found to be a time-harmonic wave.

CONCLUSIONS AND OPEN QUESTIONS

This approach to the optimal measurement problem decouples the measurement issue from the problem of forming an image. However, this work shows only how to find the single best measurement for determining whether an unknown scatterer is identical to a particular guess. It is not clear how to extend this work to find a full set of measurements that would be necessary for forming an image.

This analysis shows that the iterative time-reversal work of Prada and Fink (1994) and Prada, Thomas, and Fink (1995) provides an experimental method to obtain optimal fields. Moreover, this analysis explains the pulse-broadening seen in (Prada, Thomas, and Fink, 1995) and (Cherkaeva and Tripp, 1997): the optimal time-domain waveform is a time-harmonic one.

This analysis suggests that the commonly-used pings and chirps are not optimal from the point of view of providing the biggest signal to distinguish an unknown scatterer from a particular guess.

There are many open questions related to this work, some of which are probably easy and others hard. In particular, problems that seem to be relatively straightforward are extending this work to Maxwell's equations and to materials in which the medium parameters depend on frequency. Extending the theory to measurements made in a limited aperture is probably not difficult, but will involve detailed modeling of transducers or antennas.

More difficult problems include extending the work to dissipative media, and finding a full set of optimal measurements for forming an image. In the fixed-frequency case, it is clear that a full set of optimal measurements are the eigenfunctions of the operator $\mathcal{S}$ or $\mathcal{S} - \mathcal{S}_0$. But the eigenvalues of these operators depend continuously on frequency, which means that there is no "next best" incident field. Thus it is not clear what finite set of measurements contain the most possible "information" in a broader sense.

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