Microlocal High-Range-Resolution ISAR for Low Signal-to-Noise Environments

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ABSTRACT

We consider the problem of identification of airborne objects from high-range-resolution radar data. We use high-frequency asymptotics to show that certain features of the object correspond to identifiable features of the radar data. We study the cases of single scattering and scattering from re-entrant structures such as ducts.

This work suggests a method for target identification that circumvents the need to create an intermediate radar image from which the object's characteristics are to be extracted. As such, this scheme may be applicable to efficient machine-based radar identification programs.

Keywords: Inverse Synthetic Aperture Radar, Kirchhoff scattering, noise

1. INTRODUCTION

Current methods of identifying objects from radar data generally involve first forming an image, and then attempting to identify features of the image.[1–6] Here we propose a different approach, namely to carry out the identification directly from examination of the raw radar data.

This approach requires determining which features of the radar data correspond to which features of the object. Our approach[7] relates the singular structure (such as edges) of the target to the singular structure of the data set. Restricting our attention to the singular structure—specifically, to a certain set in phase space called the *wavefront set*—allows us to use the tools of microlocal analysis.[8–11] This strategy was first applied to imaging problems by Beylkin[12]; its uses in seismic prospecting,[13–16] X-ray tomography,[17,18] sonar,[19] and synthetic-aperture radar[20,21] are active areas of research. An approach similar to the one we pursue here, in which we use microlocal analysis not to do imaging but instead to study the connection between features of the target and the data, was considered for the X-ray tomography problem by Quinto.[22]

We begin in section 2 by examining the general properties of radar scattering and developing mathematical models for the measured data. These models involve Fourier integral operators with kernels that are oscillatory integrals; it is this that makes it possible to study these models with the techniques of microlocal analysis. Next we present an overview of the microlocal concepts and theorems that are relevant to our investigation (section 3.1). These two sections serve to introduce our notation, assumptions, and terminology. In particular, we assume throughout that the target's rotational acceleration is negligible. Section 3 contains our main results: the calculation of the wavefront sets for the single-scattering case and for multiple scattering from re-entrant structures such as ducts.

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2. RADAR DATA

High-range-resolution (HRR) radar systems transmit the equivalent of a short electromagnetic pulse and measure the time delay of the corresponding waveform reflected from a target. This provides an estimate of the target's range, and, more generally, the range to individual target substructures.

To obtain different views of the target, radar systems can use multiple pulses that interrogate the target as it rotates and sequentially presents different aspects to the radar. Such systems are known as inverse synthetic-aperture radar, or ISAR, systems.

Ultimately, the behavior of radar data is determined by scattered-field solutions to the wave equation. Since radar systems transmit and receive radio waves, we should generally examine the electromagnetic (vector) wave equation. For simplicity, however, we will examine the scalar wave equation and assume that the components of the electromagnetic field each satisfy[23]

$$\left(\nabla^2 - c^{-2}\,\partial_t^2\right) u(t, \boldsymbol{y}) = 0 \tag{1}$$

in the region exterior to the scattering object Ω. We write the total field as a sum of the incident and scattered fields $u = g + u^{\mathrm{s}}$, where the Green's function g represents the field due to a point source at $\boldsymbol{x}$, the position of the radar. Specifically, g is given by[24]

$$g(t, \boldsymbol{y} - \boldsymbol{x}) = \frac{\delta(t - |\boldsymbol{y} - \boldsymbol{x}|/c)}{4\pi|\boldsymbol{y} - \boldsymbol{x}|} = \int \frac{\mathrm{e}^{-\mathrm{i}\omega(t - |\boldsymbol{y}-\boldsymbol{x}|/c)}}{8\pi^2|\boldsymbol{y} - \boldsymbol{x}|}\,\mathrm{d}\omega \tag{2}$$

and satisfies

$$\left(\nabla^2 - c^{-2}\partial_t^2\right) g(t, \boldsymbol{y} - \boldsymbol{x}) = -\delta(t)\delta(\boldsymbol{y} - \boldsymbol{x}). \tag{3}$$

The dependence of g on the source position $\boldsymbol{x}$ induces a similar dependence in the scattered field u^s, which we write as $u^s(t, \boldsymbol{y}, \boldsymbol{x})$.

In section 2.1, we develop a mathematical model for HRR radar data and explain the fundamental role played by the single-scattering approximation. We examine the case of scattering from a re-entrant structure such as a duct in section 2.2.

2.1. The Kirchhoff approximation

The Kirchhoff approximation[14] is a geometric optics approximation. We use it to obtain an expression for the scattered field as follows. First we multiply (1) by $g(t - t', \boldsymbol{x} - \boldsymbol{y})$ and (3) by $u(t', \boldsymbol{y})$, subtract the resulting equations, and apply Green's theorem to the region exterior to the scattering object Ω, and use the outgoing radiation conditions to eliminate the contributions to the integral from infinity. The result is

$$u^s(t, \boldsymbol{x}, \boldsymbol{x}) \;=\; \int_{\partial\Omega} \Big(g(t - t', \boldsymbol{x}, \boldsymbol{y})\partial_{\boldsymbol{\nu}} u^s(t', \boldsymbol{y}, \boldsymbol{x}) - u^s(t', \boldsymbol{y}, \boldsymbol{x})\partial_{\boldsymbol{\nu}} g(t - t', \boldsymbol{x}, \boldsymbol{y}) \Big)\, dt' dS_{\boldsymbol{y}}, \tag{4}$$

where $\partial_{\boldsymbol{\nu}}$ denotes differentiation with respect to the outward unit normal to Ω. It is on the right side of (4) that we use the geometrical optics approximation to the scattered field. In particular, the geometrical optics approximation assumes that, on the illuminated surface, the phase of the scattered field is determined by the high-frequency law of reflection (i.e., the angle of incidence is equal to the angle of reflection), and the amplitude of the scattered field is proportional to that of the incident field. The constant of proportionality is called the *reflection coefficient*; we denote it by $V(\boldsymbol{y})$. Strictly speaking, it depends also on the angle between the direction of incidence and the direction of scatter, but for typical ISAR applications these angles vary so little that we can neglect this dependence. For a rotating target, the illuminated surface also varies with angle; but again we assume that the angles vary so little that this effect can be neglected. Consequently, on the right side of (4), we use

$$u^s(t, \boldsymbol{y}, \boldsymbol{x}) \;\approx\; V(\boldsymbol{y})\frac{\delta(t - |\boldsymbol{y} - \boldsymbol{x}|/c)}{4\pi|\boldsymbol{y} - \boldsymbol{x}|} \;=\; V(\boldsymbol{y}) \int \frac{\mathrm{e}^{-\mathrm{i}\omega(t - |\boldsymbol{y}-\boldsymbol{x}|/c)}}{8\pi^2|\boldsymbol{y} - \boldsymbol{x}|}\,\mathrm{d}\omega$$

$$\partial_{\boldsymbol{\nu}} u^s(t,\boldsymbol{y},\boldsymbol{x}) \approx V(\boldsymbol{y})\,(\partial_{\boldsymbol{\nu}}|\boldsymbol{y}-\boldsymbol{x}|)\,\frac{\delta'(t-|\boldsymbol{y}-\boldsymbol{x}|/c)}{4\pi|\boldsymbol{y}-\boldsymbol{x}|} = V(\boldsymbol{y})\,(\partial_{\boldsymbol{\nu}}|\boldsymbol{y}-\boldsymbol{x}|)\int i\omega \frac{\mathrm{e}^{-\mathrm{i}\omega(t-|\boldsymbol{y}-\boldsymbol{x}|/c)}}{8\pi^2|\boldsymbol{y}-\boldsymbol{x}|}\,\mathrm{d}\omega \tag{5}$$

where in differentiating we have retained only the leading order term for large ω. Using these expressions in (4) and simplifying results in

$$u^s(t,\boldsymbol{x},\boldsymbol{x}) = \int_{\partial\Omega}\int i\omega \frac{\mathrm{e}^{-\mathrm{i}\omega(t-2|\boldsymbol{y}-\boldsymbol{x}|/c)}}{(8\pi^2|\boldsymbol{y}-\boldsymbol{x}|)^2} V(\boldsymbol{y}) 2\partial_{\boldsymbol{\nu}}|\boldsymbol{y}-\boldsymbol{x}|\,\mathrm{d}\omega dS_{\boldsymbol{y}} \tag{6}$$

where now $\partial\Omega$ denotes the illuminated portion of the target surface.

The value of the Kirchhoff approximation is that it removes the nonlinearity in the inverse problem: it replaces the product of two unknowns (V and u) by a single unknown (V) multiplied by known quantities. This approximation is, however, a single-scattering approximation, and an important contribution to radar scattering comes from multiple bounces associated with corner reflectors. Corner scattering has the property that it can be seen from many directions; in this respect such scattering centers behave like "point" scatterers. To model corners, we simply interpret V as an effective reflection coefficient for the corner.

The model (6) applies to a stationary target and a single incident wave that starts at position $\boldsymbol{x}$ at time t. We now assume that the incident field is a series of pulses, beginning at times $t=\theta_n, n=1,2,\ldots$, so that

$$u_n^{\mathrm{inc}}(t',\boldsymbol{y}) = \int S_{\mathrm{inc}}(\omega') \frac{\mathrm{e}^{-\mathrm{i}\omega'(t'-\theta_n-|\boldsymbol{x}-\boldsymbol{y}|/c)}}{8\pi^2|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}\omega'\,, \tag{7}$$

where

$$S_{\mathrm{inc}}(\omega) = \mathcal{F}\{s_{\mathrm{inc}}\}(\omega) = \frac{1}{2\pi}\int s_{\mathrm{inc}}(t')\,\mathrm{e}^{\mathrm{i}\omega t'}\,\mathrm{d}t' \tag{8}$$

is the Fourier transform of the signal used to establish the interrogating field transmitted to the target. We also assume that the target is translating with velocity $\boldsymbol{v}$ and rotating, so that at time t, we have $V(t,\boldsymbol{y}) = Q_{\mathrm{K}}(\mathcal{O}^{-1}(t)(\boldsymbol{y}-\boldsymbol{v}t))$, where $\mathcal{O}(t)$ denotes a rotation operator (an orthogonal matrix).

We denote by $u_n^{\mathrm{sc}}(t,\boldsymbol{x})$ the scattered field at the radar due to the nth transmitted pulse . This field induces a system signal whose Kirchhoff-approximated value we denote by $s_{\mathrm{sc}}(\boldsymbol{x},n,t)$:

$$s_{\mathrm{sc}}(\boldsymbol{x},n,t) = \int_{\partial\Omega}\int (2\mathrm{i}\omega) \frac{\mathrm{e}^{-\mathrm{i}\omega(t-\theta_n-2|\boldsymbol{x}-\boldsymbol{y}|/c)}}{(8\pi^2|\boldsymbol{x}-\boldsymbol{y}|)^2} Q_{\mathrm{K}}(\mathcal{O}^{-1}(\theta_n)(\boldsymbol{y}-\boldsymbol{v}\theta_n)) S_{\mathrm{inc}}(\omega')\partial_{\boldsymbol{\nu}}|\boldsymbol{y}-\boldsymbol{x}|\,\mathrm{d}\omega\mathrm{d}S_{\boldsymbol{y}}\,. \tag{9}$$

Here we have made the *start-stop* approximation, i.e., the target is moving sufficiently slowly that it can be treated as stationary during the time of illumination by the radar pulse. (Whether this is a good approximation depends on the length of the pulse, and the speed and size of the target.)

In (9), we neglect the overall target velocity (set $\boldsymbol{v}=0$), let $t-\theta_n \to t$, and make the change of variables $\boldsymbol{z} = \mathcal{O}^{-1}(\theta_n)\boldsymbol{y}$. These operations convert (9) into

$$s_{\mathrm{sc}}(\boldsymbol{x},n,t) = \int_{\partial\Omega}\int (2\mathrm{i}\omega) \frac{\mathrm{e}^{-\mathrm{i}\omega(t-2|\boldsymbol{x}-\mathcal{O}(\theta_n)\boldsymbol{z}|/c)}}{(8\pi^2|\boldsymbol{x}-\mathcal{O}(\theta_n)\boldsymbol{z}|)^2} Q_{\mathrm{K}}(\boldsymbol{z}) S_{\mathrm{inc}}(\omega)\partial_{\boldsymbol{\nu}}|\mathcal{O}(\theta_n)\boldsymbol{z}-\boldsymbol{x}|\,\mathrm{d}\omega\mathrm{d}S_{\boldsymbol{z}}\,. \tag{10}$$

We use the far-field approximation $|\boldsymbol{x}-\boldsymbol{w}| = |\boldsymbol{x}| - \hat{\boldsymbol{x}}\cdot\boldsymbol{w} + \mathrm{O}(|\boldsymbol{x}|^{-1})$ (with the hat denoting unit vector), the orthogonality of $\mathcal{O}$, and the notation $R=|\boldsymbol{x}|$, $\hat{\boldsymbol{R}}_n = -\mathcal{O}^{\mathrm{T}}(\theta_n)\hat{\boldsymbol{x}}$ to rewrite (10) as

$$s_{\mathrm{sc}}(\boldsymbol{x},n,t) \approx \frac{2}{(8\pi^2 R)^2}\int_{\partial\Omega}\int Q_{\mathrm{K}}(\boldsymbol{z})(\mathrm{i}\omega)S_{\mathrm{inc}}(\omega)\,\mathrm{e}^{-\mathrm{i}\omega[t-(R+\hat{\boldsymbol{R}}_n\cdot\boldsymbol{z})/c]}\boldsymbol{\nu}\cdot\hat{\boldsymbol{R}}_n\,\mathrm{d}\omega\mathrm{d}S_{\boldsymbol{z}}\,. \tag{11}$$

ISAR systems typically use a correlation receiver.[1, 25, 35, 37] This means that our model for the radar data must include the correlation process. In particular, we correlate the signal (11) with a signal of the form $s_{\mathrm{inc}}(t'-t) = \int S(\omega')\exp(-\mathrm{i}\omega'(t'-t))\,\mathrm{d}\omega'$ to obtain the output of the correlation receiver:

$$\begin{aligned}\eta_{\mathrm{K}}(\theta_n,t) &= \int s_{\mathrm{sc}}(\boldsymbol{x},n,t')\overline{s}_{\mathrm{inc}}(t'-t)\,\mathrm{d}t' \qquad (12)\\ &= \frac{2}{(8\pi^2R)^2}\int Q_{\mathrm{K}}(\boldsymbol{z})(\mathrm{i}\omega)S_{\mathrm{inc}}(\omega)\overline{S}_{\mathrm{inc}}(\omega')\,\mathrm{e}^{-\mathrm{i}\omega[t'-2(R+\hat{\boldsymbol{R}}_n\cdot\boldsymbol{z})/c]}\,\boldsymbol{\nu}\cdot\hat{\boldsymbol{R}}_n\mathrm{e}^{\mathrm{i}\omega'(t'-t)}\,\mathrm{d}\omega\mathrm{d}\omega'\mathrm{d}t'\mathrm{d}S_{\boldsymbol{z}}\,,\end{aligned}$$

where the bar denotes complex conjugation and where to save space we use only one integral sign to denote the 5-dimensional integral. In (12) we carry out the integrations over ω' and t' to obtain

$$\eta_{\mathrm{K}}(\theta_n, t) = \frac{4\pi}{(8\pi^2 R)^2} \int_{\partial\Omega} \int Q_{\mathrm{K}}(\boldsymbol{z})\,(\mathrm{i}\omega)|S_{\mathrm{inc}}(\omega)|^2\, \mathrm{e}^{-\mathrm{i}\omega[t-2(R+\hat{\boldsymbol{R}}(\theta_n)\cdot\boldsymbol{z})/c]}\boldsymbol{\nu}\cdot\hat{\boldsymbol{R}}_n\,\mathrm{d}\omega\mathrm{d}S_{\boldsymbol{z}}\,. \tag{13}$$

We introduce the notation $r_n(\boldsymbol{z}) = \hat{\boldsymbol{R}}_n \cdot \boldsymbol{z}$ and insert $\delta_{\partial\Omega}$ in order to convert the $\boldsymbol{z}$ integral to a three-dimensional one:

$$\eta_{\mathrm{K}}(\theta_n, t) = \frac{4\pi}{(8\pi^2 R)^2} \int_{\partial\Omega} \int (\mathrm{i}\omega)|S_{\mathrm{inc}}(\omega)|^2\, \mathrm{e}^{-\mathrm{i}\omega[t-2r_n(\boldsymbol{z})/c]} Q_{\mathrm{K}}(\boldsymbol{z})\delta_{\partial\Omega}(\boldsymbol{z})\boldsymbol{\nu}\cdot\hat{\boldsymbol{R}}_n\,\mathrm{d}\omega\mathrm{d}\boldsymbol{z}\,. \tag{14}$$

Equation (14) is our model for the radar data in the single-scattering case. We note that the kernel of (14) involves an oscillatory integral, which suggests that it can be analyzed with the techniques of microlocal analysis.

2.2. Dispersive scattering by re-entrant structures

For re-entrant structures with openings that can be associated with the location $\boldsymbol{z}$, the most complicated aspect of multiple scattering (i.e., the accounting) can be eliminated. This simplification is made possible by a model[1] for scattering from such structures that includes wave propagation within the duct or cavity. Here, the analysis is done by treating the re-entrant structure as a waveguide: for Q in (10), we use

$$Q_{\mathrm{d}}(\omega, \theta_n, \boldsymbol{z}) = q_{\mathrm{M}}(\theta_n, \boldsymbol{z}) \sum_m \rho_m\, \mathrm{e}^{\mathrm{i}2L(\boldsymbol{z})c^{-1}\sqrt{\omega^2 - w_m^2}}\,. \tag{15}$$

In this equation, m indexes the eigen-solutions (modes) of the waveguide problem, w_m denotes the mode cutoff frequency, ρ_m is the strength of the mode, q_{M} is proportional to the amount of energy that gets coupled into the re-entrant feature, and $L(\boldsymbol{z})$ is the distance from the mouth of the duct/cavity to a scattering center within. We denote by M the mouth of the structure and assume that $L(\boldsymbol{z})$ is constant over M and zero off M. We note that this scattering model includes dependencies on ω and θ.

We take

$$q_{\mathrm{M}}(\theta_n, \boldsymbol{z}) = A(\hat{\boldsymbol{N}}\cdot\hat{\boldsymbol{R}}_n)\varphi_{\mathrm{M}}(\boldsymbol{z})\,, \tag{16}$$

where $\hat{\boldsymbol{N}}$ is the (effective) normal to the waveguide opening, A is a coupling pattern that gives the angular dependence of the coupling strength, and φ_{M} is a function that is supported in a neighborhood of M and is further characterized in section 3.3. Equation (15) models only the contribution to the scattered field from scatterers within the waveguide; scattering from the edges of the waveguide mouth is handled separately (as in section 2.1).

In the time domain, (15) corresponds to

$$q_{\mathrm{d}}(t', \theta_n, \boldsymbol{z}) = q_{\mathrm{M}}(\theta_n, \boldsymbol{z}) \sum_m \rho_m \int \mathrm{e}^{\mathrm{i}2L(\boldsymbol{z})c^{-1}\sqrt{\omega^2 - w_m^2}}\, \mathrm{e}^{-\mathrm{i}\omega t'}\,\mathrm{d}\omega = q_{\mathrm{M}}(\theta_n, \boldsymbol{z}) \sum_m \rho_m I_m(t')\,. \tag{17}$$

Since $L(\boldsymbol{z})$ is assumed to be constant on M,

$$I_m(t') = \int \mathrm{e}^{\mathrm{i}2Lc^{-1}\sqrt{\omega^2 - w_m^2}}\, \mathrm{e}^{-\mathrm{i}\omega t'}\,\mathrm{d}\omega \tag{18}$$

is independent of $\boldsymbol{z}$. This integral can be expressed in terms of the Heaviside function H and the Bessel function J_0 as

$$I_m(t') = \int H(t'' - 2L/c) J_0\left(w_m\sqrt{t''^2 - (2L/c)^2}\right) \int \mathrm{i}\sqrt{\omega^2 - w_m^2}\, \mathrm{e}^{-\mathrm{i}\omega(t'-t'')}\,\mathrm{d}\omega\mathrm{d}t''\,. \tag{19}$$

Consequently, $I_m(t)$ is the convolution of $H(t-2L/c)J_0\left(w_m\sqrt{t^2-(2L/c)^2}\right)$ with

$$f(t) = \mathrm{i}\mathcal{F}^{-1}\left\{\sqrt{\omega^2 - w_m^2}\right\}(t)\,. \tag{20}$$

Since the downrange dimension of a typical radar image is actually travel time, we can see from (19) that the image of scattering centers located within ducts/cavities that obey this model will not be localized to a point. Instead, the associated image will be stretched and extended in the downrange dimension. The "stretching" property follows from the scaling behavior of w_m in the argument of J_0. This general behavior is a consequence of *dispersion*—waves reflected from such scattering centers exhibit a frequency-dependent time delay (as in equation (17)). In practice, such nonlocal image elements can be difficult to map to the local target structures that created them, and are usually considered to be image artifacts.

To obtain a model for radar data, we substitute for Q in (10) the expression

$$Q_{\mathrm{d}}(\omega', \theta_n, \boldsymbol{z}) = \int q_{\mathrm{d}}(t', \theta_n, \boldsymbol{z})\, \mathrm{e}^{\mathrm{i}\omega' t'}\, \mathrm{d}t'. \tag{21}$$

We carry out the computations (11) through (14) as before and, finally, in (14) we substitute expression (21). We thus obtain for the output of the correlation receiver

$$\eta_{\mathrm{d}}(\theta_n, t) = \iint K_{\mathrm{d}}(\theta_n, t, \boldsymbol{z}, t') q_{\mathrm{d}}(t', \theta_n, \boldsymbol{z})\, \mathrm{d}t' \mathrm{d}\boldsymbol{z}\,, \tag{22}$$

where

$$K_{\mathrm{d}}(\theta_n, t, \boldsymbol{z}, t') = \frac{1}{(4\pi R)^2} \int (\mathrm{i}\omega) |S_{\mathrm{inc}}(\omega)|^2 \mathrm{e}^{-\mathrm{i}\omega[t_n - t' - 2r_n/c]}\, \mathrm{d}\omega\,. \tag{23}$$

This equation is our model for radar data from structurally dispersive target elements. Again, (22) involves oscillatory integrals which can be studied with the techniques of microlocal analysis. Equation (22) is not quite in the form of a Fourier Integral Operator (because it involves multiplication in the θ_n variable as well as integration), but it could be converted into one by introducing another variable and modifying the phase of (23) appropriately.

3. WAVEFRONT SETS FOR RADAR DATA

The target features that interest us are the boundary of the scattering object and localized scattering centers such as corners. These target features we characterize by the singular structure of Q, which we describe in terms of its wavefront set.

3.1. Wavefront sets

Mathematically the singular structure of a function can be characterized by its *wavefront set*, which involves both the location $\boldsymbol{x}$ and corresponding directions $\boldsymbol{\xi}$ of singularities.[8–11]

Definition. The point $(\boldsymbol{x}_0, \boldsymbol{\xi}_0)$ is *not* in the wavefront set $\mathrm{WF}(f)$ of the function f if there is a smooth cutoff function ψ with $\psi(\boldsymbol{x}_0) \neq 0$, for which the Fourier transform $\mathcal{F}(f\psi)(\lambda\boldsymbol{\xi})$ decays rapidly (i.e., faster than any polynomial in $1/\lambda$) as $\lambda \to \infty$ for $\boldsymbol{\xi}$ uniformly in a neighborhood of $\boldsymbol{\xi}_0$.

This definition says that to determine whether $(\boldsymbol{x}_0, \boldsymbol{\xi}_0)$ is in the wavefront set of f, one should 1) localize around $\boldsymbol{x}_0$ by multiplying by a smooth function ψ supported in the neighborhood of $\boldsymbol{x}_0$, 2) Fourier transform $f\psi$, and 3) examine the decay of the Fourier transform in the direction $\boldsymbol{\xi}_0$. Rapid decay of the Fourier transform in direction $\boldsymbol{\xi}_0$ corresponds to smoothness of the function f in the direction $\boldsymbol{\xi}_0$.

Example: a point scatterer. If $Q(\boldsymbol{x}) = \delta(\boldsymbol{x})$, then $\mathrm{WF}(Q) = \{(\boldsymbol{0}, \boldsymbol{\xi}) : \boldsymbol{\xi} \neq \boldsymbol{0}\}$.

Example: a specular flash. Suppose $Q(\boldsymbol{x}) = H(\boldsymbol{x}\cdot\boldsymbol{\nu})$, where H denotes the Heaviside function. Then $\mathrm{WF}(Q) = \{(\boldsymbol{x}, \alpha\boldsymbol{\nu}) : \boldsymbol{x}\cdot\boldsymbol{\nu} = 0, \alpha \neq 0\}$.

Wavefront sets can be specified closed sets[26]:

THEOREM 3.1. *If $S = \{(\boldsymbol{x}, \boldsymbol{\xi})\}$ is a closed subset of $\mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$, then there is a function on $\mathbb{R}^n$ whose wavefront set is S.*

Our strategy is to work out explicitly how the wavefront set of Q corresponds (via (13)) to the wavefront set of η. We denote the wavefront set of Q by

$$\mathrm{WF}(Q) = \{(\boldsymbol{z}, \boldsymbol{\zeta}) : \boldsymbol{\zeta} \neq \boldsymbol{0}\}\,. \tag{24}$$

For calculating the wavefront set of η, the basic tool is the method of stationary phase; the results we need are the following theorems.[8, 9, 11]

THEOREM 3.2. *(Wavefront set of a product) Suppose*

$$\mathrm{WF}(f) + \mathrm{WF}(g) \equiv \{(\boldsymbol{x}, \boldsymbol{\xi}_f + \boldsymbol{\xi}_g) : (\boldsymbol{x}, \boldsymbol{\xi}_f) \in \mathrm{WF}(f),\ (\boldsymbol{x}, \boldsymbol{\xi}_g) \in \mathrm{WF}(g)\} \tag{25}$$

contains no points of the form $(\boldsymbol{x}, \boldsymbol{0})$. Then the wavefront set of the product fg satisfies

$$\mathrm{WF}(fg) \subseteq (\mathrm{WF}(f) + \mathrm{WF}(g)) \cup \mathrm{WF}(f) \cup \mathrm{WF}(g)\,. \tag{26}$$

Wavefront set of a function as embedded in a larger space. If we have a function f of the variable x, and we want to consider it to be a function of the variables x and y, then we can write f as the *pull-back* P^*f for the mapping $P : (x, y) \mapsto x$. Then[8] the wavefront set of P^*f is

$$\mathrm{WF}(P^*f) = \left\{((x, y); DP^{\mathrm{T}}\xi) : (x, \xi) \in \mathrm{WF}(f)\right\} = \{(x, y); (\xi, 0)) : (x, \xi) \in \mathrm{WF}(f)\}\,. \tag{27}$$

THEOREM 3.3. *(Wavefront set of an oscillatory integral) Suppose K is defined by*

$$K(\boldsymbol{x}) = \int \mathrm{e}^{\mathrm{i}\phi(\boldsymbol{\omega}, \boldsymbol{x})} a(\boldsymbol{x}, \boldsymbol{\omega})\, \mathrm{d}\boldsymbol{\omega}, \tag{28}$$

where ϕ and a satisfy the following conditions:

1. *ϕ is real-valued.*
2. *$\phi(\lambda\boldsymbol{\omega}, \boldsymbol{x}) = \lambda\phi(\boldsymbol{\omega}, \boldsymbol{x})$.*
3. *At every point $(\boldsymbol{\omega}, \boldsymbol{x})$, at least one of the derivatives $\partial_{x_j}\phi$ or $\partial_{\omega_j}\phi$ is nonzero.*
4. *There is some μ and M for which, on any compact set X, the estimate*

$$|\partial_{x_1}^{n_1}\partial_{x_2}^{n_2}\cdots\partial_{x_N}^{n_N}\partial_{\omega_1}^{m_1}\cdots\partial_{\omega_J}^{m_J} a(\boldsymbol{x}, \boldsymbol{\omega})| \leq C_{X,\boldsymbol{n},\boldsymbol{m}}(1 + |\boldsymbol{\omega}|)^{\mu - M|\boldsymbol{m}| + (1-M)|\boldsymbol{n}|} \tag{29}$$

   holds, with $|\boldsymbol{n}| = \sum n_j$.

Then the wavefront set of K satisfies

$$\mathrm{WF}(K) \subseteq \{(\boldsymbol{x}, \nabla_{\boldsymbol{x}}\phi) : \nabla_{\boldsymbol{\omega}}\phi(x) = \boldsymbol{0}\}. \tag{30}$$

THEOREM 3.4. *A Fourier integral operator $f(\boldsymbol{x}) = \int K(\boldsymbol{x}, \boldsymbol{y})g(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y}$ maps the wavefront set of g to the wavefront set of f according to the (twisted) canonical relation*

$$\Lambda' = \{[(\boldsymbol{x}; \boldsymbol{\xi}), (\boldsymbol{y}, \boldsymbol{\eta})] : (\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{\xi}, -\boldsymbol{\eta}) \in \mathrm{WF}(K)\} \tag{31}$$

In other words, $\mathrm{WF}(f)$ is the set of $(\boldsymbol{x}; \boldsymbol{\xi})$ for which $[(\boldsymbol{x}; \boldsymbol{\xi}), (\boldsymbol{y}, \boldsymbol{\eta})]$ is in Λ' for some $(\boldsymbol{y}; \boldsymbol{\eta}) \in \mathrm{WF}(g)$.

3.2. Wavefront set for the Kirchhoff model

We write (14) as

$$\eta_{\mathrm{K}}(\theta_n, t) = \iint K_{\mathrm{K}}(\theta_n, \theta', t, \boldsymbol{z})[\delta_\Omega(\boldsymbol{z}) Q_{\mathrm{K}}(\boldsymbol{z}) \boldsymbol{\nu}_{\boldsymbol{z}} \cdot \hat{\boldsymbol{R}}(\theta')] \, \mathrm{d}\boldsymbol{z} \mathrm{d}\theta' , \tag{32}$$

where $\hat{\boldsymbol{R}}(\theta') = -\mathcal{O}^{\mathcal{T}}(\theta') \cdot \boldsymbol{x}$ and

$$K_{\mathrm{K}}(\theta_n, \theta', t, \boldsymbol{z}) = \frac{1}{(4\pi R)^2} \iint (\mathrm{i}\omega) |S_{\mathrm{inc}}(\omega)|^2 \mathrm{e}^{-\mathrm{i}\omega[t - 2r_n(\boldsymbol{z})/c]} e^{i\omega'(\theta_n - \theta')} \, \mathrm{d}\omega \mathrm{d}\omega' . \tag{33}$$

Under the assumptions on S_{inc} of Theorem 3.3, equation (32) expresses η_{w} in terms of a Fourier Integral Operator applied to Q_{w}, and therefore the wavefront set of η_{w} can be calculated in terms of that of Q_{w} by Theorem 3.4.

First we calculate the canonical relation Λ' for the kernel K_{K}:

The canonical relation. We assume that $(\mathrm{i}\omega')^2 |S_{\mathrm{inc}}(\omega)|^2$ satisfies the hypothesis of theorem 3.3. The phase of K_{w} is

$$\phi = -\omega[t - 2r_n(\boldsymbol{z})/c] + \omega'(\theta_n - \theta') \tag{34}$$

and so

$$\Lambda' = \Big\{ (\theta_n, t : \sigma, \tau)(\boldsymbol{z}, \theta'; \boldsymbol{\zeta}, \sigma') : \quad t - 2r_n(\boldsymbol{z})/c = 0, \qquad \theta_n = \theta' \qquad \sigma = \partial_{\theta_n} \phi = -\tfrac{2\omega}{c} \tfrac{\partial \hat{\boldsymbol{R}}(\theta_n)}{\partial \theta_n} \cdot \boldsymbol{z} + \omega' \\ \tau = \partial_t \phi = -\omega, \qquad \boldsymbol{\zeta} = -\nabla_{\boldsymbol{z}} \phi = \tfrac{2\omega}{c} \hat{\boldsymbol{R}}(\theta_n) \qquad \sigma' = -\partial_{\theta'} \phi = \omega' \Big\} , \tag{35}$$

The wavefront set of $[\delta_\Omega(\boldsymbol{z}) Q_{\mathrm{K}}(\boldsymbol{z}) \boldsymbol{\nu}_{\boldsymbol{z}} \cdot \hat{\boldsymbol{R}}(\theta')]$. We assume that Q_{K} and $\partial\Omega$ are smooth over the (typically small) data acquisition interval. Then the only singular part of $[\delta_\Omega(\boldsymbol{z}) Q_{\mathrm{K}}(\boldsymbol{z}) \boldsymbol{\nu}_{\boldsymbol{z}} \cdot \hat{\boldsymbol{R}}(\theta')]$ is the delta function supported on $\partial\Omega$. Thus the wavefront set of $[\delta_\Omega(\boldsymbol{z}) Q_{\mathrm{K}}(\boldsymbol{z}) \boldsymbol{\nu}_{\boldsymbol{z}} \cdot \hat{\boldsymbol{R}}(\theta')]$ is simply

$$\mathrm{WF}([\delta_\Omega(\boldsymbol{z}) Q_{\mathrm{K}}(\boldsymbol{z}) \boldsymbol{\nu}_{\boldsymbol{z}} \cdot \hat{\boldsymbol{R}}(\theta')]) = \{(\boldsymbol{z}, \theta'; \boldsymbol{\zeta}, \sigma') : \ \boldsymbol{z} \in \partial\Omega, \ \theta' \text{ arbitrary}, \boldsymbol{\zeta} \propto \boldsymbol{\nu}_{\boldsymbol{z}}, \ \sigma' = 0\} \tag{36}$$

The wavefront set of η_{K}. the wavefront set of η_{w} is contained in the set

$$\Big\{ (\theta_n, t; \sigma, \tau,) : \ t - 2r_n(\boldsymbol{z})/c = 0, \boldsymbol{z} \in \partial\Omega, \ \hat{\boldsymbol{R}}(\theta_n) \propto \boldsymbol{\nu}_{\boldsymbol{z}}, \ (\sigma, \tau) \propto \Big(-(2/c)\partial_{\theta_n} \hat{\boldsymbol{R}}(\theta_n) \cdot \boldsymbol{z}, 1 \Big) \Big\} . \tag{37}$$

In particular, the wavefront set corresponding to a single point scatterer at $\boldsymbol{z}^0$ will be the curve $t_n - 2r_n(\boldsymbol{z}^0)/c = 0$ whose normal vector is $(\sigma, \tau) \propto \Big((2/c)\partial_{\theta_n} \hat{\boldsymbol{R}}(\theta_n) \cdot \boldsymbol{z}^0, 1 \Big)$.

3.3. Wavefront sets for scattering from re-entrant structures

The dispersive-scattering model of equation (15) links the downrange artifacts of equation (19) to the target image through the $\varphi_{\mathrm{M}}(\boldsymbol{z})$ factor. We choose φ_{M} by Theorem 3.1 so that it is supported in a neighborhood of M and its wavefront set is

$$\mathrm{WF}(\varphi_{\mathrm{M}}) = \Big\{ (\tilde{\boldsymbol{z}}, \tilde{\boldsymbol{\zeta}}) : \tilde{\boldsymbol{z}} \in M, A(\tilde{\boldsymbol{\zeta}} \cdot \hat{\boldsymbol{N}}) \geq 0 \Big\} . \tag{38}$$

We convert (22) into a Fourier integral operator by introducing a delta function in the form $\delta(\theta_n - \theta') = (2\pi)^{-1} \int \exp[\mathrm{i}\omega'(\theta_n - \theta')] \mathrm{d}\omega'$:

$$\eta_{\mathrm{d}}(\theta_n, t) = \iiint K_{\mathrm{d}}(\theta_n, \theta', t, \boldsymbol{z}, t') q_{\mathrm{d}}(t', \theta', \boldsymbol{z}) \, \mathrm{d}t' \mathrm{d}\theta' \, \mathrm{d}\boldsymbol{z} , \tag{39}$$

where

$$K_{\mathrm{d}}(\theta_n, \theta', t, \boldsymbol{z}, t') = \frac{1}{(4\pi R)^2} \iint (\mathrm{i}\omega) |S_{\mathrm{inc}}(\omega)|^2 \mathrm{e}^{-\mathrm{i}\omega[t - t' - 2r_n/c]} \, \mathrm{e}^{\mathrm{i}\omega'(\theta_n - \theta')} \mathrm{d}\omega \mathrm{d}\omega' . \tag{40}$$

3.3.1. The canonical relation.

We compute the canonical relation from Theorem 3.4:

$$\begin{aligned}\Lambda' \;=\; & \Big\{(\theta_n, t:\sigma,\tau)(t',\theta',\boldsymbol{z};\tau',\sigma',\boldsymbol{\zeta}) : t - t' - 2r_n(\boldsymbol{z})/c = 0, \qquad \sigma = \partial_{\theta_n}\phi = -\frac{2\omega}{c}\frac{\partial\hat{\boldsymbol{R}}(\theta_n)}{\partial\theta_n}\cdot\boldsymbol{z} + \omega' \\ & \theta_n = \theta', \qquad \tau = \partial_t\phi = -\omega, \qquad \tau' = -\partial_{t'}\phi = \omega \qquad \boldsymbol{\zeta} = -\nabla_z\phi = \frac{2\omega}{c}\hat{\boldsymbol{R}}(\theta_n) \qquad \sigma' = -\partial_{\theta'}\phi = \omega'\Big\},\end{aligned} \tag{41}$$

3.3.2. Wavefront set of q_d.

We compute this wavefront set in steps, first finding the wavefront sets for the factors and then assembling them with the help of Theorem 3.2.

Wavefront set of q_M. The wavefront set of $q_\mathrm{M}(\theta_n, \boldsymbol{z}) = A(\hat{\boldsymbol{N}}\cdot\hat{\boldsymbol{R}}_n)\varphi_\mathrm{M}(\boldsymbol{z})$ is obtained from Theorem 3.2:

$$\mathrm{WF}(q_\mathrm{M}) \subseteq [\mathrm{WF}(A) + \mathrm{WF}(\varphi_\mathrm{M})] \cup \mathrm{WF}(A) \cup \mathrm{WF}(\varphi_\mathrm{M})\,. \tag{42}$$

The coupling pattern A, however, is assumed to be smooth; its wavefront set is therefore empty. Consequently, the wavefront set of q_M is simply the pull-back of $\mathrm{WF}(\varphi_\mathrm{M})$ to $\mathbb{R}^8$:

$$\mathrm{WF}(q_\mathrm{M}) = \Big\{(\theta,\boldsymbol{z};\sigma,\boldsymbol{\zeta}) : \boldsymbol{z}\in M,\ \sigma = 0,\ A(\boldsymbol{\zeta}\cdot\hat{\boldsymbol{N}}) \geq 0 \text{ and } A(\hat{\boldsymbol{N}}\cdot\hat{\boldsymbol{R}}(\theta)) > 0\Big\}. \tag{43}$$

The singularities of q_M correspond to points at the duct mouth during the time when the incident wave couples to the duct.

Wavefront set of I_m. The wavefront set of I_m can be calculated by cutting out a small interval about $\omega = w_m$ in the definition (20) of f, and then noting that the square root $(\omega^2 - w_m)^{1/2}$ behaves like ω for large ω. We can therefore apply Theorem 3.3 to $\int a(\omega)\exp[\omega(2\mathrm{i}L/c - t')]d\omega$, with $a(\omega) = \exp[(2\mathrm{i}L/c)(\sqrt{\omega^2 - w_m^2} - \omega)]$, to conclude that the wavefront set of I_m is

$$\mathrm{WF}(I_m) = \{(t',\tau') : t' = 2L/c, \tau' = a,\ \text{an arbitrary nonzero real number}\}\,. \tag{44}$$

We note that this wavefront set is independent of m. It corresponds to travel from the duct mouth to the scattering center within the duct.

Wavefront set of q_d. The wavefront set of $q_\mathrm{d}(t',\theta_n,\boldsymbol{z}) = q_\mathrm{M}(\theta_n,\boldsymbol{z})\sum I_m(t')$ can be obtained by first pulling back q_M and I_m to the space of $(t',\theta,\boldsymbol{z})$ and then applying the product theorem 3.2. We find

$$\begin{aligned}\mathrm{WF}(q_\mathrm{d}) \;\subseteq\; & \Big\{(2L/c,\theta,\boldsymbol{z};a,\sigma,\boldsymbol{\zeta}) : (\theta,\boldsymbol{z};\sigma,\boldsymbol{\zeta}) \in \mathrm{WF}(q_\mathrm{M}), a \text{ arbitrary}\Big\} \\ & \cup\Big\{(t,\theta,\boldsymbol{z};0,\sigma,\boldsymbol{\zeta}) : (\theta,\boldsymbol{z};\sigma,\boldsymbol{\zeta}) \in \mathrm{WF}(q_\mathrm{M})\Big\} \cup \Big\{(2L/c,\theta,\boldsymbol{z};a,0,\mathbf{0}) : a \text{ arbitrary}\Big\}\end{aligned} \tag{45}$$

3.3.3. Wavefront set of η_d.

The wavefront set of η_d is obtained by applying the canonical relation (41) to the set (45). We find $\mathrm{WF}(\eta_\mathrm{d})$ is contained in the set

$$\begin{aligned}\mathrm{WF}(\eta_\mathrm{d}) \subseteq \; & \Big\{(\theta_n, t;\sigma,\tau) :\ t_n = (2r_n(\boldsymbol{z}) + 2L)/c, \quad \text{where } \boldsymbol{z}\in M, \\ & A\left(\hat{\boldsymbol{R}}(\theta_n)\cdot\hat{\boldsymbol{N}}\right) \geq 0, \qquad (\sigma,\tau) \propto \left(-\tfrac{2\omega}{c}\tfrac{\partial\hat{\boldsymbol{R}}}{\partial\theta}\cdot\mathbf{z}, -1\right)\Big\}.\end{aligned} \tag{46}$$

We see that the critical curve in the θ_n–t_n plane is associated with scattering centers lying within the cavity at distance L from the mouth. The point $\boldsymbol{z}$ in the critical set corresponds to a point at the mouth of the re-entrant structure. In addition, the critical curve is present in the data only at angles for which energy couples into the dispersive structure, and for times after which the wave has reached the scattering center within.

4. EXTRACTING THE STRUCTURE FROM THE DATA

ISAR imaging methods are usually based, one way or another, on the Radon transform and its inversion.[27] These conventional imaging methods, however, are subject to the limitations discussed in the introduction. Alternative target estimation methods are often based on parametric fitting, and one such scheme uses matched filters[28] (which are known to be optimal in a certain sense). But this approach can be computationally expensive.

Our analysis suggests another approach, namely to determine the wavefront-set structure of the data and then use the inverse of the canonical relation (if it exists) to identify the target. In this approach, we must first extract the wavefront set of the data, but currently no methods are available for doing this. In the ideal case, this process would involve identifying curves in the (infinite-bandwidth) data, and the standard approach for identifying curves is to use the generalized Radon-Hough transform.[29,30] This, by itself, is not adequate in our case because the data are band-limited. To deal with the band-limited nature of the data, we propose[31] a modification of the CLEAN algorithm[32]—but in the data domain. In particular, we apply the generalized Radon-Hough transform to find the locus of the curve; the peak of the Radon-Hough transform tells us the most likely curve. Once we know the curve, we can determine the location of the scattering center responsible for that structure by microlocal analysis. Once we know the scattering center, we know the ambiguity structure in the data set, and can subtract that away. This process tends to eliminate the sidelobes. In summary, the algorithm[31] is:

1. apply the generalized Radon-Hough transform to find the greatest-energy curve in the range-aspect data;
2. from this curve, use the microlocal theory to find the associated scattering center $\boldsymbol{z}$;
3. from the scattering center, find the associated ambiguity function $\chi_{\boldsymbol{z}}$ (the ω integral in (14));
4. subtract a (correct) multiple of this ambiguity function from the data;
5. return to step 1.

The iterations are terminated when the energy of the curve found in step 1 is less than a pre-specified threshold.

Step 4) of this algorithm is problematic because the data are highly oscillatory and small errors in the location of the scattering centers cause constructive and destructive interference in the subtraction process. Incorrect data structure subtraction can lead to further errors in following iterations. To overcome this difficulty, we use a least-squares minimization criterion to pick the multiplier μ used in subtracting the ambiguity structure from the data. In particular, we choose μ as the solution to the minimization problem

$$\min_{\mu} E(\mu) = \min_{\mu} \|\eta - \mu\chi_{\boldsymbol{z}}\| = \min_{\mu} \sum_{\theta_n, t_n} |\eta(\theta_n, t_n) - \mu\chi_{\boldsymbol{z}}(\theta_n, t_n)|^2 \,. \tag{47}$$

This minimization is one-dimensional and can be carried out explicitly by differentiating with respect to μ and setting the derivative equal to zero:

$$0 = \frac{\mathrm{d}E}{\mathrm{d}\mu} = -2 \sum \left(\eta(\theta_n, t_n) - \mu\chi_{\boldsymbol{z}}(\theta_n, t_n)\right) \chi^*_{\boldsymbol{z}}(\theta_n, t_n)\,. \tag{48}$$

This equation has the solution

$$\mu = \frac{\sum \eta \chi^*_{\boldsymbol{z}}}{\sum |\chi_{\boldsymbol{z}}|^2}\,, \tag{49}$$

and so, at each step we modify the data as

$$\eta_{\text{new}} = \eta - \mu\chi_{\boldsymbol{z}}\,. \tag{50}$$

We note that μ is complex; its phase compensates for errors in the location of the scattering center.

5. CONCLUSIONS AND FUTURE WORK

We have suggested a new approach to ISAR target reconstruction that is appropriate to low signal-to-noise situations in which range alignment[33] is problematic. This method is distinct from traditional imaging techniques[36, 38, 39] in that it first fits a parametric curve directly to the data set, and the target characteristics are extracted directly from this curve. We have focused on two classes of target features, namely re-entrant structures and structures well-modeled by a single-scattering approximation. In particular, we have shown that when the single-scattering approximation is valid, the location of the target's scattering centers can be estimated directly from the data wavefront set. We have also determined the wavefront set of data involving scattering from re-entrant structures. Finally, we have suggested an approach for extracting the wavefront set from noisy, bandlimited data.

The approach discussed here has a number of possible advantages. First, this method allows us to exploit differences, in the data domain, between the target and the noise or clutter. If, for example, the motion of the target is different from the motion of the clutter, this fact can be used to separate data due to the target from data due to clutter. Second, this approach could be exploited in a number of ways for target imaging and identification. For image formation, the wavefront-set analysis suggests that reconstruction methods related to local tomography[17, 18] may be useful. In particular, analysis of wavefront sets can determine whether backprojection will provide an image free of certain artifacts.[16, 21, 34] In addition, wavefront-set analysis suggests an approach for producing artifact-free, superresolved images: remove all components of the data set except those that correspond to well-understood target features, and form an image from those components only.

The problem of extracting wavefront sets from noisy data is closely related to image processing problems such as edge detection, and these are active areas of current research.[40] The method we propose has been tested using synthetic radar data.[31] This preliminary work shows how the wavefront set analysis enables us to estimate target parameters from very noisy, bandlimited data.

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