Microlocal Analysis of GTD-based SAR models

Margaret Cheney[a] and Brett Borden[b]

[a] Department of Mathematical Sciences, Rensselaer Polytechnic Institute, Troy, New York, USA;

[b] Physics Department, Naval Postgraduate School, Monterey, California, USA

ABSTRACT

We show how to apply the techniques of microlocal analysis to the Potter-Moses attributed scattering center model, which is based on the Geometrical Theory of Diffraction (GTD). The microlocal methods enable us to determine how scattering centers will appear in the radar data. We show also how to extend the model to some multiple-scattering events, and we apply the microlocal techniques to the extended model.

Keywords: GTD, scattering models, SAR, ATR

1. INTRODUCTION

Current practice in radar-based target recognition is to attempt target identification/classification from fully formed radar images. However, constructing a recognizable target image from radar data is a very difficult task since the reflected field data are noisy and are usually collected from a very limited set of (generally unknown) target orientations.[16,22] Moreover, the image formation process involves a number of simplifying assumptions which do not always hold, with the result that images frequently contain artifacts from incorrectly modeled target reflection behavior.

Automatic classification systems, however, should be able to skip this imaging step because a fully-formed image is probably not required for machine-based target recognition. This observation, of course, begs the question of "what components of the raw data set are relevant to target identification?" In this paper we examine a systematic method for extracting structure-relevant information directly from measured radar data without the need to first construct an image of the target.

Our approach relates the singular structure (such as edges) of the target to the singular structure of the data set. Restricting our attention to the singular structure—specifically, to a certain set in phase space called the *wavefront set*—allows us to use the tools of microlocal analysis.[9,12,24] This strategy was first applied to imaging problems by Beylkin[1]; its uses in seismic prospecting,[2,5,10] X-ray tomography,[11,14] Synthetic-Aperture Radar,[6,17,18] and Inverse Synthetic-Aperture Radar[7,8] are active areas of research. An approach similar to the one we pursue here, in which we use microlocal analysis not to do imaging but instead to study the connection between features of the target and the data, was considered for the X-ray tomography problem by Quinto.[21]

In earlier work,[7,8] we calculated the singular structure of radar data arising from target structures undergoing single scattering, from point-like scattering centers undergoing multiple scattering, and from re-entrant

Further author information: M.C.: E-mail: cheney@rpi.edu, Telephone: 1 518 276 2646
B.B.: E-mail: bhborden@nps.edu, Telephone: 1 831 656 2855

structures such as ducts. In this paper, we show how our approach can be applied to the the earlier Potter-Moses attributed scattering center model,[20] which is a model that is based on the Geometrical Theory of Diffraction (GTD). The present paper, which is an initial report on work in progress, puts the Potter-Moses model into the framework of scattering theory, and thus connects it to the powerful apparatus of microlocal analysis. This approach makes it possible to study multiple-scattering extensions of the Potter-Moses model, and compute the corresponding singular structures in the radar data.

2. THE POTTER-MOSES MODEL AS A FOURIER INTEGRAL

2.1. The Potter-Moses model

The Potter-Moses GTD-based scattering center model[20] represents the electric field scattered from a collection of target structure elements by

$$E(f,\theta) = \sum_m A_m \left(\frac{jf}{f_c}\right)^{\alpha_m} e^{\beta_m \theta} e^{-j4\pi(f/c)(x_m \cos\theta + y_m \sin\theta)} \tag{1}$$

The mth term corresponds to a scattering center at location (x_m, y_m), scattering with frequency dependence $(f/f_c)^{\alpha_m}$, where $\alpha \in \{-1, -1/2, 0, 1/2, 1\}$ and f_c denotes the center frequency of the interrogating field. The *ad hoc* parameter β_m is introduced to account for scattering center aspect dependence.

2.2. Corresponding Fourier Integral

2.2.1. The frequency-domain scattering operator

We modify the notation, writing $\omega = 2\pi f$ for the angular frequency and $\boldsymbol{z}_m = (x_m, y_m, 0)$ for the mth scattering center. For this scattering center, the corresponding scattering operator or T-matrix,[25–27] which maps the incident field E^{in} to the scattered field E^{sc}, is

$$E^{sc}_m(\omega, \boldsymbol{x}) = A_m(\omega, \widehat{\boldsymbol{x} - \boldsymbol{z}_m}) G(\omega, |\boldsymbol{x} - \boldsymbol{z}_m|) E^{in}(\omega, \boldsymbol{z}_m) \tag{2}$$

Here the hat denotes a unit vector and

$$G(\omega, r) = \frac{e^{ikr}}{4\pi r} \tag{3}$$

where $k = \omega/c$. Expression (2) corresponds to non-isotropic scattering from a point scatterer at position $\boldsymbol{z}_m$.

For an incident wave due to a source at $\boldsymbol{y}$ with complex amplitude $P(\omega)$, E^{in} is of the form

$$E^{in}(\omega, \boldsymbol{z}) = P(\omega) G(\omega, |\boldsymbol{z} - \boldsymbol{y}|) \tag{4}$$

and the corresponding scattered field E^{sc} is

$$E^{sc}_m(\omega, \boldsymbol{x}) = A_m(\omega, \widehat{\boldsymbol{x} - \boldsymbol{z}_m}) G(\omega, |\boldsymbol{x} - \boldsymbol{z}_m|) G(\omega, |\boldsymbol{z}_m - \boldsymbol{y}|) P(\omega) \tag{5}$$

which, for a monostatic system ($\boldsymbol{x} = \boldsymbol{y}$) far from the scatterer ($|\boldsymbol{x}| >> |\boldsymbol{z}_m|$), becomes

$$E^{sc}_m(\omega, \boldsymbol{x}) \approx \frac{A_m(\omega, \widehat{\boldsymbol{x} - \boldsymbol{z}_m}) P(\omega) e^{i2k|\boldsymbol{x}|} e^{i2k\hat{\boldsymbol{x}}\cdot\boldsymbol{z}_m}}{(4\pi|\boldsymbol{x}|)^2} \tag{6}$$

Writing $\hat{\boldsymbol{x}} = (\cos\theta, \sin\theta, 0)$ in (6), we have

$$E_m^{sc}(\omega, \boldsymbol{x}) \approx \frac{A_m(\omega, \widehat{\boldsymbol{x} - \boldsymbol{z}_m})P(\omega)e^{i2k|\boldsymbol{x}|}}{(4\pi|\boldsymbol{x}|)^2} e^{-i2k(x_m \cos\theta + y_m \sin\theta)} \tag{7}$$

Comparing (7) with (1), we see that the Potter-Moses model has incorporated the geometrical spreading factors of (7) into A_m, and uses a specific form of $A_m(\omega, \widehat{\boldsymbol{x} - \boldsymbol{z}_m})$. This establishes the connection between the Potter-Moses model and the model (2).

2.2.2. The Fourier Integral Operators

To establish a connection between the Potter-Moses model and our microlocal approach,[7,8] we Fourier transform (5) (with $\boldsymbol{x} = \boldsymbol{y}$) into the time domain:

$$\mathcal{E}_m^{sc}(t, \boldsymbol{x}) = \int e^{-i\omega t} E_m^{sc}(\omega, \boldsymbol{x}) d\omega = \int e^{-i\omega[t - 2|\boldsymbol{x} - \boldsymbol{z}_m|/c)} \frac{A_m(\omega, \widehat{\boldsymbol{x} - \boldsymbol{z}_m})P(\omega)}{(4\pi|\boldsymbol{x} - \boldsymbol{z}_m|)^2} d\omega = F_m[\delta_{\boldsymbol{z}_m}](t, \boldsymbol{x}) \tag{8}$$

where $\delta_{\boldsymbol{z}} = \delta(\cdot - \boldsymbol{z})$ and where F_m is the integral operator

$$F_m[f](t, \boldsymbol{x}) = \int e^{-i\omega[t - 2|\boldsymbol{x} - \boldsymbol{y}|/c)} \frac{A_m(\omega, \widehat{\boldsymbol{x} - \boldsymbol{y}})P(\omega)}{(4\pi|\boldsymbol{x} - \boldsymbol{y}|)^2} d\omega f(\boldsymbol{y}) d\boldsymbol{y}. \tag{9}$$

The operator F_m is a Fourier integral operator provided the amplitude A satisfies certain smoothness conditions.[12,24] The (singly) scattered field for scatterers at $\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots$ is $\mathcal{E}^{sc}(t, \boldsymbol{x}) = \sum_m F_m[\delta_{\boldsymbol{z}_m}](t, \boldsymbol{x})$.

Having the angular dependence A_m as part of the operator F_m could cause difficulties if we were to attempt to invert for the $\boldsymbol{z}_m$ and the A_m. It causes no difficulties, however, in determining what features of the data correspond to the scatterers.

With the far-field approximation and the notation $R = |\boldsymbol{x}|$ and $\hat{\boldsymbol{x}} = (\cos\theta, \sin\theta, 0)$, the output of the correlation receiver is proportional to

$$\eta(t, \theta) = \sum_m \int e^{-i\omega(t - 2R/c + 2(x_m \cos\theta + y_m \sin\theta)/c)} A_m(\omega, \theta)|P(\omega)|^2 d\omega \tag{10}$$

3. FEATURES IN THE DATA: WAVEFRONT SETS

To locate and identify a target, we would like to locate the sharp discontinuity in electromagnetic parameters between the target and the surrounding air. Such sharp discontinuities are examples of singularities; the theory of microlocal analysis[9,12,13,23,24] was developed precisely to study such singularities and how they are mapped by Fourier integral operators. In particular, microlocal analysis provides theorems that tell us the results of applying the method of stationary phase to oscillatory integrals such as (8).

Mathematically the singular structure of a function $f(\boldsymbol{x})$ can be characterized by its *wavefront set*, which involves both the location $\boldsymbol{x}$ and corresponding directions $\boldsymbol{\xi}$ of singularities.[9,12,23,24] The wavefront set of the function $f(\boldsymbol{x})$ is thus a set in phase space (i.e., is a set of points $(\boldsymbol{x}, \boldsymbol{\xi})$), defined as follows.

Definition. The point $(\boldsymbol{x}_0, \boldsymbol{\xi}_0)$ is *not* in the wavefront set $\mathrm{WF}(f)$ of the function f if there is a smooth cutoff function ψ with $\psi(\boldsymbol{x}_0) \neq 0$, for which the Fourier transform $\mathcal{F}(f\psi)(\lambda\boldsymbol{\xi})$ decays rapidly (i.e., faster than any polynomial in $1/\lambda$) as $\lambda \to \infty$ for $\boldsymbol{\xi}$ uniformly in a neighborhood of $\boldsymbol{\xi}_0$.

This definition says that to determine whether $(\boldsymbol{x}_0, \boldsymbol{\xi}_0)$ is in the wavefront set of f, one should 1) localize around $\boldsymbol{x}_0$ by multiplying by a smooth function ψ supported only in the neighborhood of $\boldsymbol{x}_0$, 2) Fourier transform $f\psi$, and 3) examine the decay of the Fourier transform in the direction $\boldsymbol{\xi}_0$. Rapid decay of the Fourier transform in direction $\boldsymbol{\xi}_0$ corresponds to smoothness of the function f in the direction $\boldsymbol{\xi}_0$.[14]

Example: a point scatterer. If $f(\boldsymbol{x}) = \delta(\boldsymbol{x})$, then clearly the only singularity of δ is located at the origin. So only points of the form $(\mathbf{0}, \boldsymbol{\xi})$ can be in the wavefront set $\mathrm{WF}(\delta)$. Which directions $\boldsymbol{\xi}$ are in the wavefront set? To find out, we take the Fourier transform of δ. Since the Fourier transform is a constant, which does not decay rapidly in any direction, all directions $\boldsymbol{\xi}$ are in the wavefront set. Thus we have $\mathrm{WF}(\delta) = \{(\mathbf{0}, \boldsymbol{\xi}) : \boldsymbol{\xi} \neq \mathbf{0}\}$. (Note that $\boldsymbol{\xi} = \mathbf{0}$ is never a direction, and is thus never in the wavefront set.)

Example: a delta function along a plane. Suppose $f(\boldsymbol{x}) = \delta(\boldsymbol{x} \cdot \hat{\boldsymbol{n}})$. Then clearly the only singularities are located on the set $\boldsymbol{x} \cdot \hat{\boldsymbol{n}} = 0$. To determine which directions are in the wavefront set, we write $\delta(\boldsymbol{x} \cdot \hat{\boldsymbol{n}}) \propto \int \exp(i\omega \hat{\boldsymbol{n}} \cdot \boldsymbol{x}) d\omega = \int \int \exp(i\boldsymbol{\xi} \cdot \boldsymbol{x}) \delta(\boldsymbol{\xi} - \omega \hat{\boldsymbol{n}}) d\boldsymbol{\xi}\ d\omega$ which implies that the Fourier transform of $\delta(\boldsymbol{x} \cdot \hat{\boldsymbol{n}})$ is proportional to $\int \delta(\boldsymbol{\xi} - \omega \hat{\boldsymbol{n}})\ d\omega$. In other words, the Fourier transform is zero (and thus decays rapidly) unless $\boldsymbol{\xi}$ is proportional to $\hat{\boldsymbol{n}}$. In the direction $\hat{\boldsymbol{n}}$, it does not decay. Thus the wavefront set is $\mathrm{WF}(f) = \{(\boldsymbol{x}, \omega \hat{\boldsymbol{n}}) : \boldsymbol{x} \cdot \hat{\boldsymbol{n}} = 0, \omega \neq 0\}$.

Example: a specular flash. Suppose $f(\boldsymbol{x}) = H(\boldsymbol{x} \cdot \hat{\boldsymbol{n}})$, where H denotes the Heaviside function. The analysis is the same as for $\delta(\boldsymbol{x} \cdot \hat{\boldsymbol{n}})$, except that the Fourier transform decays like $|\boldsymbol{\xi}|^{-1}$ in direction $\hat{\boldsymbol{n}}$. Thus $\mathrm{WF}(f) = \{(\boldsymbol{x}, \omega \hat{\boldsymbol{n}}) : \boldsymbol{x} \cdot \hat{\boldsymbol{n}} = 0, \omega \neq 0\}$.

For calculating the wavefront set of η, the basic tool is the method of stationary phase; the result we need is the following theorem.[9, 12, 24]

THEOREM 3.1. *(Wavefront set of an oscillatory integral) Suppose K is defined by*

$$K(\boldsymbol{x}) = \int e^{i\phi(\boldsymbol{\omega}, \boldsymbol{x})} a(\boldsymbol{x}, \boldsymbol{\omega})\ d\boldsymbol{\omega}, \tag{11}$$

where ϕ and a satisfy the following conditions:

1. *ϕ is real-valued.*
2. *$\phi(\lambda \boldsymbol{\omega}, \boldsymbol{x}) = \lambda \phi(\boldsymbol{\omega}, \boldsymbol{x})$.*
3. *At every point $(\boldsymbol{\omega}, \boldsymbol{x})$, at least one of the derivatives $\partial_{x_j} \phi$ or $\partial_{\omega_j} \phi$ is nonzero.*
4. *There is some μ and M for which, on any compact set X, the estimate*

$$|\partial_{x_1}^{n_1} \partial_{x_2}^{n_2} \cdots \partial_{x_N}^{n_N} \partial_{\omega_1}^{m_1} \cdots \partial_{\omega_J}^{m_J} a(\boldsymbol{x}, \boldsymbol{\omega})| \leq C_{X, \boldsymbol{n}, \boldsymbol{m}} (1 + |\boldsymbol{\omega}|)^{\mu - M|\boldsymbol{m}| + (1-M)|\boldsymbol{n}|} \tag{12}$$

holds, with $|\boldsymbol{n}| = \sum n_j$.

Then the wavefront set of K satisfies

$$\mathrm{WF}(K) \subseteq \{(\boldsymbol{x}, \nabla_{\boldsymbol{x}} \phi) : \nabla_{\boldsymbol{\omega}} \phi(\boldsymbol{x}) = \mathbf{0}\}. \tag{13}$$

4. WAVEFRONT SET FOR THE POTTER-MOSES MODEL

The phase of (10) is

$$\phi_m(\omega, t, \theta) = -\omega[t - 2R/c + 2(x_m \cos\theta + y_m \sin\theta)/c]; \tag{14}$$

consequently for each m the critical set (where $\nabla_\omega \phi_m = 0$) is the set of (t, θ) such that

$$t = (2R - 2(x_m \cos\theta + y_m \sin\theta))/c; \tag{15}$$

this gives us the locations of the singularities in the data. For such a point, the associated directions are perpendicular to the critical set, namely

$$\nabla_{(t,\theta)} \phi(\omega, t, \theta) \propto (1, 2[-x_m \sin\theta + y_m \cos\theta]/c) \tag{16}$$

Assuming that the distance R from the antenna to the axis of rotation is known, we see that knowledge of a point on the singular curve means that the quantity $x_m \cos\theta + y_m \sin\theta$ is known; if the associated direction is known, that means that the quantity $-x_m \sin\theta + y_m \cos\theta$ is known. These two quantities together determine x_m and y_m. In other words, knowledge of η's wavefront set (i.e., both location and associated direction of the singularities) enables us to determine both range and cross-range position of the scattering center.

The angular dependence of the model (10) implies that the amplitude of the singularity along the curve (15) varies with θ. Where the amplitude is zero, there is no singularity, and such points are not part of the wavefront set.

The frequency dependence of the model (10) corresponds, in the ideal infinite-bandwidth case, to the strength[19] of the singularity, in the same way that

$$\delta'(t) \propto \int (i\omega) e^{i\omega t} d\omega \qquad \text{is a stronger singularity than} \qquad \delta(t) \propto \int e^{i\omega t} d\omega. \tag{17}$$

5. MULTIPLE SCATTERING WITH THE POTTER-MOSES MODEL

For a monochromatic incident wave $E^{\text{inc}}(\boldsymbol{x})$, the frequency-domain field E^{sc} scattered from N point-like scattering centers can be obtained from the Foldy-Lax[25] or T-matrix[26, 27] equations together with the model (2) for the scattering of the wave incident on each scattering center. We write E_j for the wave incident on the jth scattering center and E_j^{sc} for the wave scattered by the jth scattering center; then the multiple-scattering equations are

$$E_j^{\text{sc}}(\boldsymbol{x}) = A_j(\widehat{\boldsymbol{x} - \boldsymbol{z}_j}) G(|\boldsymbol{x} - \boldsymbol{z}_j|) E_j(\boldsymbol{z}_j) \tag{18}$$

$$E_j(\boldsymbol{x}) = E^{\text{inc}}(\boldsymbol{x}) + \sum_{i \neq j} E^{\text{sc}}(\boldsymbol{x}), \quad j = 1, 2, \ldots, N, \tag{19}$$

$$E^{\text{sc}}(\boldsymbol{x}) = \sum_j E_j^{\text{sc}}(\boldsymbol{x}) \tag{20}$$

where the ω dependence has been temporarily suppressed. Equation (18) gives the scattering operator for each individual scattering center; because bistatic scattering is now involved, the angular dependence of A_j should be thought of as modeling some shadowing effects. Equation (20) says that the scattered field is the sum of the fields scattered from each scatterer; equations (19) say that the jth local incident field is the overall incident field plus the field scattered from all the other scatterers. If the scattering strengths $A_1, A_2, \ldots, A_N$ are known, the equations (18) and (19) can be solved for the $E_j(\boldsymbol{z}_j)$; then the total field can be found from (20).

We illustrate the ideas in the case of two scatterering centers. In this case, equations (18) and (19) give rise to

$$E_1(\boldsymbol{x}) = E^{\text{inc}}(\boldsymbol{x}) + A_2(\widehat{\boldsymbol{x}-\boldsymbol{z}_2})G(|\boldsymbol{x}-\boldsymbol{z}_2|)E_2(\boldsymbol{z}_2) \tag{21}$$

$$E_2(\boldsymbol{x}) = E^{\text{inc}}(\boldsymbol{x}) + A_1(\widehat{\boldsymbol{x}-\boldsymbol{z}_1})G(|\boldsymbol{x}-\boldsymbol{z}_1|)E_1(\boldsymbol{z}_1) \tag{22}$$

Evaluating (21) at $\boldsymbol{z}_1$ and (22) at $\boldsymbol{z}_2$ gives rise to the system of equations

$$\begin{pmatrix} 1 & -A_2(\widehat{\boldsymbol{z}_1-\boldsymbol{z}_2})G(L) \\ -A_1(\widehat{\boldsymbol{z}_2-\boldsymbol{z}_1})G(L) & 1 \end{pmatrix} \begin{pmatrix} E_1(\boldsymbol{z}_1) \\ E_2(\boldsymbol{z}_2) \end{pmatrix} = \begin{pmatrix} E^{\text{inc}}(\boldsymbol{z}_1) \\ E^{\text{inc}}(\boldsymbol{z}_2) \end{pmatrix}, \tag{23}$$

where $L = |\boldsymbol{z}_2 - \boldsymbol{z}_1|$. These equations have the solutions

$$E_j(\boldsymbol{z}_j) = \frac{E^{\text{inc}}(\boldsymbol{z}_j) + A_{j'}(\widehat{\boldsymbol{z}_j-\boldsymbol{z}_{j'}})G(L)E^{\text{inc}}(\boldsymbol{z}_{j'})}{1 - A_1(\widehat{\boldsymbol{z}_2-\boldsymbol{z}_1})A_2(\widehat{\boldsymbol{z}_1-\boldsymbol{z}_2})G^2(L)}, \qquad j = 1, 2, \tag{24}$$

where $j' = 2$ if $j = 1$ and $j' = 1$ if $j = 2$. Using (24) in (20) yields

$$E^{\text{sc}}(\boldsymbol{x}) = \sum_{j=1}^{2} A_j(\widehat{\boldsymbol{x}-\boldsymbol{z}_j})G(|\boldsymbol{x}-\boldsymbol{z}_j|)\frac{E^{\text{inc}}(\boldsymbol{z}_j) + A_{j'}(\widehat{\boldsymbol{z}_j-\boldsymbol{z}_{j'}})G(L)E^{\text{inc}}(\boldsymbol{z}_{j'})}{1 - A_1(\widehat{\boldsymbol{z}_2-\boldsymbol{z}_1})A_2(\widehat{\boldsymbol{z}_1-\boldsymbol{z}_2})G^2(L)}. \tag{25}$$

A geometrical series expansion of (25), carried to triple scattering events, is

$$\begin{aligned} E^{\text{sc}}(\boldsymbol{x}) \approx \sum_{j=1}^{2} \Big[& A_j(\widehat{\boldsymbol{x}-\boldsymbol{z}_j})G(|\boldsymbol{x}-\boldsymbol{z}_j|)E^{\text{inc}}(\boldsymbol{z}_j) + A_j(\widehat{\boldsymbol{x}-\boldsymbol{z}_j})G(|\boldsymbol{x}-\boldsymbol{z}_j|)A_{j'}(\widehat{\boldsymbol{z}_j-\boldsymbol{z}_{j'}})G(L)E^{\text{inc}}(\boldsymbol{z}_{j'}) \\ & + A_j(\widehat{\boldsymbol{x}-\boldsymbol{z}_j})G(|\boldsymbol{x}-\boldsymbol{z}_j|)A_1(\widehat{\boldsymbol{z}_2-\boldsymbol{z}_1})A_2(\widehat{\boldsymbol{z}_1-\boldsymbol{z}_2})G^2(L)E^{\text{inc}}(\boldsymbol{z}_j) \Big] \end{aligned} \tag{26}$$

The first term on the right side of (26) can be interpreted as the incident field scattering from $\boldsymbol{z}_j$ and returning to the antenna at position $\boldsymbol{x}$. The second term can be interpreted as the incident field scattering first from $\boldsymbol{z}_{j'}$, propagating a distance L to the scatterer at $\boldsymbol{z}_j$, and then returning to the antenna at $\boldsymbol{x}$. Similarly the third term can be interpreted as scattering first from $\boldsymbol{z}_j$, then from $\boldsymbol{z}_{j'}$, then from $\boldsymbol{z}_j$ again, and then returning to $\boldsymbol{x}$.

In (26) we assume an incident field of the form (4) with $\boldsymbol{y} = \boldsymbol{x}$, and Fourier transform into the time domain:

$$\mathcal{E}^{\text{sc}}(t, \boldsymbol{x}) = \sum_{j=1}^{2} \left(\mathcal{E}_{j,1}(t, \boldsymbol{x}) + \mathcal{E}_{j,2}(t, \boldsymbol{x}) + \mathcal{E}_{j,3}(t, \boldsymbol{x})\right) \tag{27}$$

where

$$\mathcal{E}_{j,1}(t, \boldsymbol{x}) = \int e^{-i\omega(t-2|\boldsymbol{x}-\boldsymbol{z}_j|/c)} \frac{A_j(\omega, \widehat{\boldsymbol{x}-\boldsymbol{z}_j})P(\omega)}{(4\pi|\boldsymbol{x}-\boldsymbol{z}_j|)^2} d\omega \tag{28}$$

$$\mathcal{E}_{j,2}(t, \boldsymbol{x}) = \int e^{-i\omega(t-[|\boldsymbol{x}-\boldsymbol{z}_j|+L+|\boldsymbol{z}_{j'}-\boldsymbol{x}|]/c)} \frac{A_j(\omega, \widehat{\boldsymbol{x}-\boldsymbol{z}_j})A_{j'}(\omega, \widehat{\boldsymbol{z}_j-\boldsymbol{z}_{j'}})P(\omega)}{(4\pi)^2|\boldsymbol{z}_j - \boldsymbol{z}'^j||\boldsymbol{x}-\boldsymbol{z}_j||\boldsymbol{z}_{j'}-\boldsymbol{x}|} d\omega \tag{29}$$

$$\mathcal{E}_{j,3}(t, \boldsymbol{x}) = \int e^{-i\omega(t-[2|\boldsymbol{x}-\boldsymbol{z}_j|+L]/c)} \frac{A_j(\omega, \widehat{\boldsymbol{x}-\boldsymbol{z}_j})A_1(\omega, \widehat{\boldsymbol{z}_2-\boldsymbol{z}_1})A_2(\omega, \widehat{\boldsymbol{z}_1-\boldsymbol{z}_2})P(\omega)}{(4\pi|\boldsymbol{x}-\boldsymbol{z}_j||\boldsymbol{z}_{j'}-\boldsymbol{z}_j|)^2} d\omega \tag{30}$$

The term (28) corresponds to single scattering, the term (29) to double scattering, and the term (30) to triple scattering.

From (27) we compute the output of the correlation receiver; it is proportional to

$$\eta(t,\boldsymbol{x}) = \sum_{j=1}^{2} [\eta_{j,1}(t,\boldsymbol{x}) + \eta_{j,2}(t,\boldsymbol{x}) + \eta_{j,3}(t,\boldsymbol{x})] \tag{31}$$

where

$$\eta_{j,1}(t,\boldsymbol{x}) = \int e^{-i\omega(t-2|\boldsymbol{x}-\boldsymbol{z}_j|/c)} A_j(\omega, \widehat{\boldsymbol{x}-\boldsymbol{z}_j})|P(\omega)|^2 d\omega \tag{32}$$

$$\eta_{j,2}(t,\boldsymbol{x}) = \int e^{-i\omega(t-[|\boldsymbol{x}-\boldsymbol{z}_j|+L+|\boldsymbol{z}_{j'}-\boldsymbol{x}|]/c)} A_j(\omega, \widehat{\boldsymbol{x}-\boldsymbol{z}_j}) A_{j'}(\omega, \widehat{\boldsymbol{z}_j-\boldsymbol{z}_{j'}})|P(\omega)|^2 d\omega \tag{33}$$

$$\eta_{j,3}(t,\boldsymbol{x}) = \int e^{-i\omega(t-[2|\boldsymbol{x}-\boldsymbol{z}_j|+L]/c)} A_j(\omega, \widehat{\boldsymbol{x}-\boldsymbol{z}_j}) A_1(\omega, \widehat{\boldsymbol{z}_2-\boldsymbol{z}_1}) A_2(\omega, \widehat{\boldsymbol{z}_1-\boldsymbol{z}_2})|P(\omega)|^2 d\omega. \tag{34}$$

With the far-field approximation, $R = |\boldsymbol{x}|$, and $\hat{\boldsymbol{x}} = (\cos\theta, \sin\theta, 0)$, the corresponding equations are

$$\tilde{\eta}(t,\theta) = \sum_{j=1}^{2} [\tilde{\eta}_{j,1}(t,\theta) + \tilde{\eta}_{j,2}(t,\theta) + \tilde{\eta}_{j,3}(t,\theta)] \tag{35}$$

where the tildes mean that the far-field approximation has been taken, and where

$$\tilde{\eta}_{j,1}(t,\theta) = \int e^{-i\omega(t-2R/c+2(x_j\cos\theta+y_j\sin\theta)/c)} A_j(\omega, \widehat{\boldsymbol{x}-\boldsymbol{z}_j})|P(\omega)|^2 d\omega \tag{36}$$

$$\tilde{\eta}_{j,2}(t,\theta) = \int e^{-i\omega(t-2(R+L)/c+[(x_1+x_2)\cos\theta+(y_1+y_2)\sin\theta]/c)} A_j(\omega, \widehat{\boldsymbol{x}-\boldsymbol{z}_j}) A_{j'}(\omega, \widehat{\boldsymbol{z}_j-\boldsymbol{z}^{j'}})|P(\omega)|^2 d\omega \tag{37}$$

$$\tilde{\eta}_{j,3}(t,\boldsymbol{x}) = \int e^{-i\omega(t-2(R+L)/c+2(x_j\cos\theta+y_j\sin\theta)/c)} A_j(\omega, \widehat{\boldsymbol{x}-\boldsymbol{z}_j}) A_1(\omega, \widehat{\boldsymbol{z}_2-\boldsymbol{z}_1}) A_2(\omega, \widehat{\boldsymbol{z}_1-\boldsymbol{z}_2})|P(\omega)|^2 d\omega. \tag{38}$$

6. WAVEFRONT SETS FOR MULTIPLE SCATTERING

The wavefront set of η can be found from Theorem 3.1; the singular sets are obtained as critical points of the phase and the associated directions are the normals to these curves. In particular, in the near-field case we have singularities along the curves

1. (singe scattering) $t = 2|\boldsymbol{x}-\boldsymbol{z}_1|/c)$ and $t = 2|\boldsymbol{x}-\boldsymbol{z}_2|/c$
2. (double scattering) $t = [|\boldsymbol{x}-\boldsymbol{z}_1| + L + |\boldsymbol{z}_2-\boldsymbol{x}|]/c$
3. (triple scattering) $t = [2|\boldsymbol{x}-\boldsymbol{z}_1| + L]/c$ and $t = [2|\boldsymbol{x}-\boldsymbol{z}_2| + L]/c$

The entire curves may not be visible; their amplitude and strength depends on the A_j.

With the far-field approximation, the singularities are along the curves

1. (singe scattering) $t = 2R/c - 2(x_1\cos\theta + y_1\sin\theta)/c)$ and $t = 2R/c - 2(x_2\cos\theta + y_2\sin\theta)/c)$

2. (double scattering) $t = 2(R+L)/c - [(x_1+x_2)\cos\theta - (y_1+y_2)\sin\theta]/c)$
3. (triple scattering) $t = 2(R+L)/c - 2(x_1\cos\theta + y_1\sin\theta)/c$ and $t = 2(R+L)/c - 2(x_2\cos\theta + y_2\sin\theta)/c$

and the associated (normal) directions are

1. (singe scattering) $(1, 2[-x_1\sin\theta + y_1\cos\theta]/c)$ and $(1, 2[-x_2\sin\theta + y_2\cos\theta]/c)$
2. (double scattering) $(1, [-(x_1+x_2)\sin\theta + (y_1+y_2)\cos\theta]/c)$
3. (triple scattering) $(1, 2(-x_1\sin\theta + y_1\cos\theta)/c)$ and $(1, 2(-x_2\sin\theta + y_2\cos\theta)/c$

It may be possible to recognize these sets of curves in the data, and thus recognize multiple-scattering events. Again the curve locations together with their normal directions can be used to find the scattering center positions.

7. CONCLUSIONS

We have shown how the Potter-Moses attributed scattering center model can be put in a more general mathematical framework to which the techniques of microlocal analysis can be applied. We have also shown how the Potter-Moses model can be extended to include some multiple scattering effects.

The multiple-scattering theory developed above has the shortcoming that only one angle is used to parametrize each scattering event. A more complete model could be developed, in which each scattering event depends not only on the direction of scattering but also on the direction of incidence.

ACKNOWLEDGMENTS

This work was supported by the Office of Naval Research and by the Air Force Office of Scientific Research* under agreement number F49620-03-1-0051. It was also partially supported by SAIC under a contract from the Air Force Research Laboratory.

REFERENCES

1. Beylkin G 1985 Imaging of discontinuities in the inverse scattering problem by inversion of a causal generalized Radon transform *J. Math. Phys.* **26** 99–108
2. Beylkin G and Burridge R 1990 Linearized inverse scattering problems in acoustics and elasticity *Wave Motion* **12** 15–52
3. Borden B 1999 *Radar imaging of Airborne Targets* (Bristol: Institute of Physics)
4. Borden B 2002 Mathematical problems in radar inverse scattering *Inverse Problems* **18** R1–R28

5. Bleistein N, Cohen J K, and Stockwell J W 2000 *The Mathematics of Multidimensional Seismic Inversion* (New York: Springer)
6. M. Cheney, "Synthetic-aperture assessment of a dispersive surface", International Journal of Imaging Systems and Technology 14 (2004) 28-34
7. M. Cheney and B. Borden, "Microlocal structure of inverse synthetic aperture radar data", Inverse Problems, 19 (2003) 173-194.
8. M. Cheney and B. Borden, "Microlocal High-Range-Resolution ISAR for Low Signal-to-Noise Environments", Proceedings of SPIE AeroSense Conference, Orlando, April 2003.
9. Duistermaat J J 1996 *Fourier Integral Operators* (Boston: Birkhauser)
10. DeHoop M and Stolk C C 2002 Microlocal analysis of seismic inverse scattering in anisotropic, elastic media *Comm. Pure Appl. Math.* **55** 261–301
11. Faridani A, Finch D V, Ritman E L, and Smith K T 1997 Local Tomography II *SIAM J. Appl. Math.* **57** 1095–127
12. Grigis A and Sjöstrand J 1994 *Microlocal Analysis for Differential Operators: An Introduction (London Mathematical Society Lecture Note Series vol 196)* (Cambridge: Cambridge University Press)
13. Hörmander L 1983, *The Analysis of Linear Partial Differential Operators I* (New York: Springer)
14. Katsevich A I 1997 Local tomography for the generalized Radon transform *SIAM J. App. Math.* **57** 1128–62
15. Louis A and Quinto E T 2000 Local tomographic methods in SONAR *Surveys on Solution Methods for Inverse Problems* ed D Colton, H W Engl, A K Louis, J R McLaughlin and W Rundell (New York: Springer)
16. Mensa D L 1981 *High Resolution Radar Imaging* (Dedham, MA: Artech House)
17. Nolan C J and Cheney M 2002 Synthetic aperture inversion *Inverse Problems* **18** 221–36
18. Nolan C J and Cheney M 2003, "Synthetic Aperture Inversion for Arbitrary Flight Paths and Non-Flat Topography", IEEE Trans. on Image Processing **12** 1035-1043.
19. Petersen B E 1983, *Introduction to the Fourier Transform and Pseudo-Differential Operators* (Boston: Pitman).
20. Potter L C and Moses R L 1997, "Attributed scattering centers for SAR ATR", IEEE Trans. Image Processing **6** 79–91.
21. Quinto E T 1993 Singularities of the X-ray transform and limited data tomography in $\mathbb{R}^2$ and $\mathbb{R}^3$ *SIAM J. Math. Anal.* **24** 1215–25
22. Sullivan R J 2000 *Microwave Radar Imaging and Advanced Concepts* (Boston: Artech House)
23. Taylor M 1981 *Pseudodifferential Operators*(Princeton: Princeton University Press)
24. Treves F 1980 *Introduction to Pseudodifferential and Fourier Integral Operators* vol 1 and 2 (New York: Plenum)
25. Tsang L, Kong J A and Shin R T 1985 *Theory of Microwave Remote Sensing* (New York: Wiley)
26. Varadan V K and Varadan V V, eds. 1980 *Acoustic, electromagnetic and elastic wave scattering – focus on the T-matrix approach* (New York: Pergamon Press)
27. Waterman P C 1965 Matrix formulation of electromagnetic scattering *IEEE Proc.* **53** 805–812.