Dimension and Multiplicity of Graded Rings and Modules

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1 Introduction

In this paper, $R = \oplus R_n$ is a $\mathbb{Z}$-graded commutative ring with 1, where $R_n = 0$ for $n < 0$, and $R_0$ is equal to a field that we call $k$. We further assume that $R$ is finitely generated as a $k$-algebra by a set of homogeneous elements of positive degree (which also makes $R$ Noetherian). The category we are interested in has as objects all $\mathbb{Z}$-graded $R$-modules (with possibly a finite number of nonzero negative graded pieces) that are finitely generated as graded $R$-modules. The morphisms are graded ring homomorphisms of degree zero, and we call this category $C(R)$. For a module $M \in C(R)$, $M = \oplus_i M_i$, and we shall see that $M_j$ is a finite dimensional vector space over $k$ for every $j$.

In the case of the “standard” graded ring $R$ (i.e. $R$ is finitely generated entirely by degree 1 elements as an $R_0$-algebra), a module $M \in C(R)$ has multiplicity given by the leading term in its Hilbert polynomial - an integer valued polynomial which agrees with the function $H_M(n) = \text{vdim}_k(M_n)$, for $n >> 0$.

For a non-standard grading on $R$, we don’t necessarily get a Hilbert polynomial for every $M \in C(R)$, as there may be an infinite number of $j$ such that $\text{vdim}_k(M_j) = 0$. For example, take $M = R = k\lbrack x\rbrack$ where $\deg(x) > 1$. So our goal for this paper is to define a more general multiplicity for $C(R)$ and investigate its properties. We are guided largely by previous work done by Avramov and Buchweitz [1], Maiorana [7], Smoke [8], and Serre’s work in the local algebra case [9].

In fact, many of our results on multiplicity parallel the local case almost identically, and it was our goal to make explicit how the transition from the local case to $C(R)$ is accomplished.

Chapter 2 reviews basic ideas from commutative algebra in the ungraded context that we will need throughout the paper. In chapters 3 and 4 we provide basic definitions and theorems in the graded context, all the while working to define a more geometric dimensional invariant for $M \in C(R)$, which we call $D(M)$. Several equivalent definitions for $D(M)$ are laid out in chapter 5, with equality of these different definitions proved in corollary 5.6 and theorem 5.17.

From there, we use chapter 6 to discuss homological algebra in $C(R)$, with particular attention given to Euler characteristics, Euler-Poincare series, and Koszul complexes. Given a sequence of homogeneous elements which form a system of parameters for $M$ (what we call a “sop”), the alternating sum of the Betti numbers of the Koszul complex define what we call the Koszul multiplicity. Observe that the multiplicity depends on the choice of “sop.”

Chapter 7 is dedicated to investigating the properties of this multiplicity and showing that it may be computed in a variety of different ways (theorem 7.9). Two such ways to compute multiplicity which do not use a Koszul complex are via Samuel polynomials and Euler-Poincare series.

Our main theorem (7.15) states that for $M \in C(R)$, given any choice of a “sop for $M$, the ratio of the multiplicity of $M$ to the product of degrees of elements in the “sop is independent of the choice of “sop.

2 A Review of Commutative Algebra (Ungraded Case)

While the main results of this paper deal with graded algebraic objects, we begin by presenting some standard ideas from commutative algebra without considering a grading. We largely follow the exposition of Serre [9], highlighting some results from chapter II, and Eisenbud [3].

2.1 Length and Ideals of Definition

Recall that for a ring $A$ and an $A$-module $M$, $M$ is said to be a Noetherian $A$-module if it satisfies the ascending chain condition on its submodules. That is, for any chain of $A$-submodules of $M$: $M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n \subseteq \cdots$ there exist a positive integer $i$ such that $M_j = M_{j+1}$ for every $j \geq i$. An Artinian $A$-module $M$ is one which satisfies the descending chain condition on submodules (i.e. every descending chain of submodules will eventually stabilize).

The ring $A$ is said to be an Artinian(resp. Noetherian) ring if it is an Artinian(resp. Noetherian) module over itself.

The following proposition reviews a few facts about Artinian and Noetherian modules.

**Proposition 2.1.**
i) Suppose $P$, $M$, and $N$ are $A$-modules with $0 \to P \to M \to N \to 0$ a short exact sequence. Then, $M$ is an Artinian(Noetherian) $A$-module if and only if $P$ and $N$ are Artinian(Noetherian) $A$-modules.

ii) If $M$ is a finitely generated $A$-module and $A$ is an Artinian(Noetherian) ring then $M$ is an Artinian (Noetherian) $A$-module.

**Definition 2.2.** An $A$-module is called **simple** if its only $A$-submodules are itself and the zero submodule.

**Proposition 2.3.** Let $N$ be a simple $A$-module. Then, any nonzero element $n \in N$ is a generator for $N$ and $N \cong A/P$, as an $A$-module, where $P$ is a maximal ideal in $A$.

**Proof.** If $N$ is simple, then for any nonzero $n \in N$, $\langle n \rangle$ is a submodule of $N$. Since $n \neq 0$, the definition of simple implies that $\langle n \rangle = N$. Let $n \in N$ be a nonzero element in $N$. Let $\phi : A \to N$ be defined $\phi(a) = an$. Since $N$ is generated by $n$, this map is surjective. By the first isomorphism theorem for modules, $A/\ker \phi \cong N$ as $A$-modules.

Since $N$ has no nontrivial submodules, $A/\ker \phi$ has no ideals except 0 and itself. The fourth isomorphism theorem implies that this occurs if and only if $\ker \phi$ is maximal. □

Throughout this paper we use the symbol "⊂" to indicate strictly proper containment, whereas "⊆" indicates proper containment or equality.

**Definition 2.4.** Suppose $M$ is an $A$-module with $A$-submodules $M_1$. The chain of submodules $0 = M_n \subset \cdots \subset M_1 \subset M_0 = M$ is said to be an $A$-composition series of length $n$ if each $M_j/M_{j+1}$ is a nonzero simple $A$-module.

**Definition 2.5.** The **length** of an $A$-module $M$, denoted $\ell_A(M)$, is defined to be the least length of an $A$-composition series for $M$. If $M$ has no $A$-composition series, then $\ell_A(M) = \infty$.

**Example 2.6.**

i) Let $V$ be an $n$-dimensional vector space over a field $k$. Define $\text{vdim}_k(V)$ to be the dimension of $V$ as a vector space over $k$.

It is clear that no subspace of dimension larger than 1 can be simple in $V$, since removing one element from the basis for the subspace will produce a new, nonzero subspace contained in it. For the same reason, if there are two subspaces $W_1, W_2$, the quotient $W_1/W_2$ is simple if and only if $W_2$ has codimension 1 in $W_1$.

So for example, if $\{e_1, \ldots, e_n\}$ is a basis for $V$, the chain $0 \subset (e_1) \subset (e_1, e_2) \subset \cdots \subset (e_1, \ldots, e_{n-1}) \subset (e_1, \ldots, e_n) = V$ is a $k$-composition series for $V$. Any other $k$-composition series for $V$ will come from a different choice of basis, but the length will always be the same. Also,

$$\ell_k(V) = \text{vdim}_k(V).$$

ii) Let $p$ be a prime number in $\mathbb{Z}$, then $\mathbb{Z}/p\mathbb{Z}$ is simple as a $\mathbb{Z}$-module.

iii) Let $n \geq 1$ be an integer and let $p$ be a prime number, then $\ell_\mathbb{Z}(\mathbb{Z}/p^n\mathbb{Z}) = n$.

**Lemma 2.7.**

Suppose $M$, $N$, and $P$ are $A$-modules.

i) If $M$ admits an $A$-composition series, then every $A$-composition series for $M$ has the same $A$-length, i.e. length is a well defined function.

ii) An $A$-module $M$ admits an $A$-composition series (has finite $A$-length) if and only if $M$ is both an Artinian and Noetherian $A$-module.

iii) If $0 \to P \to M \to N \to 0$ is a short exact sequence of $A$-modules, then $M$ admits an $A$-composition series if and only if $P$ and $N$ admit $A$-composition series.
Theorem 2.8. **Length adds over short exact sequences**

Suppose $P$, $M$, and $N$ are $A$-modules where $0 \to P \to M \to N \to 0$ is a short exact sequence. Then, if $M$ admits an $A$-composition series, so do $P$ and $N$, and $\ell_A(M) = \ell_A(P) + \ell_A(N)$.

**Proof.** If $M$ admits an $A$-composition series, then by the previous lemma, so do $P$ and $N$. Suppose $\ell_A(P) = j$ and $\ell_A(N) = k$. There exist $A$-composition series $0 = P_1 \subset \cdots \subset P_0 = P$ and $0 = N_k \subset \cdots \subset N_0 = N$. If we consider the image of each $P_i$ in $M$, we get a corresponding chain of $A$-submodules in $M$ such that each inclusion is maximal. Similarly, since $N \cong M/P$, this isomorphism gives a corresponding chain of $A$-submodules in $M/P$ such that each inclusion is maximal.

Then, by the fourth isomorphism theorem for modules we get a one to one correspondence taking the chain in $M/P$ to a chain $P \subset \cdots \subset M$. This is a chain of length $k$ and again the maximality of each inclusion is preserved, so successive quotients are simple.

If we put the two chains together, we get a chain of length $j+k$, starting at 0 and ending at $M$, where each successive quotient is simple, i.e. we have a composition series of length $j+k$ for $M$. Therefore, by part i of the previous lemma, every $A$-composition series of $M$ has this length and so the theorem is proved.

**Corollary 2.9.** Suppose the following is an exact sequence of $A$-modules, where each $M_i$ admits an $A$-composition series, $0 \to M_1 \to \cdots \to M_n \to 0$. Then, $\sum_{i=1}^n (-1)^i \ell_A(M_i) = 0$.

**Example 2.10.** Let $n$ be a positive integer and say its prime decomposition is $n = p_1^{a_1} \cdots p_r^{a_r}$. By the Chinese remainder theorem, $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/p_1^{a_1}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p_r^{a_r}\mathbb{Z}$. By the previous theorem and example, $\ell_\mathbb{Z}(\mathbb{Z}/n\mathbb{Z}) = \ell_\mathbb{Z}(\mathbb{Z}/p_1^{a_1}\mathbb{Z}) + \cdots + \ell_{\mathbb{Z}}(\mathbb{Z}/p_r^{a_r}\mathbb{Z}) = a_1 + \cdots + a_r$.

**Definition 2.11.** The **annihilator** of the $A$-module $M$ in $A$ is $\text{Ann}_A(M) = \{a \in A \mid am = 0, \text{ for every } m \in M\}$.

**Theorem 2.12.** A **characterization of finite length $A$-modules** (Theorem 2.14, Corollary 2.17 [3])

Let $M$ be an $A$-module. The following conditions are equivalent:

i) $M$ has finite $A$-length.

ii) $M$ is both an Artinian and Noetherian $A$-module.

If $A$ is a Noetherian ring and $M$ is a finitely generated $A$-module, then items i) and ii) are also equivalent to:

iii.) All the primes of $A$ that contain $\text{Ann}_A(M)$ are maximal.

iv.) $A/\text{Ann}_A(M)$ is an Artinian ring, so that the only primes in $A/\text{Ann}_A(M)$ are maximal and there are only a finite number of them.

Let $A$ be a Noetherian ring. $A$ is certainly a finitely generated $A$-module, so we apply the above theorem and observe that $A$ is an Artinian ring if and only if all of the primes in $A$ are maximal and there are only a finite number of them.

**Definition 2.13.** **Ideal of Definition (Ungraded)**

A proper ideal $\mathfrak{I}$ of a ring $A$ is said to be an ideal of definition for an $A$-module $M$ if $\ell_A(M/\mathfrak{I}M)$ is finite.

**Lemma 2.14.** Let $X$ be a finitely generated $B$-module, where $B$ is an Artinian ring (equivalently, $B$ is Noetherian and of finite $B$-length). Then, $\ell_B(X) < \infty$.

In particular, let $A$ be a Noetherian ring and let $M$ be any finitely generated $A$-module (not necessarily of finite $A$-length). Let $\mathfrak{I}$ an ideal of definition for $A$, then $\mathfrak{I}$ is also an ideal of definition for $M$, i.e.

$\ell_A(A/\mathfrak{I}) < \infty \Rightarrow \ell_A(M/\mathfrak{I}M) < \infty$.

**Proof.** By finite generation, there is an integer $n > 0$ and a surjective homomorphism of $B$-modules which sends $B^n \to X$. Then, by proposition 2.1, $X$ is necessarily an Artinian $B$-module (since $B^n$ is an Artinian $B$-module). By the same reasoning, $X$ is also a Noetherian $B$-module. So, $X$ is both a Noetherian and Artinian $B$-module and lemma 2.7 implies $\ell_B(X) < \infty$.

The second statement of the lemma comes from setting $X = M/\mathfrak{I}M$. Then if $B = A/\mathfrak{I}$, $B$ must be Artinian since $\mathfrak{I}$ was assumed to be an ideal of definition for $A$. Now, $X = M/\mathfrak{I}M$ is a finitely generated $B$-module, so the hypotheses of the first statement are met and we conclude that $\ell_B(X) < \infty$. However, every $B$-submodule of $X$ is also an $A$-submodule of $X$, and vice versa. Thus $\ell_A(X) < \infty$ as well.
Definition 2.16. Suppose $\mathfrak{m}$ is an ideal of definition for $M$. Since $A/\mathfrak{m}$ is a field it has $A$-length equal to 1 and so $\mathfrak{m}$ is an ideal of definition for $A$. By the lemma above, $\mathfrak{m}$ is an ideal of definition for $M$.

Moreover, since $M/\mathfrak{m}M$ is finitely generated over $A$, the $A/\mathfrak{m}$-module structure on $M/\mathfrak{m}M$ is defined by its $A$-module structure, and the set of $A$-submodules of $M/\mathfrak{m}M$ is equal to the set of $A/\mathfrak{m}$-submodules of $M/\mathfrak{m}M$. Now we apply example 2.6 to get,

$$\text{vdim}_{A/\mathfrak{m}}(M/\mathfrak{m}M) = \ell_A(M/\mathfrak{m}M) < \infty.$$ 

Example 2.15. 

i) Let $m$ be a maximal ideal of the Noetherian ring $A$, and let $M$ be a finitely generated $A$-module, then $m$ is an ideal of definition for $M$. Since $A/m$ is a field it has $A$-length equal to 1 and so $m$ is an ideal of definition for $A$. By the lemma above, $m$ is an ideal of definition for $M$.

ii) Let $M = A = A_0[x_0, \cdots, x_n]$ where $A_0$ is an Artinian ring. The ideal $J = (x_0, \cdots, x_n)$ is an ideal of definition for $M$ since $A_0[x_0, \cdots, x_n]/(x_0, \cdots, x_n) \cong A_0$, an Artinian (and hence Noetherian) ring, which must have finite length by lemma 2.7.

2.2 Filtrations and the Artin Rees Theorem

As in Serre [9] chapter II A, we make the following definitions.

Definition 2.16. Suppose $A$ is a ring and $M$ is an $A$-module. Let

$$\cdots \subseteq F^n(M) \subseteq F^{n-1}(M) \subseteq \cdots \subseteq F^1(M) \subseteq F^0(M) = M$$

be a filtration of $M$ by submodules $F^i(M)$ and let $J$ be a proper ideal in $A$. The filtration $F(M)$ is $J$-bonne (or $J$-good or $J$-stable) if $F^{n+1}(M) \supseteq JF^n(M)$ for every $n \geq 0$, and if $F^{i+1}(M) = JF^i(M)$, for $i >> 0$.

Definition 2.17. For any ring $A$ with ideal $J$ and any $A$-module $M$, the $J$-adic filtration is defined,

$$\cdots J^n M \subseteq J^{n-1} M \subseteq \cdots \subseteq J^1 M \subseteq J^0 M = M.$$

If $I$ is any ideal of $A$ with $M$ an $A$-module, then the $I$-adic filtration is $I$-bonne.

If we take the $J$-adic filtration of $M$ and intersect it with some $A$-submodule of $M$, we do not generally get back the $I$-adic filtration of the submodule, however we have the following theorem to address this situation.

Theorem 2.18. Artin-Rees ([9], II-9, Theorem 12)

Let $M$ be a finitely generated $A$-module where $A$ is Noetherian. Let $P$ be an $A$-submodule of $M$ and $J$ a proper ideal in $A$. Then, there exists an $m_0 \geq 0$ such that

$$P \cap J^{m+m_0} M = J^m(P \cap J^{m_0} M),$$

for every $m \geq 0$.

In other words, the filtration $F^n(P) = P \cap J^n M$ is $J$-bonne.

Theorem 2.19. Suppose that $0 \to P \to M \to N \to 0$ is a short exact sequence of $A$-modules and that $J$ is an ideal of definition for $M$. Then $J$ is also an ideal of definition for $N$ and $P$.

Proof. There are short exact sequences

$$0 \to P/(P \cap J^n M) \to M/J^n M \to N/J^n N \to 0,$$

for every $n \geq 0$. Thus, since $\ell_A(M/J^n M)$ is finite, so is $\ell_A(N/J^n N)$ and $\ell_A(P/(P \cap J^n M))$, for every $n \geq 0$. So $J$ is an ideal of definition for $N$.

Now, by the Artin-Rees theorem, there is an $m_0$ such that

$$J^m(P \cap J^{m_0} M) = P \cap J^{m_0 + m} M,$$

for every $m \geq 0$. Since $J^m(P \cap J^{m_0} M) \subseteq J^m P$, for every $m \geq 0$, there is a surjection

$$P/J^m(P \cap J^{m_0} M) \to P/J^m P,$$

for every $m \geq 0$.

Thus, there is a surjection

$$P/(P \cap J^{m_0 + m} M) \to P/J^m P,$$

for every $m \geq 0$. So, $J$ is an ideal of definition for $P$. 

$\square$
2.3  Associated Primes

Definition 2.20. The spectrum of a ring \( A \) is the set of all prime ideals of \( A \), it is denoted \( \text{Spec}(A) \). For an ideal \( \mathfrak{I} \) in \( A \), we denote the set of primes in \( A \) which contain \( \mathfrak{I} \) by \( \mathcal{V}(\mathfrak{I}) \).

From the perspective of algebraic geometry, the \( \mathcal{V}(\mathfrak{I}) \) make up the closed sets in the Zariski topology on \( \text{Spec}(A) \).

For primes \( \mathfrak{I} \) and \( \mathfrak{J} \), we have \( \mathcal{V}(\mathfrak{I} \cap \mathfrak{J}) = \mathcal{V}(\mathfrak{I}) \cap \mathcal{V}(\mathfrak{J}) \) and so \( \mathcal{V}(\mathfrak{I}^n) \) for positive integers \( n \). Also, \( \mathcal{V}(\sum \mathfrak{I}_i) = \cap \mathcal{V}(\mathfrak{I}_i) \).

For the rest of this section, assume that \( A \) is a Noetherian ring.

Definition 2.21. Let \( M \) be an \( A \)-module and \( \mathfrak{p} \) a prime such that the localization \( M_\mathfrak{p} \) is nonzero. The collection of all such primes forms a set called the support of \( M \), denoted \( \text{Supp}(M) \).

Proposition 2.22. (Corollary 2.7 [3]) If \( M \) is a finitely generated \( A \)-module, and \( \mathfrak{p} \) is a prime ideal, then \( \mathfrak{p} \in \text{Supp}(M) \) if and only if \( \mathfrak{p} \) contains \( \text{Ann}_A(M) \).

Example 2.23. For any finitely generated Artinian \( A \)-module \( X \), all primes in \( \text{Supp}(X) \) are maximal and there are only a finite number of them by theorem 2.12.

Definition 2.24. A prime \( \mathfrak{p} \) of \( A \) is associated to \( M \) if \( \mathfrak{p} \) is the annihilator of an element of \( M \). The set of all associated primes to \( M \) is called \( \text{Ass}_A(M) \). Note that every associated prime to \( M \) contains \( \text{Ann}_A(M) \).

If \( \mathfrak{I} \) is an ideal of \( A \), we say that the associated primes to the \( A \)-module \( A/\mathfrak{I} \) are the associated primes of \( \mathfrak{I} \).

From the definition we see that \( \mathfrak{p} \) is an associated prime to \( M \) if and only if \( A/\mathfrak{p} \) is isomorphic to an \( A \)-submodule of \( M \).

Proposition 2.25. Let \( M \) be a finitely generated \( A \)-module and \( \mathfrak{I} \) an ideal in \( A \).

i) If \( M \neq 0 \), then \( \text{Ass}_A(M) \) is a finite set and is nonempty. The set \( \text{Ass}_A(M) \) contains all the primes minimal among primes containing \( \text{Ann}_A(M) \) (i.e. primes in \( \text{Supp}(M) \)). These minimal primes are sometimes called the isolated primes, whereas the other associated primes may be called the embedded primes.

ii) \( \mathfrak{I} \) is an ideal of definition for \( M \) if and only if all elements of the finite set \( \mathcal{V}((\text{Ann}_A(M) + \mathfrak{I})) \) are maximal ideals.

Proof:

i) See theorem 3.1 in Eisenbud [3].

ii) By definition, \( \mathfrak{I} \) is an ideal of definition for \( M \) if and only if \( \ell_A(M/\mathfrak{I}M) < \infty \). Now, \( (\mathfrak{I} + \text{Ann}_A(M))M = \mathfrak{I}M \) both as a set and as an \( A \)-module. Therefore, \( M/\mathfrak{I}M = M/(\mathfrak{I} + \text{Ann}_A(M))M \) and so \( \ell_A(M/(\mathfrak{I} + \text{Ann}_A(M))M < \infty \).

Theorem 2.12 implies that this is equivalent to having only a finite number of primes in \( A/\text{Ann}_A(M/(\mathfrak{I} + \text{Ann}_A(M)))M \) which are all maximal. Now apply the fourth isomorphism theorem for rings, and the theorem is proved.

\[ \square \]

2.4  Samuel polynomial

Here we define the Samuel polynomials. In the next section, we will see that the degree of this polynomial may provide a useful invariant of rings.

Definition 2.26. Let \( f \) be a function from \( \mathbb{Z} \to \mathbb{Z} \). Such a function is said to be polynomial-like if there exists a polynomial \( p_f(t) \in \mathbb{Q}[t] \) such that \( f(n) = p_f(n) \) for all sufficiently large integers \( n \).
**Definition 2.27.** Let \( f \) be a polynomial-like function, and \( n \) an integer. Define the **difference operator** \( \Delta \), to act on \( f \) by \( \Delta f(n) = f(n + 1) - f(n) \). Let \( \Delta^r \) denote repeated application of the difference operator \( \Delta \), exactly \( r \) times.

**Proposition 2.28.** (Lemma 1 pg. 20 [9])
Let \( f \) and \( g \) be nonzero polynomial-like functions.

- \( \Delta \) is a linear operator on polynomial-like functions. That is, \( \Delta(f + g) = \Delta(f) + \Delta(g) \), and \( \Delta(c \cdot f) = c \cdot \Delta(f) \), for any scalar \( c \).

- \( \Delta f \) is polynomial-like, and thus, so too is \( \Delta^r f \) for any integer \( r \geq 1 \).

- The set of all functions of the form \( Q_i(t) = \binom{t}{i} \) for \( i \geq 0 \) forms a basis for \( \mathbb{Q}[t] \) over \( \mathbb{Z} \). If \( p_f(t) = a_d t^d + \cdots + a_0 \) in \( \mathbb{Q}[t] \) is the degree \( d \) polynomial which agrees with \( f \) for all sufficiently large integer inputs, then we can alternately write \( p_f(t) = \Sigma_{i=0}^d c_i Q_i(t) \) for integers \( c_i \). By definition, \( Q_d(t) = \binom{t}{d} = \frac{t(t-1)\cdots(t-d+1)}{d!} \), whose only degree \( d \) term is \( \frac{t^d}{d!} \). Now, by comparing the two ways to write \( p_f(t) \), we see that the leading coefficient \( a_d \) must equal \( \frac{t^d}{d!} \). In particular, we have that \( d! \cdot a_d \) is an integer. Also, when \( f(n) \geq 0 \) for all sufficiently large \( n \), then \( a_d \) is a positive integer.

**Proposition 2.29.** Let \( f \) be a polynomial-like function, with \( p_f(t) \) as in the previous proposition. Let \( u \) be a positive integer, then for all \( n \) sufficiently large we have

\[
\begin{align*}
\Delta^u f(n) = \begin{cases} \\
\frac{d!}{(d-u)!} a_d n^{d-u} + \text{lower order terms} & u < d \\
0 & u = d \\
\end{cases} \\
\end{align*}
\]

\( i) \) \( \Delta^u f(n) = \begin{cases} \\
\frac{d!}{(d-u)!} a_d n^{d-u} + \text{lower order terms} & u < d \\
0 & u = d \\
\end{cases} \\
\]

\( ii) \) \( \sum_{j=0}^{u} (-1)^j \binom{u}{j} f(n-j) = \Delta^u f(n-u). \)

**Proof.** Part \( i) \) is explained on pg. 20 of Serre [9]. We prove \( ii) \) by induction on \( u \). For \( u = 1 \), \( \Delta f(n-1) = f(n) - f(n-1) = \binom{0}{0} f(n-0) - \binom{1}{1} f(n-1) \), so the two sides agree for any sufficiently large \( n \). Now suppose the result holds for \( u \) and all \( n \) sufficiently large, then prove the result for \( u+1 \).

We have

\[
\begin{align*}
\Delta^{u+1} f(n-(u+1)) &= \Delta \Delta^u f(n-1 - u) \\
&= \Delta \sum_{j=0}^{u} (-1)^j \binom{u}{j} f(n-1-j) \\
&= \sum_{j=0}^{u} (-1)^j \binom{u}{j} \Delta f(n-1-j) \\
&= \sum_{j=0}^{u} \binom{u}{j} \left[ (-1)^j f(n-j) + (-1)^{j+1} f(n-(j+1)) \right] \\
\end{align*}
\]

Rewrite the above sum with \( u+2 \) summands: with the \( 0^{th} \) term of this sum is \( f(n) \), the \( u+1^{st} \) term of the sum is \( (-1)^{u+1} \binom{u}{u} f(n-(u+1)) \), and for any \( j \) between 1 and \( u \), the \( j^{th} \) term of the sum is \( (-1)^j \binom{u}{j} f(n-j) + (-1)^j \binom{u}{j-1} f(n-(j+1)) \). Of course, \( \binom{u}{j} + \binom{u}{j} = \binom{u+1}{j} \), and so we conclude that

\[
\begin{align*}
\Delta^{u+1} f(n-(u+1)) &= \sum_{j=0}^{u+1} (-1)^j \binom{u+1}{j} f(n-j) \\
\end{align*}
\]

\( \square \)

**Definition 2.30.** **The Samuel function**
Let \( A \) be a Noetherian ring and \( M \) a finitely generated \( A \)-module. Suppose \( \mathcal{I} \) is an ideal of definition for \( M \), and \( \mathcal{F}(M) \) is an \( \mathcal{I} \)-bonne filtration of \( M \). Given these hypotheses, the following function from \( \mathbb{Z} \) to \( \mathbb{Z} \) is well-defined:

\[
p(\mathcal{F}(M), \mathcal{I}, n) = \ell_A(M/\mathcal{F}^n(M))
\]

for \( n \geq 0 \).
Lemma 2.34. Suppose that it was already proved in theorem 2.19 that using the theorem above, for \( n > 0 \), \( F \). Using the Artin-Rees Theorem, less than \( \deg (R) \) where \( N \) of finitely generated \( A \).

Example 2.32. Consider \( R = k[x,y]/(y^2 - x^3) \) as a module over itself. One may compute \( p(R, m, n) = \frac{2}{11}n^3 + L.O.T \).

Next, we discuss exactly how the Samuel functions for any \( \mathfrak{I} \)-bonne filtration can be compared to that for the \( \mathfrak{J} \)-adic filtration. We use \( p(M, \mathfrak{I}, n) \) to denote the Samuel function for the \( \mathfrak{I} \)-adic filtration of \( M \).

Theorem 2.33. ([9], II.4, Lemma 3 and Proposition 9) Suppose \( M, A, \) and \( \mathfrak{I} \) are as before. Let \( F(M) \) be an \( \mathfrak{I} \)-bonne filtration of \( M \).

i) For \( n > 0 \), \( p(M, \mathfrak{I}, n) = p(\mathfrak{I}(M), \mathfrak{I}, n) + R(n) \), where \( R(n) \) is a polynomial whose leading coefficient is positive, and whose degree is strictly less than the degree of the polynomial function determined by \( p(M, \mathfrak{I}, n) \).

ii) If \( (\mathfrak{I} + \text{Ann}_A(M))/\text{Ann}_A(M) \) is generated by \( r \) elements as an ideal in \( A/\text{Ann}_A(M) \), then for \( n > 0 \), \( p(M, \mathfrak{I}, n) \) is a polynomial function of degree less than or equal to \( r \), and \( \Delta^r p(M, \mathfrak{I}, n) \leq \ell_A(M/\mathfrak{I}M) \).

Lemma 2.34. ([9], II.4, Proposition 10) Suppose that \( 0 \to P \to M \to N \to 0 \) is a short exact sequence of finitely generated \( A \)-modules, and that \( \mathfrak{I} \) is an ideal of definition for \( M \). Then, \( \mathfrak{I} \) is an ideal of definition for \( N \) and \( P \), and, for \( n > 0 \),

\[
p(M, \mathfrak{I}, n) + R(n) = p(N, \mathfrak{I}, n) + p(P, \mathfrak{I}, n),
\]

where \( R(n) \) is a polynomial function of \( n \) whose leading coefficient is positive, and whose degree is strictly less than \( \deg (p(P, \mathfrak{I}, n)) \).

Proof. It was already proved in theorem 2.19 that \( \mathfrak{I} \) is an ideal of definition for \( P \) and \( N \).

If \( F^0(P) = P \cap \mathfrak{I}M \), then for every \( n \geq 0 \), there is a short exact sequence

\[
0 \to P/F^0(P) \to M/\mathfrak{I}M \to N/\mathfrak{I}N \to 0.
\]

Using the Artin-Rees Theorem, \( F(P) \) is an \( \mathfrak{I} \)-bonne filtration of \( P \).

Therefore, first, \( p(M, \mathfrak{I}, n) = p(N, \mathfrak{I}, n) + p(F(P), n) \), since length adds over short exact sequences. Then, using the theorem above, for \( n > 0 \), \( p(F(P), n) = p(P, \mathfrak{I}, n) - R(n) \), where \( R(n) \) is a polynomial whose leading coefficient is positive, and whose degree is strictly less than the degree of the polynomial function determined by \( p(P, \mathfrak{I}, n) \).

Proposition 2.35. (Proposition 11 chapter II.4 [9]) The degree of \( p(M, \mathfrak{I}, n) \) depends only on \( V(\mathfrak{I} + \text{Ann}_A(M)) = V(\mathfrak{I}) \cap \text{Supp}(M) \).
Proof. Since $\mathfrak{J}$ is an ideal of definition for $M$, $\mathcal{V}(\mathfrak{J}+\text{Ann}_A(M))$ is a finite set consisting of only maximal ideals. We may assume that $\text{Ann}_A(M) = 0$ so suppose that $\mathcal{V}(\mathfrak{J}) = \{m_1, \ldots, m_r\}$. Let $\mathfrak{J}'$ be an ideal of $A$ such that $\mathcal{V}(\mathfrak{J}) = \mathcal{V}(\mathfrak{J}')$. We need to show that for sufficiently large integers $n$, $\deg(p(M, \mathfrak{J}, n)) = \deg(p(M, \mathfrak{J}', n))$.

Since $\mathcal{V}(\mathfrak{J}') \subseteq \mathcal{V}(\mathfrak{J})$, the corollary to proposition 2 in chapter 1 of Serre [9] implies that there exists an integer $m > 0$ such that $\mathfrak{J}'^m \subseteq \mathfrak{J}'$. So for every $n \geq 0$, $\mathfrak{J}'^m \subseteq \mathfrak{J}'^n$. Thus, $M/\mathfrak{J}'^nM \to M/\mathfrak{J}'^mM$, so $\ell_A(M/\mathfrak{J}'^nM) \leq \ell_A(M/\mathfrak{J}'^mM)$, since length adds over short exact sequences. By definition, $p(M, \mathfrak{J}, mn) \geq p(M, \mathfrak{J}', n)$ for each $n \geq 0$.

We also know that each of these functions is polynomial-like so there exist polynomials with rational coefficients which agree with each function for sufficiently large $n$ (Remember that we give these polynomials the same names as the functions that they represent.) Since the function value $p(M, \mathfrak{J}, mn)$ is greater than or equal to $p(M, \mathfrak{J}', n)$ for every $n \geq 0$, the degree of the polynomial $p(M, \mathfrak{J}, n)$ is greater than or equal to the degree of the polynomial $p(M, \mathfrak{J}', n)$ for sufficiently large $n$.

Now, since $\mathcal{V}(\mathfrak{J}) = \mathcal{V}(\mathfrak{J}')$, we switch the roles of $\mathfrak{J}$ and $\mathfrak{J}'$ in the argument above to get the reverse inequality and finish the proof. 

\begin{definition}
  \textbf{d}_1(M, \mathfrak{J})
  
  Let $\mathfrak{J}$ be an ideal of definition for $M$, then $d_1(M, \mathfrak{J})$ is the degree of the polynomial function of $n$ that calculates $p(M, \mathfrak{J}, n)$, for $n >> 0$. Define $d_1(0, \mathfrak{J}) = -\infty$.
\end{definition}

When $M$ is an $A$-module and $A$ is a local Noetherian ring, the following proposition shows that $d_1(M, \mathfrak{J})$ is independent of the choice of ideal of definition $\mathfrak{J}$. When $d_1$ only depends on the $A$-module $M$ and not on the ideal of definition chosen, we just write $d_1(M)$.

\begin{proposition}
  Let $A$ be a local Noetherian ring with maximal ideal $\mathfrak{m}$ and let $M$ a finitely generated $A$-module with ideal of definition $\mathfrak{J}$. Then, the degree of $p(M, \mathfrak{J}, n)$ is equal to the degree of $p(M, \mathfrak{m}, n)$.
\end{proposition}

Proof. Recall that $\mathfrak{m}$ is an ideal of definition for $M$, as was shown in example 2.15. Proposition 2.35 implies that the degree of $p(M, \mathfrak{J}, n)$ depends only on $\mathcal{V}(\mathfrak{J} + \text{Ann}_A(M))$. So all we need to show is that $\mathcal{V}(\mathfrak{J} + \text{Ann}_A(M)) = \mathcal{V}(\mathfrak{m} + \text{Ann}_A(M))$, and $\mathcal{V}(\mathfrak{m} + \text{Ann}_A(M))$ is simply equal to $\mathfrak{m}$.

Now, part ii) in proposition 2.25 implies that since $\mathfrak{J}$ is an ideal of definition, there are only a finite number of primes in $A$ that contain $\mathfrak{J} + \text{Ann}_A(M)$ and all such primes are maximal. Since this is a local ring, there can only be one, namely $\mathfrak{m}$.

\section{Krull Dimension and a Dimension Theorem}

\begin{definition}
The \textbf{Krull dimension}, or just dimension, of an $A$-module $M$, denoted $\dim_A(M)$, is the greatest integer $D$ such that there exists a strictly increasing chain

$$p_0 \subset \cdots \subset p_D$$

of prime ideals in $A$, where $\text{Ann}_A(M) \subseteq p_0$. Recall that the ring $A$ is not considered to be a prime ideal of itself. For the zero module, we define $\dim_A(0) = -\infty$.

The Krull dimension of a ring $B$ is defined to be the Krull dimension of $B$ as a module over itself, and we write $\dim(B)$ rather than $\dim_B(B)$.
\end{definition}

\begin{example}
  If $M$ is a finitely generated $A$-module, then $\dim_A(M) = 0$ if and only if $M$ is an Artinian $A$-module, using Theorem 2.12 iii).
\end{example}

\begin{definition}
The \textbf{height} of a prime ideal $\mathfrak{p}$ is denoted $\text{ht}(\mathfrak{p})$, it is defined as the largest integer $n$ such that a chain of prime ideals $p_0 \subset \cdots \subset p_n = \mathfrak{p}$ exists.

Observe that the height of a prime ideal $\mathfrak{p}$ in $A$ is equal to the Krull dimension of the localization $A_\mathfrak{p}$.
\end{definition}

\begin{definition}
  $s_1(M)$
  
  Let $(A, \mathfrak{m})$ be a local Noetherian ring with $M$ a finitely generated $A$-module. Then, $s_1(M)$ is the smallest integer $n$ such that there exist $x_1, \ldots, x_n \in \mathfrak{m}$ with $M/(x_1, \ldots, x_n)M$ of finite length.

  In section 5.3 of this paper, we prove a theorem which might be considered a graded analogue to the following.
Theorem 2.42. (Theorem 1 III-B in [9])
For a local Noetherian ring $A$ and finitely generated $A$-module $M$ we have,
\[ d_1(M) = s_1(M) = \dim_A(M). \]

3 Introductory Results in the Graded Context

In this chapter we present some of the foundational ideas for graded algebraic objects, which are the main focus of this paper. Note carefully remark 3.5 which establishes the notation that will be used throughout this paper.

Definition 3.1. A ring $A$ is a $\mathbb{Z}$-graded ring if there exist abelian subgroups $A_n$ of $A$ such that $A = \bigoplus_{n \in \mathbb{Z}} A_n$ and $A_n \cdot A_m \subseteq A_{n+m}$, for any integers $n$ and $m$.

Elements of the subgroup $A_n$ are called homogeneous elements of degree $n$. Note that $0 \in A_n$ for every $n$. For every $a \in A$, $a$ may be written uniquely as $a = \Sigma_{n \in \mathbb{Z}} a_n$, where $a_n \in A_n$ and $a_j = 0$ for $|j|$ sufficiently large. The $a_n$ are called homogeneous components of $a$.

Definition 3.2. For a graded ring $A$ and $A$-module $M$, $M$ is said to be a $\mathbb{Z}$-graded module if there exist abelian subgroups $M_n$ of $M$ such that $M = \bigoplus_{n \in \mathbb{Z}} M_n$ and $A_n \cdot M_m \subseteq M_{n+m}$, for any integers $n$ and $m$.

Throughout this paper we say that a module $M$ (or ring $R$) is positively graded if $M_j = 0$ whenever $j < 0$ (resp. $R_j = 0$ whenever $j < 0$). In our notation, $R$ will be generally used for positively graded rings.

Definition 3.3. Let $A$ be a $\mathbb{Z}$-graded ring and let $M$ and $N$ be graded $A$-modules. Suppose $\psi : M \to N$ is an $A$-module homomorphism. We say that $\psi$ is a graded homomorphism of degree $d$ if for every integer $n$, $\psi(M_n) \subseteq N_{n+d}$.

Definition 3.4. Let $A$ be a $\mathbb{Z}$-graded ring and let $M$ be a $\mathbb{Z}$-graded $A$-module. If $d \in \mathbb{Z}$, define a graded abelian group $M(d)$ by setting $M(d)_j = M_{d+j}$ for every $j \in \mathbb{Z}$. The reader can check that $M(d)$ is a graded $A$-module, for every $d \in \mathbb{Z}$.

Remark 3.5. From this point on, we reserve the variables $A$ and $R$ for the following:

i) Let $A$ be a Noetherian ring which is $\mathbb{Z}$-graded.

ii) Let $R$ be a positively graded ring, where $R_0 = k$ is a field and $R$ is finitely generated by homogeneous elements of positive degree as a $k$-algebra.

As frequently as possible, we will try to remind the reader of this, but it should be assumed that this is the case for any ring $A$ or $R$, from here on out.

For such a graded ring $R$, the superfluous ideal $R_+$ is defined to be $R_+ = \bigoplus_{i \geq 1} R_i$. This is a maximal ideal (since $R/R_+ \cong k$, a field), in fact, it is the unique graded maximal ideal for $R$. We borrow the notation typical for local rings and call the superfluous ideal $m$ from now on.

We regard $R_0 = k$ as being a graded $R$-module concentrated in degree zero, i.e. $k_n = 0$ for each $n \neq 0$ and $k_0 = k$. If $r \in R_n$, the $R$-multiplication on $k$ is defined $r \cdot k = 0$ for $n \neq 0$ and is just the $k$-multiplication for $r \in R_0 = k$. The following categories make up the environments in which we will be working.

Definition 3.6. The Category $\mathcal{C}(-)$
Suppose $B$ is any $\mathbb{Z}$-graded ring. The category $\mathcal{C}(B)$ has objects finitely generated $\mathbb{Z}$-graded $B$-modules. The morphisms of $\mathcal{C}(B)$ are the $B$-module homomorphisms which are graded of degree zero (i.e. degree-preserving).

Lemma 3.7. If $B$ is a $\mathbb{Z}$-graded ring, and $M \in \mathcal{C}(B)$, then for every $d \in \mathbb{Z}$, $M(d) \in \mathcal{C}(B)$.
3.1 Basic Results in $\mathcal{C}(A)$

Again, let’s suppose that the ring $A$ is a $\mathbb{Z}$-graded, Noetherian ring. The $A$-module $M$ is in $\mathcal{C}(A)$ if and only if $M$ is a finitely generated $A$-module which is also $\mathbb{Z}$-graded on $A$ (i.e. $A_m \cdot M_n \subset M_{n+m}$). Many of the results in this section do not require $A$ to be Noetherian, but nevertheless we will assume that $A$ is a Noetherian ring.

**Definition 3.8.** An ideal $\mathfrak{I} \subseteq A$ is a graded ideal if it is generated by homogeneous elements.

The following lemma characterizes $\mathbb{Z}$-graded ideals.

**Lemma 3.9.** An ideal $\mathfrak{I}$ in a $\mathbb{Z}$-graded ring $A$ is graded if and only if for all $f \in \mathfrak{I}$, every homogeneous component of $f$ is also in $\mathfrak{I}$.

The following lemma characterizes graded ideals which are prime.

**Lemma 3.10.** For a proper graded ideal $\mathfrak{I} \subset A$, $\mathfrak{I}$ is prime if and only if for all homogeneous elements $x$, $y \in A$ with $xy \in \mathfrak{I}$, we have $x \in \mathfrak{I}$ or $y \in \mathfrak{I}$.

**Proof.** The forward direction is immediate.

Suppose $\mathfrak{I}$ is $\mathbb{Z}$-graded with the property that for all homogeneous elements $x$ and $y$ with $xy \in \mathfrak{I}$, we have that either $x$ or $y$ is in $\mathfrak{I}$, and $\mathfrak{I} \neq A$.

Suppose $s$ and $t$ are two elements of $A$ such that $st \in \mathfrak{I}$. For sake of contradiction, let us assume that neither $s$ nor $t$ is in $\mathfrak{I}$. Since $A$ is graded, we may write $s$ and $t$ in terms of their homogeneous components, $s = \sum s_i$ and $t = \sum t_j$, where all but a finite number of terms in each sum are nonzero.

Since neither $s$ nor $t$ are elements of $\mathfrak{I}$, there exists integers $d$ and $e$ such that the homogeneous components $s_d \notin \mathfrak{I}$ and $t_e \notin \mathfrak{I}$. Since all but a finite number of homogeneous components of $s$ and $t$ are nonzero, we may assume that $d$ and $e$ are the largest integers with $s_d \notin \mathfrak{I}$ and $t_e \notin \mathfrak{I}$.

The homogeneous component of $st$ in degree $d+e$ has the form $(st)_{d+e} = \sum_{i+j=d+e} s_i t_j$. By assumption $st \in \mathfrak{I}$, and because $\mathfrak{I}$ is graded each homogeneous component of $st$ must be in $\mathfrak{I}$. In particular $(st)_{d+e} \in \mathfrak{I}$.

For each term in the sum $\sum_{i+j=d+e} s_i t_j$, either $i > d$ or $j > e$, or $i = d$ and $j = e$. By the maximality assumption on $d$ and $e$, we have that $s_i t_j \notin \mathfrak{I}$ whenever $i > d$ or $j > e$. It immediately follows that $s_d t_e \notin \mathfrak{I}$ also. But $s_d$ and $t_e$ are homogeneous elements whose product is in $\mathfrak{I}$, and by hypothesis we conclude that either $s_d$ or $t_e$ is an element of $\mathfrak{I}$, a contradiction. \qed

**Lemma 3.11.** For a finitely generated graded module $M$ over a graded ring $A$, $\text{Ann}_A(M)$ is a graded ideal of $A$.

**Proof.** Let $a \in \text{Ann}_A(M)$, by using the grading on $A$ we may decompose $a$ into its homogeneous components, say $a = \sum_{j \in \mathbb{Z}} a_j$ where $a_j \in A_j$. For sake of contradiction, assume $a_k \in A_k$ is a homogeneous component of $a$ such that $a_k \cdot m \neq 0$ for some $m \in M$. We may assume that $m$ is a homogeneous element of $M$, say $m \in M_n$.

We have that $a \cdot m = 0$ but $a_k \cdot m \neq 0$. Thus, $a_k \cdot m = -\sum_{j \neq k} a_j \cdot m$. Now by considering the degree of each side of this equation, we have that $\deg(a_k) = \deg(a_j)$ for every $j$. Since $a$ was decomposed into its homogeneous components, this is only possible if $a_j = 0$ for every $j \neq k$, in which case $a = a_k$ and so $a_k \in \text{Ann}_A(M)$.

Note that, by definition, $\text{Ann}_A(M) = \text{Ann}_A(M(d))$, for every $d \in \mathbb{Z}$.

**Definition 3.12.** For a prime $p$ in a graded ring, define $p^*$ as the largest, graded ideal contained in $p$; i.e. $p^*$ is the ideal generated by all homogeneous elements of $p$.

**Lemma 3.13.** Let $A$ be a $\mathbb{Z}$-graded ring, $p \in \text{Spec}(A)$. Then, $p^* \in \text{Spec}(A)$.

**Proof.** $p^*$ is graded so we apply lemma 3.10. If $x$ and $y$ are homogeneous elements such that $xy \in p^* \subseteq p$ then by the primeness of $p$, either $x$ or $y$ is in $p$. But both $x$ and $y$ are homogeneous and thus by definition, either $x$ or $y$ must be in $p^*$. \qed
Proposition 3.14. Let $A$ be a $\mathbb{Z}$-graded ring with $M$ a finitely generated graded $A$-module.

i) If $p \in \text{Ass}_A(M)$, then $p$ is a graded ideal of $A$ and it is the annihilator of a homogeneous element in $A$.

ii) If $\mathcal{I}$ is a graded ideal in $A$, then all primes in $A$ that are minimal as primes containing $\mathcal{I}$ are graded.

iii) The minimal elements of $\text{Supp}(M)$ are graded. (These primes are called “the minimal primes of $M$”.)

Proof: i) We proceed as in [3], proposition 3.12 pg. 99.

Let $p \in \text{Ass}_A(M)$, then for some $m \in M$, $p = \text{Ann}_A(m)$, and $m \neq 0$, by definition. Let $f \in p$, and suppose $f = \sum_{j=1}^{s} f_j$ is the unique decomposition of $f$ into its homogeneous components, $f_j$, in $A$.

Suppose the degree of $f_j$ is the (possibly negative) integer $d_j$, and that $d_a < d_b$ for $a < b$. Similarly, decompose $m$ as $\sum_{i=1}^{t} m_i$, where $\deg(m_i) = e_i$, and $e_a < e_b$ whenever $a < b$.

Observe that $f_1 m_1 = 0$, since $f_1 m_1$ is the unique term of lowest degree in the product $fm$, and $fm = 0$.

To prove that $p$ is graded, we need to show that $f_i \in p$ for every $i$. By induction on $s$, it suffices to show that $f_1 \in p$.

Now we will use induction on $t$: the number of homogeneous components of $m$. If $t = 1$, then $m = m_1$ and $0 = fm = \sum_{j=1}^{s} f_j m_1 = f_1 m_1 + \text{higher degree terms}$. By the comment above, $f_1 m_1 = 0$, but $m_1 = m$, so we are done.

Now let $t > 1$, and suppose for all $n \in M$ with fewer than $t$ nonzero homogeneous components, such that $Ann_A(n)$ is a prime ideal, we have that $Ann_A(n)$ is a graded ideal.

Suppose that $f_1$ is not an element of $p$, thus $f_1 m \neq 0$ and $p$ is not a graded ideal.

Observe that $f_1 m = \sum_{i=2}^{s} f_i m_i$ since $f_2 m_i = 0$. Let $\mathcal{I} = Ann_A(f_1 m)$. $\mathcal{I}$ is a proper ideal since $f_1 m \neq 0$. $\mathcal{I}$ is a prime ideal, for if $a, b \in A$ and $ab \in \mathcal{I}$, then $a f_1 m = 0$; so that $a f_1 \in p$. Now, $f_1 \notin p$, so $a \in p$ or $b \in p$; say $a \in p$, then $am = 0$ so $a f_1 m = 0$ and $a \in \mathcal{I}$.

Thus by our induction hypothesis, $\mathcal{I}$ is a graded prime ideal. Also, $p \subseteq \mathcal{I}$. Now, $p$ cannot be equal to $\mathcal{I}$ since $\mathcal{I}$ is graded and $p$ is not. So, $p \subseteq \mathcal{I}$ is a proper containment. There exists $g \in \mathcal{I}$ such that $g \notin p$. By definition, $gf_1 m = 0$ so that $gf_1 \in p$, but $p$ is prime so $f_1 \in p$, a contradiction.

Since $p$ is graded, $p m_i = 0$ for each $i$, then $p \subseteq \bigcap_i Ann_A(m_i) \subseteq Ann_A(m) = p$, and so $p = \bigcap_i Ann_A(m_i)$.

Now the product of ideals is always contained in the intersection, so $\Pi_i Ann_A(m_i) \subseteq \bigcap_i Ann_A(m_i) = p$.

Since $p$ is prime, there exists an $i$ such that $Ann_A(m_i) \subseteq p$ and therefore $Ann_A(m_i) = p$.

ii) The primes of $A$ minimal over $\mathcal{I}$ are in one-one correspondence with the minimal primes of $A/\mathcal{I}$; since $\mathcal{I}$ is graded, so is $A/\mathcal{I}$. Since $A$ is Noetherian, so is $A/\mathcal{I}$, and every minimal prime in $A/\mathcal{I}$ is an associated prime of $A/\mathcal{I}$ using proposition 2.24, thus is graded by i). So, the corresponding prime ideal in $A$, minimal over $\mathcal{I}$, is also graded.

Alternatively, we could avoid 2.24 for part ii), by noting that if $A$ is graded and $p$ is minimal amongst prime ideals in $A$, then since $p^*$ is prime and contained in $p$, it must be equal to $p$, so $p$ is graded.

iii) : This follows from ii), with $\mathcal{I} = Ann_A(M)$. \qed

3.2 Dimension and Localization in $C(A)$

Here we prove a lemma for a $\mathbb{Z}$-graded ring $A$ which states the relation between the height of a prime ideal $p \in A$ and the graded height of $p^*$, defined below.

Recall from section 3.1 that $p^*$ is the largest graded prime contained in $p$. Later we use the results of this section to prove a theorem which states that the Krull dimension of a positively graded ring is equal to its graded Krull dimension which we define presently.

Recall from section 2.5 that that the height of a prime ideal $p$, graded or not, is the longest length of a chain of primes $p_0 \subset \cdots \subset p_n = p$. Here we define the graded height of a graded prime ideal $p$ in the
Lemma 3.20. If \( A \) is a ring, by requiring that all chains of primes as above consist of graded prime ideals. We denote the graded height of the graded prime \( p \) by \( \text{ht}(p) \).

One notes that for every graded prime \( p \), \( \text{ht}(p) \geq *\text{ht}(p) \).

Definition 3.15. The graded Krull dimension of a graded \( A \)-module \( M \), denoted \(*\text{dim}_A(M)\), is the greatest \( D \) such that there exists a strictly increasing chain

\[
p_0 \subset \ldots \subset p_D
\]
of graded prime ideals in \( R \) such that \( \text{Ann}_A(M) \subseteq p_0 \). For the zero module, we define \(*\text{dim}(0) = -\infty\).

One observes that, for any graded \( A \)-module \( M \),

- \(*\text{dim}_A(M) \leq \text{dim}_A(M)\).

Also, since \( \text{Ann}_A(M) = \text{Ann}_A(M(n)) \), for every \( n \in \mathbb{Z} \),

- \( \text{dim}_A(M) = \text{dim}_A(M(n)) \), for every \( n \in \mathbb{Z} \).
- \(*\text{dim}_A(M) = *\text{dim}_A(M(n)) \), for every \( n \in \mathbb{Z} \).

Example 3.16. The graded Krull dimension of \( K_0[t, t^{-1}] \)

Let \( K_0 \) be a field and consider \( K_0[t, t^{-1}] \), where \( \deg(t) > 0 \) (i.e. the Laurent polynomial ring over the field \( K_0 \)).

Observe that the set \( D = \{ t^i : i \in \mathbb{N} \cup \{0\} \} \) is multiplicatively closed with 1 and does not contain 0. We will localize \( K_0[t] \) at \( D \). One sees that \( D^{-1}K_0[t] \cong K_0[t, t^{-1}] \) (these rings are isomorphic as graded rings). By basic algebra, \( K_0 \) being a field implies that \( K_0[t] \) is a PID, and the localization of a PID is itself a PID, so we conclude that \( D^{-1}K_0[t] \) is a PID. Also observe that every PID which is not a field has Krull dimension 1. Consider the element \( 1 + t \) of \( K_0[t, t^{-1}] \). This element clearly has no inverse in \( K_0[t, t^{-1}] \) as \( \frac{1}{1+t} \) is a series with a non-finite number of nonzero terms. Thus we see that \( K_0[t, t^{-1}] \) is not a field, but it is a PID and so \( \text{dim}(K_0[t, t^{-1}]) = 1 \). Observe also that every nonzero homogeneous element in \( K_0[t, t^{-1}] \) has an inverse in \( K_0[t, t^{-1}] \).

Suppose \( p \) is a graded prime ideal in \( K_0[t, t^{-1}] \). If \( p \neq 0 \), then there exists a nonzero homogeneous element \( s \) of \( p \). But then \( s \) has an inverse \( s^{-1} \) in \( K_0[t, t^{-1}] \). Then, \( 1 = ss^{-1} \) is an element of \( p \) and so \( p = K_0[t, t^{-1}] \), but this is a contradiction since prime ideals are by definition proper ideals.

Thus, the only graded prime is 0 and \(*\text{dim}(K_0[t, t^{-1}]) = 0 \). Therefore,

\[ *\text{dim}(K_0[t, t^{-1}]) = 0 \text{ while } \text{dim}(K_0[t, t^{-1}]) = 1 \ . \]

Definition 3.17. A graded field is a \( \mathbb{Z} \)-graded ring in which every nonzero, homogeneous element is invertible.

Proposition 3.18. Let \( K \) be a graded field, then \( K_0 \) is a field and either \( K_i = 0 \) for every nonzero \( i \) or \( K \cong K_0[t, t^{-1}] \) as a graded ring, where \( \deg(t) > 0 \).


Definition 3.19. Let \( M \in \mathcal{C}(A) \), let \( S \) be a multiplicatively closed set of homogeneous elements in \( A \) which contains 1. Call such a multiplicatively closed subset a “graded MCS in \( A \)”. Define \( S^{-1}M \) to be the usual localization of \( M \). If \( x \) is a homogeneous element of \( M \), and \( x/a \in S^{-1}M \), we define \( \deg(x/a) = \deg(x) - \deg(a) \), and we may define a grading on \( S^{-1}M \) by letting \( (S^{-1}M)_i = \{ x/a \in S^{-1}M : x \text{ is homogeneous, } \deg(x/a) = i \} \). Given this grading, we call \( S^{-1}M \) the graded localization of \( M \).

One then obtains the graded ring of fractions, \( S^{-1}A \). Note that \( S^{-1}M \) is a graded \( S^{-1}A \)-module.

Standard facts about localization in the ungraded case quickly yield:

Lemma 3.20. If \( M \in \mathcal{C}(A) \), and \( S \) is a graded MCS in \( A \), then...
a. $S^{-1}M \in \mathcal{C}(S^{-1}A)$.

b. If $A$ is a Noetherian ring then $S^{-1}A$ is a Noetherian ring.

c. There is a one-one, inclusion-preserving correspondence between the prime ideals in $R$ that are disjoint from $S$, and the prime ideals in $S^{-1}R$ given by $p \mapsto S^{-1}p$; moreover, correspondence restricts to a one-one correspondence between the graded prime ideals in $R$ disjoint from $S$ and the graded prime ideals in $S^{-1}R$, and further restricts to a one-one correspondence between the ideals (all graded) in $\text{Ass}_{S^{-1}A}(M)$ that are disjoint from $S$, and the ideals (also all graded) in $\text{Ass}_{S^{-1}A}S^{-1}M$.

Lemma 3.21. Let $A$ be a $\mathbb{Z}$-graded Noetherian ring, $p \in \text{Spec}(A)$, and define $p^*$ as the largest, graded prime ideal contained in $p$. Let $\text{ht}(p) = d$.

i) If $q \in \text{Spec}(A)$ and $p^* \subseteq q \subseteq p$ then either $q = p$ or $q = p^*$.

ii) For any prime $p \in \text{Spec}(A)$, there exists a chain of primes $q_0 \subset \cdots \subset q_d = p$, such that $q_0, \cdots q_{d-1}$ are all graded.

iii) If $p$ is graded then there exists a chain of graded prime ideals such that

$$p_0 \subset p_1 \subset \cdots \subset p_d = p,$$

i.e. The graded height of a graded prime equals the usual height.

iv) If $p$ is not graded ($p^*$ is a proper subset of $p$), then

$$\text{ht}(p) = \text{ht}(p^*) + 1 = \ast \text{ht}(p^*) + 1.$$  

Proof.  

i) If $p$ is graded then $p^* = p$ and thus $q = p^* = p$.

So, suppose instead that $p$ is ungraded, that is, $p^*$ is a proper subset of $p$. We know that there is an inclusion preserving bijection between the set of prime ideals of $A$ which contain $p^*$ and the set of prime ideals of $A/p^*$. Thus,

$$p^* \subseteq q \subseteq p \leftrightarrow p^*/p^* \subseteq q/p^* \subseteq p/p^*.$$  

Replacing $A$ by $A/p^*$, which is again a graded ring, we may assume that $A$ is a domain and $p^* = 0$.

Now we have $0 \subseteq q \subseteq p$. We define $S = \{s \in A : s$ is homogeneous, $s \neq 0\}$. This is a graded MCS, not containing 0, so we may construct the localization $S^{-1}A$ which is a $\mathbb{Z}$-graded ring as previously discussed. Since $p^* = 0$, $p$ can’t have any nonzero, homogeneous elements so it must be true that $S \cap p = \emptyset$. By the basic properties of localization, there is a bijection as follows:

The set of prime ideals, graded or not, in $S^{-1}A \leftrightarrow$

The set of prime ideals of $A$, graded or not, which do not intersect $S$.

$S^{-1}A$ is a $\mathbb{Z}$-graded ring and by definition of $S$, every homogeneous element of $S^{-1}A$ has an inverse. So $S^{-1}A$ is a graded field by definition. So, by example 3.16, $S^{-1}A$ must have Krull dimension equal to 0 or 1.

However, $0 \subseteq S^{-1}q \subseteq S^{-1}p$ is a chain of primes in $S^{-1}A$ and to satisfy the dimension being 1 or 0 we must have that either $S^{-1}q = 0$ or $S^{-1}q = p$. But using the bijection of primes in the localization to primes in $A$, we conclude that either $q = 0$ or $q = p$.

ii) This proof follows that of Bruns and Herzog [2].
Now suppose $d \geq 2$. Then there exists a chain of prime ideals

$$p_0 \subset p_1 \subset \cdots \subset p_d = p,$$

which cannot be lengthened and still end in $p$.

Now observe that the $ht(p_{d-1}) = d - 1$, for if there is a chain of primes ending in $p_{d-1}$ of length $e$, then there is a chain of primes ending in $p_d = p$ of length $e + 1$, then it must be true that $e + 1 \leq d$ which implies that $ht(p_{d-1}) \leq d - 1$. However $p_0 \subset \cdots \subset p_{d-1}$ has length $d - 1$ so $ht(p_{d-1}) \geq d - 1$. Therefore, $ht(p_d) = d - 1$.

Thus by the induction hypothesis, there is a chain $q_0 \subset \cdots \subset q_{d-2} \subset p_{d-1} \subset p_d$ of primes with all $q_i$ graded.

**Case 1:** $p$ is not graded.

If we star each side of $q_{d-2} \subset p_{d-1}$ we get that $q_{d-2} \subset p_{d-1}^\ast$. But since $q_{d-2}$ is graded, we have that $q_{d-2} = q_{d-2}^\ast$ and thus,

$$q_{d-2} \subset p_{d-1}^\ast \subset p_d.\)$$

Where the last inequality is strict because $p_d$ is not graded by hypothesis.

Now suppose that $p_d^\ast = q_{d-2}$. Then, $q_{d-2} = p_d^\ast \subset p_{d-1} \subset p_d$. But this is a contradiction of part a of this lemma, since $p_{d-1} \neq q_{d-2}$, nor does it equal $p_d$.

Therefore, $p_d^\ast \neq q_{d-2}$ and thus $q_{d-2} \subset p_d^\ast \subset p_d$. By taking $q_{d-1} = p_d^\ast$ we conclude our proof by induction in this case.

**Case 2:** $p$ is graded.

Since $q_{d-2} \subset p_d$ and both are graded, there exists a nonzero homogeneous element $a \in p_d$ which is not an element of $q_{d-2}$. Define $I = q_{d-2} + \langle a \rangle \subset p_d$, a graded ideal. Now, there exists a prime $q_{d-1}$ minimal over $I$, contained in $p_d$. By lemma, ?? we know that $q_{d-1}$ is graded since $I$ is graded. We have that $q_{d-2} \subset I \subset q_{d-1} \subset p_d$; furthermore, $q_{d-1}/q_{d-2}$ is a minimal prime over $I/q_{d-2}$.

We now use the principal ideal theorem:

The ring $A/q_{d-2}$ is a domain and, the ideal $I/q_{d-2}$ is a principal ideal generated by the nonzero element $a + q_{d-2}$. Since $q_{d-1}/q_{d-2}$ is minimal over $I/q_{d-2}$, $ht(q_{d-1}/q_{d-2}) = 1$ in $A/q_{d-2}$, by the principal ideal theorem.

We have the chain $q_{d-2} \subset I \subset q_{d-1} \subset p_d$. Suppose $p_d = q_{d-1}$, then $q_{d-2} \subset p_{d-1} \subset p_d = q_{d-1}$. If we pass to the quotient we get the chain $0 = q_{d-2}/q_{d-2} \subset p_{d-1}/q_{d-2} \subset q_{d-1}/q_{d-2}$. So, $ht(q_{d-1}/q_{d-2}) \geq 2$. But this contradicts the fact that $ht(q_{d-1}/q_{d-2}) = 1$, as shown above. Therefore, $q_{d-1} \subset p_d$.

Thus, we have the chain $q_0 \subset \cdots \subset q_d = p$ with $q_0, \cdots, q_{d-1}$ all graded.

iii) Suppose $p$ is graded. By part ii of this lemma, there exists graded primes $q_0, \cdots, q_{d-1}$ such that $q_0 \subset \cdots \subset q_{d-1} \subset p$. We know that $^\ast ht(p) \leq ht(p) = d$. But, since $p$ is graded, the chain above is graded and thus $^\ast ht(p) = ht(p)$.

iv) Suppose $p$ is not graded. We must have that $ht(p) \geq ht(p^\ast) + 1$, since $p^\ast \subset p$.

By part ii of this lemma, there is a chain of primes $q_0 \subset \cdots \subset q_{d-1} \subset p_d = p$, where each $q_i$ is graded.

Thus, $q_0 \subset \cdots \subset q_{d-2} \subset p^\ast$ since $q_{d-1} \subset p \Rightarrow q_{d-1} \subset p^\ast \Rightarrow q_{d-2} \subset p$. So we have that $ht(p^\ast) \geq d - 1 = ht(p) - 1$ and $ht(p) \leq ht(p^\ast) + 1$.

Therefore, $ht(p) = ht(p^\ast) + 1$. Moreover, since $p^\ast$ is graded, we have by part iii of this lemma that $ht(p^\ast) = ^\ast ht(p^\ast)$. So,

$$ht(p) = ht(p^\ast) + 1 = ^\ast ht(p^\ast) + 1.$$
3.3 Elementary EGA II Results

We summarize here some definitions and results from Grothendieck’s EGA II which will be useful in this paper [5].

In this section $S$ is a positively graded ring (i.e. with $S_n = 0$ for $n < 0$), but we do not assume that $S_0$ is a field (as is the case for the ring $R$). We use Grothendieck’s notation throughout. Observe that an $S$-module $M$ need not be positively graded.

First, some definitions.

- If $d > 0$, $S^{(d)} = \oplus_{n \geq 0} S_{nd}$. This is also a positively graded ring.
- If $d > 0$, and $k$ is an integer such that $0 \leq k \leq d - 1$, $M^{(d,k)} = \oplus_{n \in \mathbb{Z}} M_{nd+k}$. Note that $M^{(d,k)}$ is an $S^{(d)}$-module.
- Homomorphisms of graded $S$-modules and graded rings are degree zero $S$-module homomorphisms or ring homomorphisms.
- A graded $S$-module $F$ is a free graded $S$-module if and only if it is isomorphic as a graded $S$-module to a direct sum of $S$-modules of the form $S(n)$ ($n$ may vary over $\mathbb{Z}$).

Lemma 3.22. (2.1.3, 2.1.4, 2.1.5 of [5]:) Let $S$ be a positively graded ring. Let $E$ be a subset of $S$ consisting entirely of homogeneous elements.

a. $E \subseteq S_+$ generates $S_+$ as an $S$-module if and only if $E$ generates $S$ as an $S_0$-algebra.

b. $S_+$ is a finitely generated ideal in $S$ if and only if $S$ is a finitely generated $S_0$-algebra.

c. $S$ is Noetherian if and only if $S_0$ is Noetherian and $S$ is a finitely generated $S_0$-algebra.

Proof.

a. $\Rightarrow$:

Suppose $E \subseteq S_+$ generates $S_+$ as an $S$-module.

Let’s proceed by induction on degree. Let $r \in S_1$ be homogeneous. Clearly $r$ is an element of $S_+$. Thus, by our hypothesis, $\exists e_i \in E, r_i \in S$ such that $r = \Sigma_i r_i e_i$. For each $i$, we have that deg($r_i e_i$) = deg($r_i$) + deg($e_i$). Since $E \subseteq S_+$, every element in $E$ has degree greater than or equal to 1. Thus, for every $i$, deg($r_i$) = 0 and deg($e_i$) = 1. So, $r_i \in S_0$ for every $i$ and $r$ is in the $S_0$-algebra generated by $E$.

Now suppose that all homogeneous elements of $S$ of degree less than or equal to $k - 1$ are members of the $S_0$-algebra generated by $E$.

Then, for $r \in S_k$, there exists $r_j \in S$ and $e_j \in E$ such that $r = \Sigma_j r_j e_j$. Then, deg($r_j$) + deg($e_j$) = $k$ for every $j$.

Since deg($e_j$) $\geq 1$ for each $j$, we have that deg($r_j$) < $k$ for each $j$. By the induction hypothesis, each $r_j$ may be written as a sum of monomials over $E$ with coefficients in $S_0$, but then $r = \Sigma_j r_j e_j$ may be written in the same form.

$\Leftarrow$: Suppose $E$ generates $S$ as an $S_0$-algebra. Then, if $s \in S_+$ there exists an $n$ and elements $e_1, \ldots, e_n \in E$ such that $s$ may be written $s = \Sigma_I r_I e^I$, where $I = (i_1, \ldots, i_n)$ is a multi-index with at least one component $i_k \neq 0$, $e^I = e_1^{i_1} \cdots e_n^{i_n}$, $r_I \in S_0$, with $r_I = 0$ for all but finitely many $I$. Thus it is enough to show that a monomial $e^I$, as just described, is in the $S$-span of $E$. If $i_k$ is a component of $I$ that is not zero, then $e^I = (e_1^{i_1} \cdots e_{i_k-1} \cdots e_n^{i_n}) e_k$, which is in the $S$-span of $E$.

b. Observe that $S_+$ being a finitely generated ideal in $S$ is equivalent to $S_+$ being a finitely generated $S$-module. Then we only need to take the set $E$ from part a to be a finite set generating $S_+$ as an ideal.

c. $\Rightarrow$: Suppose $S$ is Noetherian. Consider $S$ as a module over itself, by the Noetherian hypothesis, we have that all $S$-submodules (ideals) of $S$ are finitely generated. $S_+$ is an ideal by definition, so it is a finitely generated $S$-module. By part b of this theorem, this implies that $S$ is a finitely generated $S_0$-algebra. It remains to show that $S_0$ is Noetherian; this follows since $S_0$ is isomorphic to the quotient $S/S_+$.

$\Leftarrow$: This follows from the Hilbert Basis Theorem.
Proposition 3.23. (Lemma 2.1.6 of [5]). Let $M$ be a finitely generated graded $S$-module. Then,

a. For all $n \in \mathbb{Z}$, $M_n$ is a finitely generated $S_0$-module, and there exists an $n_0 \in \mathbb{Z}$ such that $M_k = 0$, for $k \leq n_0$.

b. There exists an $n_1 \in \mathbb{Z}$ and a positive integer $h$ such that for every $n \geq n_1$, $M_{n+h} = S_h M_n$.

c. For every $(d,k)$, where $d$ is a positive integer and $k$ is an integer, $0 \leq k \leq d-1$, $M^{(d,k)}$ is a finitely generated $S^{(d)}$-module.

d. For every positive integer $d$, $S^{(d)}$ is a finitely generated $S_0$-algebra.

e. There exists a positive integer $h$ such that $S_{m'h} = (S_h)^m$, for every $m \geq 0$.

f. For every positive integer $n$, there exists a nonnegative integer $m_0$ such that $S_m \subseteq S_{m_0}^n$, for $m \geq m_0$.

Given this proposition, we can define the Poincaré series of a finitely generated graded $S$-module, if $S_0$ is an Artinian ring.

First, we denote the ring $\mathbb{Z}[[t]][t^{-1}]$ by $\mathbb{Z}(t)$; thus an element of $\mathbb{Z}(t)$ is a formal Laurent series $f(t)$ with integer coefficients; there always exists an $n \in \mathbb{Z}$ with $t^n f(t) \in \mathbb{Z}[t]$.

Definition 3.24. Suppose $S$ is a positively graded ring, with $S_0$ Artinian, and $M$ is a finitely generated graded $S$-module. Then the Poincaré series of $M$ is the formal Laurent series with integer coefficients

$$P_M(t) = \sum_{i \in \mathbb{Z}} \ell_{S_0}(M_i) t^i,$$

where $\ell_{S_0}(M_i)$ is the length of the finitely generated module $M_i$ over the Artinian ring $S_0$. Furthermore, since there is an integer $n_0$ such that $M_j = 0$ for $j < n_0$, $P_M(t) \in \mathbb{Z}(t)$.

Sometimes the Poincaré series is called the Hilbert series, or the Hilbert-Poincaré series.

Example 3.25. Find the Poincaré series of the ring $k[x_0, \ldots, x_r]$, where $k$ is a field and the degree of each $x_i$ is 1.

By definition, $k[x_0, x_1, \ldots, x_r]_d$ is the set of all monomials in $r+1$ variables whose degree is $d$. An arbitrary element looks like $x_0^{a_0} x_1^{a_1} \cdots x_r^{a_r}$, where $a_0 + a_1 + \cdots + a_r = d$. We recognize that calculating the dimension of the degree $d$ component is equivalent to the counting problem, "How many ways can $d$ chips be placed in $r+1$ buckets?" One can readily check the answer is

$$\binom{r + d}{r}.

The function $1/(1 - t)$ has power series equal to $1 + t + t^2 + t^3 + \cdots$ and recall the power series for $1/(1-t)^{r+1}$ is given by $\sum_{d=0}^{\infty} \binom{d+r}{r} t^d$. For a quick check of this fact, one can take successive derivatives of $1/(1-t)$ and observe the pattern.

Thus, the $d^{th}$ coefficient of the function $1/(1-t)^{r+1}$, written in power series form, is identical to the dimension of the $d^{th}$ component of the graded polynomial ring in $r$ variables over $K$.

Example 3.26. Find the Poincaré series of the ring $k[x_0, \ldots, x_r]$, where $k$ is a field and each $x_i$ has $\deg(x_i) = d_i > 0$.

Observe that in the graded decomposition of $k[x_0, \ldots, x_r]$, there is a nonzero graded component only for the following degrees: one for each $d_0$ through $d_r$, all multiples of each $d_i$ from $d_0$ through $d_r$, and all products of all multiples of $d_0$ through $d_r$ i.e. the Poincaré series will have the form, $(1 + t^{d_0} + t^{d_0+2} + \cdots)(1 + t^{d_1} + t^{d_1+2} + \cdots)(1 + t^{d_r} + t^{d_r+2} + \cdots)$. Then,

$$P(k[x_0, \ldots, x_r]) = \frac{1}{\Pi_i (1 - t^{d_i})}.$$
More generally, we have the following theorem.

**Theorem 3.27.** (The Hilbert-Serre Theorem) Let \( M \in C(R) \). Suppose that \( R \) is generated as a \( k \)-algebra by elements \( x_1, \ldots, x_n \) of positive degrees \( d_1, \ldots, d_n \). Then,

\[
P_M(t) = \frac{q(t)}{\prod_{i=1}^{n}(1-t^{d_i})},
\]

where \( q(t) \in \mathbb{Z}[t, t^{-1}] \).

Furthermore, if \( M \) has no elements of negative degree, \( q(t) \in \mathbb{Z}[t] \).

We will need the idea of a tensor product of graded modules over a graded ring, and present here Grothendieck’s definition.

If \( M \) and \( N \) are graded \( S \)-modules, define the graded abelian group

\[
M \otimes_{\mathbb{Z}} N
\]
defined by

\[
(M \otimes_{\mathbb{Z}} N)_k = \oplus_{i+j=k} M_i \otimes_{\mathbb{Z}} N_j,
\]

where \( k \in \mathbb{Z} \).

Let \( P \) be the subgroup of \( M \otimes_{\mathbb{Z}} N \) generated by elements of the form \((xs \otimes y) - (x \otimes sy)\), where \( x \in M \), \( s \in S \) and \( y \in N \). One can show that \( P \) is the graded subgroup of \( M \otimes_{\mathbb{Z}} N \) generated by elements \((x_i s_j \otimes y_k) - (x_i \otimes s_j y_k)\), where \( x_i \in M_i \), \( s_j \in S_j \), and \( y_k \in N_k \) and \( i, j, k \) are arbitrary. Thus we may define the graded \( S \)-module

\[
M \otimes_S N = (M \otimes_{\mathbb{Z}} N)/P.
\]

More elementary facts about Poincaré series are collected below. The graded ring \( S \) is as in Definition 3.23.

- Suppose \( S \) and \( \tilde{S} \) are two positively graded rings, with \( S_0 = \tilde{S}_0 \). In addition suppose that a graded abelian group \( M \) is simultaneously in \( C(S) \) and \( C(\tilde{S}) \). Then whether we consider \( M \) as an \( S \)-module or as a \( \tilde{S} \)-module, its Poincaré series does not change.
- If \( M \in C(S) \), then so is \( M(n) \), for every \( n \in \mathbb{Z} \), and

\[
P_{M(n)}(t) = t^{-n}P_M(t).
\]
- If \( 0 \to P \to M \to N \to 0 \) is a short exact sequence in \( C(S) \), then

\[
P_M(t) = P_P(t) + P_N(t).
\]
- If \( M, N \in C(S) \), then \( P_{M \otimes_{S_0} N}(t) = P_M(t)P_N(t) \).

**Example 3.28.** Let \( R = k[x_1, \ldots, x_n] \) where the degree of each \( x_i \) equals 1, and let \( f \) be a degree \( d \) homogeneous polynomial in \( R \). For the reader familiar with algebraic geometry, the ring \( R/(f) \) is the coordinate ring of a degree \( d \) hypersurface in projective space. Then, there is a short exact sequence in \( C(R) \),

\[
0 \to R(-d) \xrightarrow{f} R \to R/(f) \to 0,
\]

using the additivity of the Poincaré series and that \( P_{R(-d)}(t) = t^dP_R(t) \), we have

\[
P_{R/(f)}(t) = P_R(t) - P_{R(-d)}(t)
\]

\[
= \frac{1}{(1-t)^n} - \frac{t^d}{(1-t)^n}
\]

\[
= \frac{(1-t)(1+t+\cdots+t^{d-1})}{(1-t)^n}
\]

\[
= \frac{1+t+\cdots+t^{d-1}}{(1-t)^{n-1}}.
\]
Notice that the Krull dimension of \( R/(f) \) is \( n - 1 \) and that this is also the order of the pole at \( t = 1 \) of the Poincare series - this is not a coincidence as we shall see later.

Suppose now that we have a non-standard grading on \( R \), say that the degree of \( x_i \) is equal to \( d_i \). We compute

\[
P_{R/(f)}(t) = P_R(t) - P_{R(-d)}(t)
\]

\[
= \frac{1}{(1 - t^{d_1}) \cdots (1 - t^{d_n})} - \frac{t^d}{(1 - t)(1 + t + \cdots + t^{d-1})}
\]

\[
= \frac{1}{(1 - t)(1 + t + \cdots + t^{d-1})} \left[ \frac{1 + t + \cdots + t^{d-1} - t^d}{1 + t + \cdots + t^{d-1}} \right].
\]

4 In the Category \( C(R) \)

Lemma 4.1. Let \( X \in C(R) \), and ignoring the grading on \( X \), suppose that \( X \) is also a simple \( R \)-module. Then, there exists an integer \( d \) and a graded isomorphism of graded \( R \)-modules \( X \cong k(-d) \)

Proof. Note that \( k(-d) \) is a graded \( R \)-module by defining \( r \cdot 0 = 0 \) for every \( r \in m \).

By definition of simple \( R \)-module, the only \( R \)-submodules of \( X \) (graded or otherwise) are the zero submodule and itself. Let \( x \in X \) and suppose that \( x \) is a homogeneous element of degree \( d \). Certainly \( Rx \) is an \( R \)-submodule of \( X \), moreover it is a graded \( R \)-submodule since it is generated by a homogeneous element. Since \( x \) was picked to be nonzero, we have that \( Rx = X \).

Suppose \( \phi : R \rightarrow X(d) \) is given by mapping an element \( r \in R \) to \( r \cdot x \) in \( X(d) \). Then, \( \phi \) preserves degree and is a graded morphism, it is also surjective since \( X \) is generated as an \( R \)-module by \( x \). By the first isomorphism theorem for modules, \( R/\ker(\phi) \cong X(d) \). Since \( \phi \) is a graded morphism, \( \ker(\phi) \) is a graded \( R \)-submodule of \( R \) (a graded ideal) and this isomorphism is also a graded isomorphism.

By definition, \( \ker(\phi) = \text{Ann}_R(x) \). Also, since \( X \) is simple, \( X(d) \) is too, so the above isomorphism implies that the only ideals of \( R/\text{Ann}_R(x) \) are zero and itself. Therefore, \( \text{Ann}_R(x) \) is a maximal, graded ideal in \( R \) and so \( \text{Ann}_R(x) = m \) and \( R/m \) is by definition isomorphic to \( k \) as an object in \( C(R) \). Thus, we have a graded isomorphism in \( C(R) \), \( k \cong X(d) \), or by shifting the grading on each side, \( k(-d) \cong X \). Observe that this proof did not require the restriction to \( C^<(R) \).

\[\square\]

Lemma 4.2. Graded Nakayama’s Lemma

Let \( M \in C(R) \). If \( M \neq 0 \), then \( mM \neq M \).

Proof. By proposition 3.23 there exists an \( i \) which is the smallest integer such that \( M_i \neq 0 \). If \( m = 0 \), then \( mM_m \neq M_m \), since \( M \) was taken to be nonzero. If \( m \neq 0 \), then \( M_i \nsubseteq mM_m \) since the degree of any element of \( m \) is strictly larger than zero. Thus, \( mM_m \neq M_m \).

\[\square\]

Theorem 4.3. Let \( X \in C(R) \), \( X \neq 0 \), and \( \ell_X(X) < \infty \). Then, the dimension of \( X \) as a vector space over \( k \), \( \text{vdim}_k(X) \), is equal to \( \ell_X(X) \).

Proof. We will use induction on \( \ell_X(X) \). Recall that the \( R \)-submodules in an \( R \)-composition series for \( X \) need not necessarily be graded, and that \( \ell_X(X) \) is computed with such a composition series.

Suppose \( \ell_X(X) = 1 \), then \( X \) is a simple \( R \)-module. It is also a graded \( R \)-module and by lemma 4.1, \( X \cong k(t) \) for some integer \( t \). Therefore, \( \text{vdim}_k(X) = 1 \), so the length and dimension are equal.

Now suppose the hypothesis holds true for \( R \)-modules with length strictly less than \( d = \ell_X(X) > 1 \), i.e. any graded \( R \)-module in \( C(R) \) with length strictly less than \( d \) has its \( R \)-length equal to its \( k \)-dimension.

The sequence \( 0 \rightarrow mM \rightarrow X \rightarrow X/mX \rightarrow 0 \) is a short exact sequence of \( R \)-modules and so \( \ell_X(mX) + \ell_X(X/mX) = \ell_X(X) < \infty \).
By graded Nakayama’s lemma, $mX \neq X$ which implies that $\ell_R(mX) < \ell_R(X)$. We apply the induction hypothesis, $\ell_R(mX) = \text{vdim}_k(mX)$. Also recall example 2.15, which implies that $\ell_R(X/mX) = \text{vdim}_k(X/mX)$. Therefore, we have:

$$\ell_R(X) = \ell_R(mX) + \ell_R(X/mX)$$

$$= \text{vdim}_k(mX) + \text{vdim}_k(X/mX)$$

$$= \text{vdim}_k(X).$$

Lemma 4.4. Let $M \in \mathcal{C}(R)$. For any positive integer $n$, $M/m^nM$ is a finite dimensional vector space over $k$.

Proof. Let $i$ be an integer greater than or equal to 0. $M$ is a Noetherian module, so all of its $R$-submodules are finitely generated over $R$, in particular consider the $R$-submodules $m^iM$ and $m^{i+1}M$.

We have that $m^iM/m^{i+1}M$ is a finitely generated $R$-module. Say $\{z_1, \ldots, z_i\}$ is an $R$-generating set for $m^iM$. Let $z \in m^iM/m^{i+1}M$, then $z = \Sigma_i r_j z_j + m^{i+1}M$, where $r_j \in R$. Without loss of generality, pick a $j$ such that $r_j z_j \neq 0$. Since $z_j \in m^iM$, $\deg(r_j) = 0$, else $r_j z_j$ would be zero in the quotient. Thus, $r_j \in k$ and $\{z_1, \ldots, z_i\}$ is a $k$-spanning set of $m^iM/m^{i+1}M$, so that $m^iM/m^{i+1}M$ is a finite dimensional vector space over $k$.

Finally, for any fixed, positive integer $n$, we have the following chain of $R$-submodules of $M$:

$$m^nM \subseteq m^{n-1}M \ldots \subseteq mM \subseteq M.$$

By above, each consecutive quotient in this chain is a finite dimensional vector space over $k$. Therefore, $M/m^nM$ is a finite dimensional vector space over $k$.

Definition 4.5. If $w_1, \ldots, w_t \in m$ are homogeneous elements, the graded subring of $R$ generated by $k$ and $w_1, \ldots, w_t$ will be denoted by $k(w_1, \ldots, w_t)$. The superfluous ideal in $k(w_1, \ldots, w_t)$ will be denoted by $m(w)$.

Proposition 4.6. Suppose that $\mathcal{I} \subseteq R$ is an ideal in $R$ generated by homogeneous elements $w_1, \ldots, w_t \in m$. Then, if $M$ is in $\mathcal{C}(R)$,

i) For every $n \geq 0$, $\mathcal{I}^nM = m(w)^nM$.

ii) $M/\mathcal{I}M$ is in $\mathcal{C}(k)$ if and only if $M/m(w)M$ is also in $\mathcal{C}(k)$.

Proof. If $\alpha = (\alpha_1, \ldots, \alpha_t)$ is a multi-index of nonnegative integers, we say that the degree of $\alpha$ is $\alpha_1 + \cdots + \alpha_t$. The “monomial” $w^\alpha$ is equal to $w_1^{\alpha_1} \cdots w_t^{\alpha_t}$. Since $m(w) \subseteq \mathcal{I}$, $m(w)^nM \subseteq \mathcal{I}^nM$, for every $n \geq 0$.

On the other hand, if $x \in \mathcal{I}^nM$, then $x = \sum_{\deg(\alpha) = n} r_\alpha x_\alpha$, with $r_\alpha \in R$ and $x_\alpha \in M$. But this is equal to $\sum_{\deg(\alpha) = n} w^\alpha(r_\alpha x_\alpha)$, with $r_\alpha x_\alpha \in M$, and is thus an element of $m(w)^nM$.

Part ii) follows immediately.

Proposition 4.7. A Characterization of Graded Ideals of Definition in $\mathcal{C}(R)$

Suppose that $\mathcal{I} \subseteq R$ is an ideal in $R$ generated by homogeneous elements $w_1, \ldots, w_t \in m$. Then, if $M$ is in $\mathcal{C}(R)$, the following are equivalent:

i) $M$ is in $\mathcal{C}(k(w_1, \ldots, w_t))$.

ii) $M/\mathcal{I}M$ is in $\mathcal{C}(k)$.

iii) $M/\mathcal{I}^nM$ is in $\mathcal{C}(k)$, for every $n \geq 0$.

iv) There exists an $n_0 > 0$ such that $m^{n_0}M \subseteq \mathcal{I}M$.

v) There exists an $n_0 > 0$ such that $m^{n_0}M \subseteq \mathcal{I}^jM$, for every $j \geq 1$. 

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there is a nonzero homogeneous element $m$ of graded ideals of definition in $C$, which is exactly saying that $M/\mathfrak{M} \in \mathcal{C}(k)$, the second item on the list. So this truly is a characterization of graded ideals of definition in $\mathcal{C}(R)$.

The implications $i) \Rightarrow ii)$ and $i) \Rightarrow iii)$ follow from the previous two propositions: apply 4.4 to $k\langle w_1, \ldots, w_t \rangle$ instead of $R$, and $m(w)$ instead of $m$, then use 4.5.

For $ii) \Rightarrow i)$: suppose that $M/\mathfrak{M}$ is a finite dimensional vector space over $k$, say $x_1, \ldots, x_n$ are homogeneous elements of $M$ such that $\bar{x}_1, \ldots, \bar{x}_n$ is a basis for $M/\mathfrak{M}$ over $k$ (where $\bar{x}_i = x_i + \mathfrak{M})$. Suppose that there is a nonzero homogeneous element $m_0 \in M$ of least positive degree with respect to not being in the span of $x_1, \ldots, x_n$ as a $k(w_1, \ldots, w_t)$-module. Now, we know that

$$m_0 + \mathfrak{M} = \sum_i a_i \bar{x}_i,$$

with $a_i \in k$ for every $i$.

Thus, $m_0 - \sum a_i x_i \in \mathfrak{M}$, so that there are elements $z_1, \ldots, z_t \in M$ such that

$$m_0 - \sum a_i x_i = \sum w_j z_j.$$

Since the degree of $w_j \in R$ is positive, the degree of $z_j \in M$ is strictly less than the degree of $m_0$, for every $j$. Thus each $z_j$ is in the $k\langle w_1, \ldots, w_t \rangle$-span of $x_1, \ldots, x_n$. But then, $m_0$ is in the $k\langle w_1, \ldots, w_t \rangle$-span of $x_1, \ldots, x_n$ as well, a contradiction.

To see that $ii)$ is equivalent to $iv)$, begin by assuming that there is an $n_0$ such that $m^{n_0} M \subseteq \mathfrak{M}$. By lemma 4.4, $M/m^{n_0} M$ is a finite dimensional vector space over $k$. Moreover, there is a surjective $k$-vector space homomorphism taking $M/m^{n_0} M$ onto $M/\mathfrak{M}$. Since dimension splits over short exact sequences, $M/\mathfrak{M}$ is a finite dimensional vector space over $k$.

On the other hand, suppose $x_1, \ldots, x_n$ are homogeneous elements of $M$ such that $\bar{x}_1, \ldots, \bar{x}_n$ form a basis of $M/\mathfrak{M}$ as a $k$-vector space (where $\bar{x}_i = x_i + \mathfrak{M}$). Let $n_0 - 1$ be the maximum degree of the elements $x_1, \ldots, x_n$. Then, $(M/\mathfrak{M})_t = 0$, for $t \geq n_0$. Thus, if $z \in m^{n_0} M$ is a homogeneous element, $z$ has degree greater than or equal to $n_0$, so $z \in \mathfrak{M}$.

Finally, $iv)$ and $v)$ are equivalent: suppose that $n_0$ is such that $m^{n_0} M \subseteq \mathfrak{M}$. Assume that $j \geq 1$, and that $m^{n_0+j} M \subseteq \mathfrak{M}^j$. Then

$$m^{n_0+j} M = \mathfrak{M}^{n_0} m^{n_0+j} M \subseteq m^{n_0+j} \mathfrak{M} = \mathfrak{M}^{n_0+j} M \subseteq \mathfrak{M}^j M = \mathfrak{M}^{j+1} M.$$

As a Corollary (though this could also be easily proved directly):

**Corollary 4.8.** **With the hypotheses of the above theorem, if $\mathfrak{M}$ is a graded ideal of definition for $M$, then $\mathfrak{M}$ is a graded ideal of definition for $M(n)$, for every $n \in \mathbb{Z}$.**

### 4.1 Associated graded rings and modules

We use the notation of the last section.

If $w_1, \ldots, w_t$ are homogeneous elements of $m$ that generate an ideal $\mathfrak{M}$ in $R$, define

$$H(w) = \oplus_{n \geq 0} m(w)^n/m(w)^{n+1};$$

this is the associated graded ring for $m(w) \subseteq k(w_1, \ldots, w_t)$.

This is a typical construction to get a “standard” graded ring out of any filtered ring (e.g. [9] pg.12), of course in our case $R$ already has a grading on it which $H(w)$ inherits, but let’s consider the grading $H(w)$ has regardless of the grading on $R$; for example, when we say “an element $\xi$ of $H(w)$ has degree $s$ ” we mean $\xi \in m(w)^s/m(w)^{s+1}$, not that a coset representative for $\xi$ is in $R_s$.

Note that $H(w)$ is generated as a graded $k$-algebra by a $k$-basis for $m(w)/m(w)^2$, so is finitely generated as a $k$-algebra by elements of degree one (the “standard” case).

Define the associated graded module

$$\text{gr}_2(M) = \oplus_{n \geq 0} \mathfrak{M}^n M/\mathfrak{M}^{n+1} M;$$
this is a graded $H(w)$-module using the same conventions as for $H(w)$: forget the inherited grading and grade according to the direct sum defining $gr_3(M)$ above.

More generally, suppose that

$$
\cdots \subseteq F^n(M) \subseteq F^{n-1}(M) \subseteq \cdots \subseteq F^1(M) \subseteq F^0(M) = M
$$

is a filtration of $M$ by submodules $F^i(M)$ in $C(R)$. The associated graded module for the filtration is

$$
gF(M) = \oplus_{n \geq 0} F^n(M)/F^{n+1}(M).
$$

Again, we forget the grading that these associated graded modules inherit from $M; for example, consider $M(\mathfrak{a})$ as a graded $H(w)$-module with the elements of degree $s$ being $\mathfrak{a}^s M/\mathfrak{a}^{s+1} M$.

**Proposition 4.9.** Suppose that $M$ is in $C(R)$ and $\mathfrak{a}$ is a graded ideal of definition for $M$. Let $F(M)$ be an $\mathfrak{a}$-bonne filtration of $M$. Suppose also that $w_1, \ldots, w_t$ are homogeneous elements that generate $\mathfrak{a}$ as an ideal in $R$. Then,

i) $\mathfrak{a}^n M \subseteq F^n(M)$, for every $n \geq 0$.

ii) $F^n(M)/F^{n+1}(M)$ is a finite dimensional vector space over $k$, for every $n \geq 0$.

iii) $M$ is in $C(k\langle w_1, \ldots, w_t \rangle)$, via restriction to the subring $k\langle w_1, \ldots, w_t \rangle$ of $R$, and $F(M)$ is also an $m(w)$-bonne filtration of $M$ as a $k\langle w_1, \ldots, w_t \rangle$-module.

iv) The associated graded module

$$
gF(M) = \sum_{n \geq 0} F^n(M)/F^{n+1}(M)
$$

is a finitely generated graded $H(w)$-module.

**Proof.** i) This follows from the definition of $\mathfrak{a}$-bonne.

ii) There are short exact sequences of vector spaces over $k$ for every $n \geq 0$

$$
0 \to F^n(M)/F^{n+1}(M) \to M/F^n(M) \to M/F^{n+1}(M) \to 0,
$$

and by part i), we get a surjection

$$
M/\mathfrak{a}^n M \to M/F^n(M) \to 0.
$$

Now, our characterization of graded ideals of definition 4.7 gives that $M/\mathfrak{a}^n M \subseteq C(k)$ for all $n > 0$, and so the surjection gives that $\text{vdim}_k(M/F^n(M)) < \infty$ for all $n > 0$. Finally, since vector space dimension is additive over short exact sequences, we get that $\text{vdim}_k(F^n(M)/F^{n+1}(M)) < \infty$. Item iii) also follows from from our characterization of graded ideals of definition.

iv) Suppose that $n_0$ is such that $\mathfrak{a} F^n(M) = F^{n+1}M$, for $n \geq n_0$. Then, note that $gF(M)$ is generated by a $k$-basis for

$$
\sum_{i=0}^{n_0} F^i(M)/F^{i+1}(M),
$$

as an $H(w)$-module.

\[ \square \]

**Corollary 4.10.** Suppose that $M$ is in $C(R)$, $\mathfrak{a}$ is an ideal of definition for $M$ and $w_1, \ldots, w_t$ are homogeneous elements that generate $\mathfrak{a}$ as an ideal in $R$. Then, $gr_3(M)$ is a finitely generated graded $H(\bar{w})$-module; in fact, this module is generated by a $k$-basis for $M/\mathfrak{a}M$, as an $H(\bar{w})$-module.
4.2 Samuel Polynomial in $\mathcal{C}(R)$

In this section we show that for $M \in \mathcal{C}(R)$, the degree of the Samuel polynomial of $M$ (defined in section 2.4) is a well defined invariant of $M$.

Recall Theorem 4.3 which stated that if a module in $\mathcal{C}(R)$ has finite length, then the $R$-length of the module is equal to the $k$-vector space dimension. Also, our characterization of graded ideals of definition has that $\ell_{R}(M/\mathfrak{J}^{n}M) < \infty$ for a graded ideal of definition $\mathfrak{J}$, and all $n > 0$. As a corollary, we have the following:

**Corollary 4.11.** For $M \in \mathcal{C}(R)$ and $\mathfrak{J}$ a graded ideal of definition for $M$, the Samuel function

$$p(M, \mathfrak{J}, n) = \ell_{R}(M/\mathfrak{J}^{n}M) = \text{vdim}_{k}(M/\mathfrak{J}^{n}M),$$

for $n \geq 0$.

Regardless of the fact that we are working in $\mathcal{C}(R)$, the usual machinery for the Samuel function will run through.

Let $M$ be in $\mathcal{C}(R)$ with $\mathfrak{J}$ an ideal of definition, but forget the grading for a moment. In chapter 2, we demonstrated the following facts about the Samuel Polynomial in the ungraded setting:

- The Samuel function $p(M, \mathfrak{J}, n)$ is polynomial-like for sufficiently large $n$.
- The degree of the Samuel polynomial, $p(M, \mathfrak{J}, n)$, depends only on $\mathcal{V}(\mathfrak{J} + \text{Ann}_{R}(M))$. Moreover, $\mathcal{V}(\mathfrak{J} + \text{Ann}_{R}(M))$ is a finite set consisting of only maximal ideals.

**Proposition 4.12.** Let $M \in \mathcal{C}(R)$ with $\mathfrak{J}$ a graded ideal of definition for $M$. The set $\mathcal{V}(\mathfrak{J} + \text{Ann}_{R}(M))$ equals \{ $m$ \}.

**Proof.** Forgetting the grading, we have that $\mathcal{V}(\mathfrak{J} + \text{Ann}_{R}(M))$ is a finite set consisting of only maximal ideals. Then, the only primes containing $\mathfrak{J} + \text{Ann}_{R}(M)$ are maximal in $R$, thus each must be minimal over $\mathfrak{J} + \text{Ann}_{R}(M)$.

Now let’s consider the grading. Certainly $\mathfrak{J} + \text{Ann}_{R}(M)$ is a graded ideal in $R$. Lemma 3.14 implies that minimal primes over graded ideals are in turn graded, so the elements of $\mathcal{V}(\text{Ann}_{R}(M) + \mathfrak{J})$ are graded. Thus, they are all maximal ideals and graded, but there is only one maximal graded ideal in $R$: $m$! 

**Corollary 4.13.** For $M \in \mathcal{C}(R)$ with graded ideal of definition $\mathfrak{J}$, the degree of the Samuel polynomial $p(M, \mathfrak{J}, n)$ is independent of the choice of $\mathfrak{J}$.

Thus, for $M \in \mathcal{C}(R)$, and every graded ideal of definition $\mathfrak{J}$, we may call the degree of any Samuel polynomial $p(M, \mathfrak{J}, n)$, $d_{1}(M)$, instead of $d_{1}(M, \mathfrak{J})$.

It is worthwhile to point out that the Samuel polynomial, unlike the Poincaré series, does not depend on the gradings on $R$ and $M$. There are probably stronger lemmas than that below, but this serves our purpose.

**Lemma 4.14.** Suppose that $R$ and $\hat{R}$ are two positively graded Noetherian rings with $R_{0} = \hat{R}_{0} = k$, a field, such that $R = \hat{R}$ as rings when we forget the gradings. In this case, the superfluous ideal $m$ is graded whether considered as an ideal in $R$ or in $\hat{R}$. Suppose that $\mathfrak{J}$ is a graded ideal in $R$, and let $\hat{\mathfrak{J}}$ be the same ideal in $\hat{R}$, $\mathfrak{J}$ may be ungraded in $\hat{R}$. Then $\mathfrak{J}$ is an ideal of definition in $R$ if and only if $\hat{\mathfrak{J}}$ is an ideal of definition, possibly ungraded, in $\hat{R}$. Moreover, since $\mathfrak{J} \subseteq m$, $\hat{\mathfrak{J}} \subseteq \hat{m}$. Also, suppose that $M$ and $\hat{M}$ are in $\mathcal{C}(R), \mathcal{C}(\hat{R})$, respectively, and that $M = \hat{M}$ when we forget the gradings. Then, $p(\mathfrak{J}, M, n) = p(\hat{\mathfrak{J}}, \hat{M}, n)$ for every $n$.

**Corollary 4.15.** For $M \in \mathcal{C}(R)$, and $\mathfrak{J}$ a graded ideal of definition generated by homogeneous elements of positive degree $x_{1}, \ldots, x_{n}$, we have

$$p(M, \mathfrak{J}, n) = p(M, \hat{m}(\vec{x}), n).$$

**Proof.** This is immediate using proposition 4.6 which states that $m(\vec{x})^{n}M = \mathfrak{J}^{n}M$ for all $n \geq 0$. 

Note that the following examples of the Samuel polynomial do not depend on the particular grading imposed on the ring $R$. 

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Example 4.16.
i) Consider $R = k[x_0, \ldots, x_u]$ as a module over itself, where $k$ is a field and the degree of every $x_i$ is equal to one. Clearly $R$ is a graded prime ideal in $A$. Where the first map is multiplication by $i$.

Clearly $R$ is a graded prime ideal in $A$. When we calculated the Poincare series for $V$.

Consider $V = x_i$, with superfluous ideal $m$ (a graded ideal of definition) consisting of all polynomials in the $x_i$’s with no constant term. So the Samuel function maps $n \to \operatorname{vdim}_k(R/m^n)$, which simply counts the “total number” of monomials of degree less than $n$.

When we calculated the Poincare series for $V$, we showed that $\operatorname{vdim}_k(k[x_0, \ldots, x_u]) = \binom{u+d}{u}$. Thus, $p(m, n) = \sum_{d=0}^{n-1} \binom{u+d}{u}$.

As $d$ varies, we see that this formula is just summing entries of the $u^\text{th}$ diagonal of Pascal’s triangle. Then, $p(m, n) = \sum_{d=0}^{n-1} \binom{u+d}{u} = \binom{u+1}{u+1}$. After a bit of simplification, we have that

\[ p(m, n) = \frac{1}{(u+1)!} \cdot n^{u+1} + L.O.T. \]

ii) Let $S = k[x_0, \ldots, x_u]$ have $\deg(x_i) = 1$ for each $i$, and $R = S/(f)$ where $f \in S_d$.

We have a graded short exact sequence

\[ 0 \to S/m^n \xrightarrow{f} S/m^{n+d} \to S/(m^{n+d} + (f)) \to 0. \]

Where the first map is multiplication by $f$ and the second map sends an element $s + m^{n+d} \to s + (m^{n+d} + (f))$.

Now, define $\bar{m} = (x_0, \ldots, x_u)$ where $x_i = x_i + (f)$. Then, $R/\bar{m}^{n+d} = (S/(f))/m^{n+d}(S/(f))$, and for all $n$ we have $\bar{m}^{n+d}(S/(f)) = [m^{n+d} + (f)]/(f)$. Thus, by the third isomorphism theorem, $R/\bar{m}^{n+d} \cong S/(m^{n+d} + (f))$ for each $n$.

By additivity we have $\operatorname{vdim}_k(R/\bar{m}^{n+d}) = \operatorname{vdim}_k(S/m^{n+d}) - \operatorname{vdim}_k(S/m^n)$. We compute,

\[ p(R, \bar{m}, n + d) = \binom{n+u+d}{u+1} - \binom{n+u}{u+1} = \frac{d}{u!} n^u + L.O.T. \]

5 More Commutative Algebra in the Graded Context

We review some definitions and theorems from commutative algebra, putting them in the graded context.

5.1 Basic definitions and theorems

Definition 5.1. Let $M$ be a finitely generated $A$-module where $A$ is Noetherian and $\mathbb{Z}$-graded.

\[ \mathcal{V}(M) = \{ p \mid p \text{ is a graded prime ideal in } R \text{ containing } \operatorname{Ann}(M) \}. \]

Recall that we have shown in section 2 that if $M \neq 0$, $\operatorname{Ass}_A(M) \neq \emptyset$; in proposition 3.13, we’ve seen that $\operatorname{Ass}_A(M)$ consists entirely of graded prime ideals and each may be written as the annihilator of a homogeneous element of $M$.

Now, if $M \neq 0$, and $x$ is a homogeneous element of $M$ of degree $d(x) \geq 0$, whose annihilator $\operatorname{Ann}_A(x) = p$ is a graded prime ideal in $A$, then there exists an injection of graded $A$-modules

\[ (A/p)(-d(x)) \to M \]

defined by $r + p \mapsto rx$. Conversely, if $p$ is a graded prime ideal of $A$ such that there exists an injection of graded $A$-modules $i : (A/p)(-d) \to M$, for some $d \geq 0$, then $p = \operatorname{Ann}(i(1 + p))$ is an associated prime of $M$.

Modifying a proof in [9], section 1, in the ungraded case we note:

Theorem 5.2. ([9], I) Let $M \in \mathcal{C}(A)$, $M \neq 0$. 

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a. There exist a finite filtration of $M$ by graded submodules

$$0 = F^{N+1}(M) \subset F^N(M) \subset \ldots \subset F^{i+1}(M) \subset F^i(M) \subset \ldots \subset F^0(M) = M,$$

a set of graded prime ideals $p_i$ and integers $d_i \geq 0$, for $0 \leq i \leq N$, with $F^i(M)/F^{i+1}(M) \cong (A/p_i)(-d_i)$ as graded $A$-modules, for every $i$.

b. For any filtration $F$ of $M$ satisfying the conditions of a. above, define $S_F = \{p_i \mid 1 \leq i \leq N\}$. Then, $\text{Ass}_A(M) \subseteq S_F \subseteq^{*} \mathcal{V}(M)$, and all three of these sets have the same minimal elements.

c. The set of minimal elements in $\text{Ass}_A(M)$ is equal to the set of minimal primes of $M$.

Proof: We only indicate the modifications in the graded case.

Set $M_0 = 0$.

For a): $\text{Ass}_A(M) \neq \emptyset$ implies, using the discussion above, that there is a graded prime ideal $p_1$ and an integer $d_1$ such that $A/p_1(-d_1)$ is isomorphic to a graded $A$-submodule $M_1$ of $M$. If $M_1 \neq M$, the argument is repeated for the graded $A$-module $M/M_1$, and then proceeds in a recursive process as in the citation, producing an ascending filtration $0 \subset M_1 \subset \ldots \subset M_i \subset M_{i+1} \subset \ldots$ of graded submodules of $M$ and integers $d_i$ such that $M_{i+1}/M_i$ is isomorphic as a graded $A$-module to $A/p_i(-d_i)$. The filtration must eventually yield $M_N = M$ at some point, since $M$ is a Noetherian $A$-module. Of course just change the indices to get the descending filtration as desired.

The proof of b) is exactly as in the ungraded case, and c) we have already noted (Proposition 2.24).

5.2 Dimension in $C(R)$

Theorem 5.3. Let the ring $S$ be a positively graded Noetherian ring, but don’t assume that $S_0$ is a field. Let $M \in C(S)$. Let $d_{S}(M)$ denote the Krull dimension of $M$ and $*d_{S}(M)$ denote the graded Krull dimension of $M$ as defined in section 3.2. Then,

i) $d_{S}(S) = *d_{S}(S)$.

ii) $d_{S}(M) = *d_{S}(M)$.

Proof.

i) Let $I$ be any proper ideal in $S$ then $I \cap S_0$ is a proper ideal in $S_0$; the proof of this is straightforward.

Now suppose that $I$ is prime in $S$, one sees also that $I \cap S_0$ is prime in $S_0$. We also have that $(I \cap S_0) \oplus S_+$ is a graded prime ideal in $S$, since $S$ is positively graded: here is one place that the positive grading is used.

Now let $N$ be a maximal ideal in $S$, which may or may not be graded and has the property that $\text{ht}(N) = d_{S}(S)$. Note that $N^* \subseteq (N \cap S_0) \oplus S_+$, and both of these are prime ideals in $S$.

Suppose $N$ is graded. Then $*\text{ht}(N) = \text{ht}(N) = d_{S}(S)$ using lemma 3.21, and by definition the graded height of $N$ is less than or equal to $*d_{S}(S)$. Therefore,

$$d_{S}(S) \leq *d_{S}(S).$$

Suppose $N$ is not graded. Then, by lemma 3.21, $\text{ht}(N) = d_{S}(S) = \text{ht}(N^*) + 1$. Since $N^* \subset (N \cap S_0) \oplus S_+$ (see the paragraph below), and both are graded prime ideals, we must have $*d_{S} \geq *\text{ht}((N \cap S_0) \oplus S_+) \geq *\text{ht}(N^*) + 1 = \text{ht}(N^*) + 1$. Therefore, again,

$$d_{S}(S) \leq *d_{S}(S).$$

Here we show that if $N^* = (N \cap S_0) \oplus S_+$, then $N$ is graded. To see this, let $\zeta \in N$. Using the grading on $S$, rewrite $\zeta$ as $\zeta = \zeta_0 + \zeta_1 + \ldots + \zeta_t$ where $\deg(\zeta_i) = i$. Now, $\zeta_1 + \cdots + \zeta_t \in S_+ \subseteq N^* \subseteq N$, but this implies that $\zeta_0 \in N$ since $\zeta \in N$. Therefore, $\zeta \in (N \cap S_0) \oplus S_+ = N^*$, and thus $N^* = N$. 

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Definition 5.4. Let \( \mathfrak{m} \) be a system of parameters for \( \mathfrak{m} \) and we have already noted that \( \dim S = \dim_R(M) \).

We have also shown that \( \text{Ann}_S(M) \) is a graded ideal in \( S \), therefore \( S/\text{Ann}_S(M) \) is a graded ring. Moreover, the positive grading on \( S \) implies that \( S/\text{Ann}_S(M) \) is positively graded, and so by part i), \( \dim_S(M) = \dim_R(M) \).

Theorem 5.5. Smoke’s Dimension Theorem (Theorem 5.5 of [8])

Let \( M \in \mathcal{C}(R) \) and suppose that \( M \) positively graded. If \( d(M), s(M) \) are defined as above, we have

\[
d(M) = s(M) = \dim_R(M) < \infty.
\]

The same theorem is true without assuming \( M \) is positively graded:

Corollary 5.6. Given \( M \in \mathcal{C}(R) \), if \( d(M), s(M) \) are defined as above, we have

\[
d(M) = s(M) = \dim_R(M) < \infty.
\]

Proof: If \( M \in \mathcal{C}(R) \), using the Grothendieck lemma, there is an \( n_0 \in \mathbb{Z} \) such that \( M(n_0) \) is positively graded. Since we have \( d(M(n_0)) = d(M), \dim_R(M(n_0)) = \dim_R(M) \) and \( s(M(n_0)) = s(M) \), the corollary follows.

Definition 5.7. If \( M \neq 0 \) is in \( \mathcal{C}(R) \), then we know \( \dim_R(M) = \dim_R(M) := D \). A sequence \( y_1, \ldots, y_D \) of homogeneous elements of \( \mathfrak{m} \subset R \), such that \( M \) is a finitely generated \( k(y_1, \ldots, y_D) \)-module is called a \text{*system of parameters*} for \( M \), as an \( R \)-module.

Henceforth we abbreviate \text{*"system of parameters"*} by \text{*"sop"*}. Note that by definition, a \text{*sop} for \( M \) is also a \text{*sop} for \( M(n) \), for every \( n \in \mathbb{Z} \) (and vice versa).
Example 5.8. Let $S = k[x_0, \ldots, x_u]$ have the standard grading, i.e. $	ext{deg}(x_i) = 1$ for each $i$. Let $f$ be a homogeneous polynomial of degree $d$. Without loss of generality, assume that $f(0,0,\ldots,1) \neq 0$. Since $k[x_0, \ldots, x_u] = k[x_0, \ldots, x_{u-1}][x_u]$, without loss of generality we may write $f = x_u^d + g_1 x_u^{d-1} + \cdots + g_{d-1} x_u + g_d$ where $g_i \in k[x_0, \ldots, x_{u-1}]$, where $g_d \neq 0$. Since $f$ is homogeneous of degree $d$, we know that each $g_i$ is homogeneous and that $\text{deg}(g_i) = i$ for each $i$.

Then, in the ring $R = S/(f)$, we have that $\bar{x}_u^d = -(\bar{g}_0 \bar{x}_u^{d-1} + \cdots + \bar{g}_{d-1} \bar{x}_u + \bar{g}_d)$. Then, $R$ is finitely generated over $k(\bar{x}_0, \ldots, \bar{x}_{u-1})$ by $1, \bar{x}_u, \ldots, \bar{x}_u^{d-1}$, so $\bar{x}_0, \ldots, \bar{x}_{u-1}$ is a $\ast$ sop for $R$ since the Krull dimension of $R$ is $u$. Notice that this $\ast$ sop is composed of degree $1$ elements.

The following proposition is another part of the “folklore” knowledge, but the citation is the first that we know of in the literature.

Proposition 5.9. (Theorem 6.2 of [8]) Suppose that $M$ is in $C(R)$ and $M$ is positively graded. Let $D(M)$ be equal to the common value $\ast \dim(M) = d(M) = s(M)$. Then, a $\ast$ sop exists, and if $y_1, \ldots, y_{D(M)} \in \mathfrak{m}$ is a $\ast$ sop for $M$, then $y_1, \ldots, y_{D(M)}$ are algebraically independent over $k$.

As before the proposition above holds for any $M \in C(R)$:

Corollary 5.10. Suppose that $M$ is in $C(M)$. Let $D(M)$ be equal to the common value $\ast \dim(M) = d(M) = s(M)$. Then, a $\ast$ sop exists, and if $y_1, \ldots, y_{D(M)} \in \mathfrak{m}$ is a $\ast$ sop for $M$, then $y_1, \ldots, y_{D(M)}$ are algebraically independent over $k$.

Proof. There is an $n_0 \in \mathbb{Z}$ such that $M(n_0)$ is positively graded. Every $\ast$ sop for $M(n_0)$ is a $\ast$ sop for $M$, now just apply the previous proposition.

Finally, we point out a key result which we shall use when discussing multiplicities in chapter 7.

Theorem 5.11. Suppose $M \in C(R)$, and that $D = \ast \dim(M) = d(M) = s(M)$. Suppose that $y_1, \ldots, y_D$ is a $\ast$ sop for $M$. Consider the positively graded subring $k(\bar{y}) = k(y_1, \ldots, y_D)$ of $R$. Let $\mathfrak{I}$ be the ideal in $R$ generated by $y_1, \ldots, y_D$, and let $\mathfrak{m}(\bar{y})$ be the ideal in $k(\bar{y})$ generated by $y_1, \ldots, y_D$. Of course $\mathfrak{m}(\bar{y})$ is the superfluous ideal in $k(\bar{y})$, so is a graded ideal of definition in that ring. Then $p(M, \mathfrak{I}, n)$, the Samuel polynomial for $M$ considered as an $R$-module with respect to the ideal $\mathfrak{I}$ of $R$ exists. Also, $p(M, \mathfrak{m}(\bar{y}), n)$, the Samuel polynomial for $M$ considered as a $k(\bar{y})$-module with respect to the ideal $\mathfrak{m}(\bar{y})$ exists and

$$p(M, \mathfrak{I}, n) = p(M, \mathfrak{m}(\bar{y}), n).$$

Proof. $\mathfrak{I}$ is a graded ideal of definition of $R$ since it is generated by a $\ast$ sop for $M$. The conclusions then follow directly from Propositions 4.6 and 4.7 and the definition of the Samuel polynomial.

5.3 E Pluribus Unum: One Dimensional Invariant From Many

We relate the definitions from the previous two sections to those given in Serre [9]. Recall from section 2.4 that $d_1(M)$ is the degree of the Samuel polynomial for any graded ideal of definition, and this degree is independent of the choice of ideal of definition when $M \in C(R)$. Also, we know $\ast \dim_R(M) = \dim_R(M)$.

Definition 5.12. $s_1(M)$ in $C(R)$

If $M \in C(R)$, $M \neq 0$, let $s_1(M)$ be the least $s$ such that there exist homogeneous elements $w_1, \ldots, w_s \in \mathfrak{m}$ such that $M/(w_1, \ldots, w_s)M$ is a finite dimensional graded vector space over $k$. Note that $s_1(M) = 0$ if and only if $M$ is a finite dimensional graded vector space over $k$.

We see that this definition of $s_1$ is identical to the one used in the dimension theorem 2.42, the only change being that $R$ need not be a local ring and since $M$ is in $C(R)$ we have replaced “length” with “dimension”.

Next, as a direct corollary of proposition 4.7 we have the following equality.

Corollary 5.13. If $M \in C(R)$, $s(M) = s_1(M)$.

Definition 5.14. If $\mathfrak{I}$ is a graded ideal in $R$, and $M$ is in $C(R)$, then $(0 : \mathfrak{I})_M$ is the graded submodule

$$\{m \in M \mid rm = 0, \forall r \in \mathfrak{I}\}$$

of $M$. 

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If $\mathfrak{I}$ is an ideal generated by $x_1, \ldots, x_u$, sometimes we write $(0 : x_1, \ldots, x_u)_M$ instead of $(0 : (x_1, \ldots, x_u))_M$ or $(0 : \mathfrak{I})_M$.

**Lemma 5.15.** (II, III-8, Lemma 2, a) and c) Let $M$ be in $\mathcal{C}(R)$, and $x \in \mathfrak{m}$ be a homogeneous element. Let $\mathfrak{I}$ be a graded ideal of definition for $M$.

i) $\mathfrak{I}$ is also an ideal of definition for $(0 : x)_M$ and $M/xM$.

ii) For $n > 0$, $p((0 : x)_M, \mathfrak{I}, n) - p(M/xM, \mathfrak{I}, n)$ is a polynomial function of $n$ of degree strictly less than $d_1(M)$.

iii) $s_1(M) \leq s_1(M/xM) + 1$.

**Proof.** The proof of i) and ii) follow from 2.19 and 2.30 since, whether one forgets the grading or not, there are exact sequences of $R$-modules (which are also exact sequences in $\mathcal{C}(R)$), where $d(x)$ is the degree of the homogeneous element $x$:

$$0 \to (0 : x)_M(-d(x)) \to M(-d(x)) \to xM \to 0$$

and

$$0 \to xM \to M \to M/xM \to 0.$$ 

For iii), just note if $w_1, \ldots, w_t \in \mathfrak{m}$ are homogeneous, then $(w_1, \ldots, w_t)(M/xM) = (w_1, \ldots, w_t, x)M/xM$, so $(M/xM)/(w_1, \ldots, w_t)(M/xM)$ can be identified with $M/(w_1, \ldots, w_t, x)M$ and iii) follows by definition of $s_1$.

Now, we know that $M$ has a finite number of associated primes and that they are all graded prime ideals in $R$. Certain of these associated primes are the minimal primes of $M$, and a subset $\mathcal{D}_R(M)$ of the minimal primes is defined by the requirement that $p \in \mathcal{D}_R(M)$ if and only if $\dim_R(R/p) = \dim_R(M)$, which is true if and only if $\dim_R(R/p) = \dim_R(M)$.

We first need the graded version of a lemma in [9]:

**Lemma 5.16.** (II, III-8, Lemma 2, b) If $M$ is in $\mathcal{C}(R)$, and $x \in \mathfrak{m}$ is a homogeneous element such that $x \notin p$, for every $p \in \mathcal{D}_R(M)$, then $\dim_R(M/xM) \leq \dim_R(M) - 1$ and $\dim_R(M/xM) \leq \dim_R(M) - 1$.

**Proof.** Note that $(x) + \text{Ann}_R(M) \subseteq \text{Ann}_R(M/xM)$ is an inclusion of graded ideals in $R$, so that any prime minimal over $\text{Ann}_R(M/xM)$ must be graded and contain $x$, thus must not be amongst the primes in $\mathcal{D}(M)$.

Now, we can outline the proof of:

**Theorem 5.17.** Suppose that $M$ is in $\mathcal{C}(R)$. Then,

$$(\ast) \dim_R(M) = \dim_R(M) = d_1(M) = s_1(M).$$

**Proof.** The first equality has already been proven.

For the last three, argue as in Serre.

First, $\dim_R(M) \leq d_1(M)$: one argues by induction on $d_1(M)$. If $d_1(M) = 0$, then there exists a positive integer $t_0$ such that $m^tM = m^{t_0}M$ for $t \geq t_0$. Let $N = m^{t_0}M$, so that $mN = N$. But then $N = 0$ by graded Nakayama. This means that $M$ is a finite dimensional vector space over $k$, and one must have $\dim_R(M) = 0$.

Supposing that $d_1(M) \geq 1$, $M$ is not zero and thus there is a minimal prime $p_0$ for $M$ which is both a graded ideal and an associated prime for $M$, such that $\dim_R(M) = \dim_R(R/p_0)$, yielding a graded submodule $N$ of $M$ isomorphic to $R/p_0$ as a graded $R$-module. Now, $d_1(M) \geq d_1(N)$ using 2.34, so it is enough to reduce to the case $M = N = R/p_0$, where $p_0$ is a graded prime ideal of $R$. Note that $\mathcal{D}_R(R/p_0) = \{p_0\}$, by definition.

Let $\text{Ann}_R(N) = p_0 \subset p_1 \subset \cdots \subset p_n$ be a chain of graded primes in $R$. Let us show that $n \leq d_1(R/p_0)$; then

$$(\ast) \dim_R(R/p_0) \leq d_1(R/p_0)$$

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and since \( *\dim = \dim \), we’ll be done.

If \( n = 0 \), the inequality above is clear. So suppose that \( n \geq 1 \). Then there is a homogeneous element \( x \in p_1 \) that is not in \( p_0 \). Then \( Ann_R(N/xN) \subseteq p_1 \) (recall \( N = R/p_0 \)), and so by definition,

\[
n - 1 \leq *\dim_R(N/xN).
\]

Also, one sees that, since \( p_0 \) is a prime ideal, and \( x \notin p_0 \), that \( (x : 0)_{R/p_0} = 0 \), so that using lemma 5.15,

\[
d_1(N/xN) \leq d_1(N) - 1.
\]

Using the induction hypotheses, \( *\dim_R(N/xN) \leq d_1(N/xN) \), so

\[
n \leq *\dim_R(N/xN) + 1 \leq d_1(N/xN) + 1 \leq d_1(N).
\]

Next, \( d_1(M) \leq s_1(M) \): Let \( \mathcal{I} = (w_1, \ldots, w_t) \subseteq m \), where the \( w_i \)'s are homogeneous, be such that \( M/\mathcal{I}M \) has finite dimension over \( k \). Without loss of generality we may assume that \( Ann_R(M) \subseteq \mathcal{I} \). Thus, \( \mathcal{I} \) is a graded ideal of definition for \( M \) and using theorem 2.33, the degree of the polynomial function \( p(M, \mathcal{I}, n) \) is less than or equal to \( t \). Therefore \( d_1(M) \leq s_1(M) \), since \( d_1(M) \) is independent of the choice of ideal of definition.

Finally, \( s_1(M) \leq *\dim_R(M) \): Use induction on \( n = *\dim_R(M) \), which is finite. The case where \( n = 0 \) is the case where \( M \) is Artinian as an \( R \)-module (using theorem 2.12) , so that \( s_1(M) = 0 \) as well. If \( n \geq 1 \), we know that there are a finite number of prime ideals minimal in \( \mathcal{D}(M) \) and that they are all graded primes. These minimal primes are not equal to \( m \) since \( *\dim_R(M) \geq 1 \) and thus their union can't be equal to \( m \). Therefore there is a homogeneous element \( x \in m \) such that \( x \notin p \) for every \( p \in \mathcal{D}(M) \). Using lemma 5.16,

\[
*\dim_R(M/xM) \leq *\dim_R(M) - 1.
\]

Using induction,

\[
s_1(M/xM) \leq *\dim_R(M/xM)
\]

and using lemma 5.15,

\[
s_1(M) \leq s_1(M/xM) + 1.
\]

So,

\[
s_1(M) \leq s_1(M/xM) + 1 \leq *\dim_R(M/xM) + 1 \leq *\dim_R(M).
\]

\[\square\]

In Sum:

- For \( M \in \mathcal{C}(R) \), \( *\dim_R(M) = d(M) = s(M) = s_1(M) = d_1(M) = \dim_R(M) \).
- Moreover, if \( D = *\dim_R(M) \), then \( D \) is the order of the pole of \( P_M(t) \) at \( t = 1 \).

### 6 Homological Algebra in \( \mathcal{C}(R) \)

We first review some definitions of homological algebra following the treatment in [8], though this gives the parallel for the graded case to Serre’s exposition in the nongraded case [9].

If \( V \) is a one-dimensional graded vector space such that \( V_i = 0 \), if \( i \neq 0 \), we identify \( V \) with \( k \). Using this identification we may also identify \( R \otimes_k k \) with \( R \), and we often do this throughout the paper without comment.

If \( M \) and \( N \) are graded \( R \)-modules, recall the definition of the graded tensor product \( M \otimes_R N \) from section 3.3.

Now, if \( M \in \mathcal{C}(R) \), there is an integer \( n \) such that if \( j < n \), \( M_j = 0 \) (proposition 3.23). Following the notation of [1], if \( M \neq 0 \), define

\[
\iota(M) = \inf\{j \in \mathbb{Z} \mid M_j \neq 0\},
\]

with \( \iota(0) = \infty \), by definition.

Now in Avramov and Buchweitz [1], Noetherian hypotheses are not used, but replaced by other finiteness hypotheses, implied in the Noetherian case. Since we are committed to using Noetherian hypotheses already, we’ll just continue to use these.

Then one can prove
Lemma 6.1. Suppose that $M, N \in \mathcal{C}(R)$.

a. $M \otimes_R N \in \mathcal{C}(R)$, and $\imath(M \otimes_R N) \geq \imath(M) + \imath(N)$.

b. For every $n \in \mathbb{Z}$, $\imath(M(n)) = \imath(M) - n$.

c. If $0 \to P \to M \to N \to 0$ is a short exact sequence in $\mathcal{C}(R)$ then $\imath(M) = \min\{\imath(P), \imath(N)\}$.

If $V \in \mathcal{C}(k)$ and $M \in \mathcal{C}(R)$, the $R$-module structure on $M \otimes_k V$ is defined as usual by

$$r(m \otimes v) = rm \otimes v,$$

if $v \in V$ and $m \in M$, making $M \otimes_k V \in \mathcal{C}(R)$. Similarly, the $R$-module structure on $V \otimes_k M$ is defined by

$$r(v \otimes m) = v \otimes rm,$$

no sign change depending on the degree of $r$ here!

We always regard $R$ as a graded $k$-module using $R_0 = k$.

We’ve seen that $k$ is a graded $R$-module using the rule: $r\alpha = 0$, if $r$ is homogeneous of positive degree, and $r\alpha$ is just the ordinary multiplication in $k$ if $r$ has degree 0. This is just another way of saying that we are making $k$ into a graded $R$-module using the map of graded rings

$$\epsilon : R \to k$$

defined by $\epsilon(r) = 0$, if $r$ has positive degree and $\epsilon(r) = r$ if $r$ has degree zero.

We will always consider $k$ as a graded $R$-module in this way.

Since $R_0 = k$, we have

Lemma 6.2. If $V \in \mathcal{C}(k)$, then $V \otimes_k R \in \mathcal{C}(R)$ and $\imath(V \otimes_k R) = \imath(V)$.

Since tensoring an exact sequence with a free module is exact in the ungraded category, and free modules in the graded category are also free when one forgets the grading, tensoring an exact sequence with a graded free module is also exact in the graded category:

Lemma 6.3. If $F$ is a free module in $\mathcal{C}(R)$, “$\otimes_R F$” is an exact functor $\mathcal{C}(R) \to \mathcal{C}(R)$. In particular, if $\mathcal{J}$ is a graded ideal in $R$ and $M$ is in $\mathcal{C}(R)$, then $M/\mathcal{J}M \otimes_R F$ is canonically isomorphic to (we will say “is equal to”) $(M \otimes_R F)/(\mathcal{J}M \otimes_R F)$ as an $R$-module.

Note that if $n \in \mathbb{Z}$, then $R(n)$ is naturally isomorphic as a graded $R$-module to $k(n) \otimes_k R$, and since we’ve seen in 3.3 that every finitely generated free $R$-module is isomorphic to a finite direct sum of the graded modules of the form $R(n)$ (for different $n$, possibly), this yields:

Lemma 6.4. If $F$ is a finitely generated free graded $R$-module, then

a. As graded $R$-modules, $F$ is isomorphic to $V \otimes_k R$ for some $V \in \mathcal{C}(k)$.

b. If $N$ is in $\mathcal{C}(R)$ and $F = V \otimes_k R$, then

$$F \otimes_R N \cong V \otimes_k N$$

as graded $R$-modules, and so

$$(F \otimes_R N)_j = \oplus_{r+s=j} V_r \otimes_k N_s,$$

for $j \geq 0$.

c. $\imath(F \otimes_R N) = \imath(N) + \imath(V)$, if $F = V \otimes_k R$, with $V \in \mathcal{C}(k)$ and $N \in \mathcal{C}(R)$.

We will regard $V \subseteq F = V \otimes_k R$: technically we use the inclusion $v \mapsto v \otimes 1$, but sometimes just write $v \in F$. 

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6.1 Complexes in $C(R)$ and Euler Characteristics

The definition of a complex of modules in $C(R)$ is as usual: this is a sequence $(M, \partial)$

$$
\cdots \rightarrow M_j \xrightarrow{\partial_j} M_{j-1} \xrightarrow{\partial_{j-1}} \cdots \rightarrow \partial_1 M_0
$$

of objects and morphisms in $C(R)$, such that $\partial \partial = 0$ everywhere. The sequence of morphisms $\partial$ above is called the differential for the complex.

The subscripts $j$ seem assigned ambiguously, but here’s what we mean: If $(M, \partial)$ is a complex in $C(R)$ as above, then the set of elements of $M_j$ of degree $i$ is equal to

$$(M_j)_i = \tilde{M}_{j,i}.$$

In other words, when speaking of a complex in $C(R)$, a single integer subscript denotes the sequential index of the complex, and a doubly-indexed subscript is read as “first index is the complex index, second is the graded-module index”. Hopefully this won’t be too confusing.

To further set notation, we will regard any $M$ in $C(R)$ as a “complex concentrated in degree 0”–this is the complex with all differentials equal to zero, with $M_1 = 0$, if the “complex-index” $i \neq 0$, and $M_0 = M$, for “complex-index ” 0.

The homology groups of a complex $(M, \partial)$ are defined as “ker $\partial$/im $\partial$” of course, and are also in $C(R)$:

$$H_j(M)_i = \ker(\partial : M_{j,i} \rightarrow M_{j-1,i})/\text{im}(\partial : M_{j+1,i} \rightarrow M_{j,i}).$$

**Definition 6.5.** Suppose we have two complexes of modules in $C(R)$, $(M, \partial)$ and $(N, \bar{\partial})$. The map $\varphi : M \rightarrow N$ is a graded morphism of complexes in $C(R)$ if for each $j$, $\varphi_j : M_j \rightarrow N_j$ is a graded $R$-module homomorphism in $C(R)$, and for every $j$, the diagram

$$
\begin{array}{ccc}
M_j & \xrightarrow{\partial_j} & M_{j-1} \\
\varphi_j & \downarrow & \varphi_{j-1} \\
N_j & \xrightarrow{\bar{\partial}_j} & N_{j-1}
\end{array}
$$

is commutative for all $j$.

**Definition 6.6.** Let $A, B, C$ be graded complexes in $C(R)$ with graded complex morphisms $\alpha : A \rightarrow B$ and $\beta : B \rightarrow C$. We say that $0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$ is a short exact sequence of graded complexes in $C(R)$ if $0 \rightarrow A_j \xrightarrow{\alpha_j} B_j \xrightarrow{\beta_j} C_j \rightarrow 0$ is exact for every $j$.

As usual, a short exact sequence of graded complexes in $C(R)$ gives rise to a long exact sequence on homology.

**Theorem 6.7.** Let $0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$ be a short exact sequence of graded complexes in $C(R)$, there exists a graded morphism $\omega$ of complex degree $-1$ (sometimes called the connecting homomorphism) such that the sequence

$$
\cdots \xrightarrow{\omega_{j+1}} H_j(A) \xrightarrow{\alpha} H_j(B) \xrightarrow{\beta} H_j(C) \xrightarrow{\omega_j} H_{j-1}(A) \xrightarrow{\alpha} \cdots
$$

is exact.

**Definition 6.8.** Let $(M, \partial)$ and $(N, \bar{\partial})$ be graded complexes in $C(R)$ such that for each $j$, $N_j$ is a graded $R$-submodule of $M_j$ and $\partial(n) = \bar{\partial}(n)$ for all $n \in N_j$ and for all $j$. We say that $N$ is a graded subcomplex of $M$.

In this case, the quotient $M_j/N_j$ is graded for each $j$, so we may form the graded quotient complex $(M/N, \bar{\partial})$ where $\bar{\partial}$ is defined $\bar{\partial}(m + N_j) = \partial(m) + N_{j-1}$ for $m \in M_j$. We see that this is a graded morphism since $\partial$ preserves degree.

**Definition 6.9.** The Euler characteristic of a complex $(M, \partial)$ in $C(R)$ is defined when $(M, \partial)$ is a finite length complex (i.e. there exists an $N$ such that $M_i = 0$ for $i > N$), and each $R$-submodule $M_i$ of $M$ is an element of $C(k)$ (i.e. a finite dimensional graded $k$-vector space). Given these conditions, the following sum is well defined:

$$\chi(M) \doteq \Sigma_i(-1)^i \dim_k(M_i).$$
Since vector space dimension splits over short exact sequences, we get the following lemma.

**Lemma 6.10.** Let $A \rightarrow B \rightarrow C$ be a short exact sequence of graded complexes in $\mathcal{C}(R)$ such that the Euler characteristic of each complex is defined. Then, $\chi(B) = \chi(A) + \chi(C)$.

If the two conditions for a well-defined Euler characteristic of a complex are not met, there may be a way to salvage the situation by passing to homology. We have,

**Definition 6.11.** Let $(M, \partial)$ be a complex in $\mathcal{C}(R)$ such that for every $i$, $H_i(M) \in \mathcal{C}(k)$ and for $i \gg 0$ $H_i(M) = 0$. We define the Euler characteristic of the homology to be $\chi(H(M)) = \Sigma_i (-1)^i \text{vdim} \chi(H_i(M))$.

**Theorem 6.12.** When the Euler characteristic $\chi(M)$ is defined, we have that $\chi(M) = \chi(H(M))$.

**Proof.** Let $Z_j = \ker \partial_j$ be the set of cycles in $M_j$, and $B_j = \text{Im} \partial_{j+1}$ be the set of boundaries, then by definition $H_j(M) = Z_j/B_j$.

Since $0 \rightarrow Z_j \rightarrow M_j \xrightarrow{\partial_j} B_{j-1} \rightarrow 0$ and $0 \rightarrow B_j \rightarrow Z_j \rightarrow M_j \rightarrow 0$ are exact sequences of finite dimensional $k$-vector spaces for every $j$, we have that $\text{vdim}_k(M_j) = \text{vdim}_k(B_{j-1}) + \text{vdim}_k(Z_j)$ and $\text{vdim}_k(H_j(M)) = \text{vdim}_k(B_j) + \text{vdim}_k(H_j(M))$.

From these two equations we get that $	ext{vdim}_k(M_j) = \text{vdim}_k(B_{j-1}) + \text{vdim}_k(H_j(M)) + \text{vdim}_k(B_j)$. Then,

$$\chi(M) = \Sigma_i (-1)^i \text{vdim}_k(M_i) = \Sigma_i (-1)^i \text{vdim}_k(B_i) + \text{vdim}_k(H_i(M)) + \text{vdim}_k(B_{i-1}) = \Sigma_i (-1)^i \text{vdim}_k(H_i(M)) = \chi(H(M)).$$

We do have

**Theorem 6.13.** Suppose that $A \rightarrow B \rightarrow C$ is a short exact sequence of graded complexes in $\mathcal{C}(R)$ such that the Euler characteristic of the homology of each complex is defined. Then, $\chi(H(B)) = \chi(H(A)) + \chi(H(C))$.

**Proof.** By hypothesis, each module in this sequence is a finite dimensional vector space over $k$ and there exists an $I$ such that $H_i(A) = H_i(B) = H_i(C) = 0$ for every $i > I$.

Applying the long exact sequence on homology, we get an exact sequence $0 \rightarrow H_I(A) \rightarrow H_I(B) \rightarrow H_I(C) \xrightarrow{\partial} H_{I-1}(A) \rightarrow \cdots \rightarrow H_0(C) \rightarrow 0$.

Using the additivity of vector space dimension over exact sequences which eventually terminate, we have that the alternating sum of the $k$-dimension of each homology module sums to $0$. It is plain to see that we may group together terms of the sum to get $\Sigma_i (-1)^i \text{vdim}_k(H_i(A)) - \Sigma_i (-1)^i \text{vdim}_k(H_i(B)) = 0$, and conclude the result.

**Definition 6.14.** Let $(M, \partial)$ be a graded complex in $\mathcal{C}(R)$. A graded filtration of the graded complex $M$ is a bi-graded object $F^mM$ such that the $n^{th}$ row $F^nM$ is a graded complex in $\mathcal{C}(R)$, and the $m^{th}$ column is a graded filtration of the graded module $M_m$ in $\mathcal{C}(R)$. The diagram is as follows:

$$\cdots \xrightarrow{\partial} M_1 \xrightarrow{\partial} M \xrightarrow{\partial} M \xrightarrow{\partial} M \xrightarrow{\partial} \cdots$$

$$\uparrow \uparrow \uparrow$$

$$\cdots \xrightarrow{\partial} F^1(M_1) \xrightarrow{\partial} F^1(M) \xrightarrow{\partial} F^1(M) \xrightarrow{\partial} F^1(M) \xrightarrow{\partial} \cdots$$

$$\uparrow \uparrow \uparrow$$

$$\cdots \xrightarrow{\partial} F^2(M_1) \xrightarrow{\partial} F^2(M) \xrightarrow{\partial} F^2(M) \xrightarrow{\partial} F^2(M) \xrightarrow{\partial} \cdots$$

$$\uparrow \uparrow \uparrow$$

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Theorem 6.15. If $FM$ is a graded filtration of the graded complex $M$ in $C(R)$ such that $\chi(F^iM/F^{i+1}M)$ is defined for every $i$, then $\chi(M/F^{n+1}M)$ is defined for every $n$ and

$$\chi(M/F^{n+1}M) = \sum_{j=0}^{n} \chi(F^jM/F^{j+1}M).$$

Proof. By the third isomorphism theorem for modules, we get a short exact sequence of graded complexes $0 \to F^jM/F^{j+1}M \to M/F^{j+1}M \to M/F^jM \to 0$ for each $j$. Applying lemma 6.10, we get that $\chi(M/F^{j+1}M) = \chi(F^jM/F^{j+1}M) + \chi(F^jM/F^{j+1}M)$. We can recursively apply this formula to the sum $\sum_{j=0}^{\infty} \chi(F^jM/F^{j+1}M)$, and since $F^0M$ is defined to be $M$, we obtain the result.

Also, in a similar manner, as a corollary of theorem 6.13, we have

Lemma 6.16. If $FM$ is a graded filtration of the graded complex $M$ in $C(R)$ such that $\chi(H(F^iM/F^{i+1}M))$ is defined for every $i$, then $\chi(H(M/F^{n+1}M))$ is defined for every $n$ and

$$\chi(H(M/F^{n+1}M)) = \sum_{j=0}^{n} \chi(H(F^jM/F^{j+1}M)).$$

Definition 6.17. An acyclic complex $(M, \partial)$ is a complex such that $H_j(M) = 0$, if $j > 0$.

Definition 6.18. If $(M, \partial)$ and $(G, \partial)$ are complexes in $C(R)$, then $M \otimes_R G$ is the complex defined by (the subscripts below are all “complex subscripts”):

- $(M \otimes_R G)_j = \oplus_{i=0}^{j} M_i \otimes_R G_{j-i}$
- $\partial : \oplus_{i=0}^{j} M_i \otimes_R G_{j-i} \to \oplus_{i=0}^{j-1} M_i \otimes_R G_{j-1-i}$ is defined by saying that $\partial(m \otimes g) = \partial(m) \otimes g + (-1)^i (m \otimes \partial(g))$, if $m \in M_i$ and $g \in G_{j-i}$.

Definition 6.19. A graded projective resolution of $M \in C(R)$ is an acyclic chain complex of modules in $C(R)$:

$$P : \cdots \to P_n \to P_{n-1} \to \cdots \to P_0,$$

such that each $P_i$ is in $C(R)$, is projective as an $R$-module and $H_0(P) \cong M$.

Now, if $M, N \in C(R)$, the $R$-modules $Tor^n_R(M, N)$ are defined as usual, forgetting the gradings everywhere. However, we shall see that in fact $Tor^n_R(M, N)$ has a grading on it making it an object in $C(R)$. That is, if $M$ and $N$ are in $C(R)$ we shall construct a graded free resolution of $M$ in $C(R)$. Now, applying the functor $- \otimes_R N$ to $P$ gives of course a graded chain complex in $C(R)$. The $i^{th}$ homology group of this complex is by definition $Tor^R_i(M, N)$ and is back in $C(R)$. As usual, $Tor_R^i(M, N)$ is independent of the choice of a projective resolution, graded or not; moreover, as we shall see, every projective $R$-module that is actually in $C(R)$ is a free graded $R$-module, and it’s not hard to see that any two free resolutions of $M$ in $C(R)$ are “graded”-homotopic so that the grading on $Tor_R^i(M, N)$ may be considered natural with respect to the category $C(R)$. In addition, the natural identification $Tor_R^i(M, N) = Tor^R_i(N, M)$ from the ungraded category carries over to the graded category $C(R)$.

We construct what is called a minimal graded free resolution of $M$ in $C(R)$ below.

Definition 6.20. A surjective map $\theta : L \to M$ in $C(R)$ is minimal if $L$ is free and ker $\theta \subset mL$.

Definition 6.21. A minimal resolution of $M$ in $C(R)$ is a resolution

$$\cdots \xrightarrow{\theta_3} V_1 \otimes_k R \xrightarrow{\theta_2} V_0 \otimes_k R \xrightarrow{\theta_1} M \to 0$$

of $M$ by free modules in $C(R)$ such that $\theta_0$ is a minimal surjection and $\theta_i : V_i \otimes_k R \to ker \theta_{i-1}$ is a minimal surjection for every $i > 0$. 

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Theorem 6.22. Every module $M$ in $C(R)$ has a minimal resolution.

Proof. We have shown before that $M/mM$ is a finitely generated graded $R/m$-vector space. Let $\{x_1, \ldots, x_n\}$ be a graded generating set for $M$ such that upon passing to the quotient, $\{\hat{x}_1, \ldots, \hat{x}_n\}$ generates $M/mM$ as a graded $R/m$-vector space.

Define $F_0$ to be the free graded $R$-module on the set $\{\hat{x}_1, \ldots, \hat{x}_n\}$, where $\deg(\hat{x}_i) = \deg(x_i)$ for all $i$. Let $\theta_0$ be the graded $R$-module homomorphism from $F$ to $M$ which sends $\hat{x}_i \mapsto x_i$. By construction $F_0 \in C(R)$.

For $\theta_0$ to be a minimal map we need to show that $\ker \theta_0 \subseteq mM$, note that $\theta_0$ is surjective by construction. Let $r_i$ be a homogeneous element of $R$ such that $\deg(r_i) + \deg(\hat{x}_i)$ is constant whenever $1 \leq i < n$, further, suppose that $\Sigma_i r_i \hat{x}_i \in \ker \theta_0$ so that $\Sigma_i r_i x_i = 0$. If we pass to the quotient, $\Sigma_i r_i x_i = 0$, but since the $\hat{x}_i$ were picked to be a basis, we get this if and only if $\tilde{r}_i = 0$ for each $i$. By definition, this happens if and only if $r_i \in m$ for each $i$, and therefore $\Sigma_i r_i \hat{x}_i \in mF_0$.

Now we have the short exact sequence in $C(R)$, $0 \to \ker(\theta_0) \to F_0 \xrightarrow{\theta_0} M \to 0$. All we need to do is repeat the process above, except use $\ker \theta_0$ in place of $M$. After successive applications of the same process, we get the minimal resolution $\cdots \xrightarrow{\theta_{i+1}} F_i \xrightarrow{\theta_i} F_{i-1} \xrightarrow{\theta_{i-1}} \cdots \to F_1 \xrightarrow{\theta_1} F_0 \xrightarrow{\theta_0} M \to 0$.

Finally observe that for each $i$, $F_i$ is a free $R$-module, and as we have seen, there is a graded $R$-module homomorphism $V_i \otimes_k R \cong F_i$ for some $V_i \in C(k)$, and so we may write the free resolution as in the previous definition.

\[\square\]

Theorem 6.23. Given a minimal resolution

$$\cdots \xrightarrow{\theta_2} V_1 \otimes_k R \xrightarrow{\theta_1} V_0 \otimes_k R \xrightarrow{\theta_0} M \to 0$$

of $M$, there are graded vector space isomorphisms

$$\text{Tor}_i^R(M, k) \cong V_i,$$

for every $i \geq 1$.

Proof. We begin by taking a minimal resolution of $M$,

$$\cdots \xrightarrow{\theta_2} V_1 \otimes_k R \xrightarrow{\theta_1} V_0 \otimes_k R \xrightarrow{\theta_0} M \to 0.$$ 

To compute $\text{Tor}_i^R(M, k)$, for $i \geq 1$, we need to take homology of the following sequence,

$$\cdots \xrightarrow{\theta_i \otimes 1} V_i \otimes_k R \otimes_R k \xrightarrow{\theta_{i-1} \otimes 1} V_{i-1} \otimes_k R \otimes_R k.$$ 

We know that $V_i \otimes_k R \otimes_R k \cong V_i \otimes_k k \cong V_i$ (where $\cong$ is a natural isomorphism in $C(R)$). So the above sequence has natural isomorphisms from each module to each module of the sequence

$$\cdots \to V_i \to V_0.$$ 

Let’s compute the maps in the sequence above by computing the maps $\theta_i \otimes 1$, $i \geq 1$. So, let $v$ and $r$ be homogeneous elements of $V, R$ respectively and let $\alpha \in k$. Now, we know that $\theta_i(v \otimes r) \in m(V_{i-1} \otimes_k R)$ by minimality, so

$$\theta_i(v \otimes r) = \Sigma w_k \otimes r_k,$$

where $r_k$ are homogeneous elements of positive degree in $R$ and $w_k \in V_{i-1}$. Thus, if $\alpha \in k$,

$$(\theta_i \otimes 1)(v \otimes r \otimes \alpha) = \theta_i(v \otimes r) \otimes \alpha$$

$$= (\Sigma w_k \otimes r_k) \otimes \alpha$$

$$= \Sigma w_k \otimes 1 \otimes r_k \alpha$$

$$= 0.$$ 

The last equality holds since $r_k$ is of positive degree for every $k$, and by definition of the $R$-module structure on $k$.

Therefore, we have that $\theta_i \otimes 1$ is the zero map for each $i \geq 1$, thus $\text{Tor}_i^R(M, k) = V_i$, as a graded $R$-module. 

\[\square\]
There are some corollaries to these two theorems.

First,

**Corollary 6.24.** If $M, N \in \mathcal{C}(R)$, then for every $i \geq 0$, $\text{Tor}^R_i(M, N)$ has a natural grading and with this grading, it is in $\mathcal{C}(R)$.

**Proof.** Construct a minimal graded resolution $V \otimes_k R$ of $M$ in $\mathcal{C}(R)$ by free modules in $\mathcal{C}(R)$; tensoring with $N$ over $R$ gives a complex $V \otimes_k N$ in $\mathcal{C}(R)$. Since all maps in this complex are graded homomorphisms of graded degree zero, and $R$ is Noetherian, all kernels and images of these homomorphisms are graded objects in $\mathcal{C}(R)$, with quotient groups also in $\mathcal{C}(R)$. \(\square\)

Next, if $M$ is in $\mathcal{C}(R)$, we say that $M$ is a flat $R$-module if it is flat as an $R$-module when we forget the gradings on $M$ and $R$.

**Corollary 6.25.** In $\mathcal{C}(R)$ every flat module, and thus every projective module, is also free.

**Proof.** Suppose that $M$ is a flat $R$-module. Then, since it is flat when we forget the gradings, $\text{Tor}^R_i(M, N) = 0$, for every $N$, whether graded or not, and every $i \geq 1$. Thus, using a minimal resolution $V \otimes_k R$ of $M$ as in the theorem above, we must have $V_i = 0$ for every $i \geq 1$.

But then the resolution shortens to $0 \to V_0 \otimes_k R \to M \to 0$, so that $M$ is isomorphic to a free graded $R$-module. \(\square\)

Finally, we have

**Corollary 6.26.** Given $M$ in $\mathcal{C}(R)$, the Poincaré series of the graded module $\text{Tor}^R_i(M, k)$

$$P_{\text{Tor}^R_i(M, k)}(t) = \sum_{j=-\infty}^{\infty} \text{vdim}_k(\text{Tor}^R_i(M, k))_t t^j$$

is a Laurent polynomial in $t$, for every $i$.

**Proof.** Using the minimal resolution $V \otimes_k R$ of $M$ as above, since $V_i$ is a finite-dimensional graded vector space over $k$ for every $i$, we must have, for $| j | > 0$, $\text{vdim}_k(V_i)_j = 0$. \(\square\)

### 6.1.1 Euler-Poincare Series

The following lemma and its corollaries are in Avramov and Buchweitz [1] (Lemma 7) and we give proofs here as well using minimal resolutions rather than the combinatorial arguments in [1].

**Lemma 6.27.** If $M, N \in \mathcal{C}(R)$, then $i(\text{Tor}^R_i(M, N)) \geq i + \iota(M) + \iota(N)$, for every $i \geq 0$.

**Proof.** We’ve already seen this for $i = 0$ since $\text{Tor}^R_0(M, N) = M \otimes_R N$. So, assume that $i > 0$.

Using a minimal graded resolution $V \otimes_k R$ of $M$, we compute $\text{Tor}^R_i(M, N)$ by applying $- \otimes_R N$ to obtain a complex in $\mathcal{C}(R)$

$$V \otimes_k N,$$

whose homology yields the $\text{Tor}$ groups.

We already know that since $\text{Tor}^R_i(M, N)$ is a sub quotient of $V_i \otimes_k N$, $i(\text{Tor}^R_i(M, N)) \geq i(V_i) + \iota(N)$.

Let us prove by induction on $i$ that $i(V_i) \geq i + \iota(M)$, then we will be done. In fact this will end up proving that

$$i(V_i \otimes_k N) \geq i + \iota(M) + \iota(N),$$

a stronger result.

Assume that $M \neq 0$.

First for $i = 0$, since one has an exact sequence

$$0 \to \ker \theta_0 \to V_0 \otimes_k R \to M \to 0.$$  

Thus, $V_0 \neq 0$, and $\ker \theta_0 \subseteq \mathfrak{m}(V_0 \otimes_k R)$. Since all nonzero homogeneous elements of $\mathfrak{m}$ have strictly positive degree, $\iota(\ker \theta_0) \geq \iota(\mathfrak{m}(V_0 \otimes_k R)) > \iota(V_0 \otimes_k R) = \iota(V_0)$. Thus since $\iota(V_0) = \iota(V_0 \otimes_k R) = \min \{ \iota(\ker \theta_0), \iota(M) \}$, we must have $\iota(V_0) = \iota(M)$. 

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Let \( i \geq 1 \) and assume that \( \iota(V_j) \geq j + \iota(M) \) for \( j < i \).

Now, for every \( t \geq 1 \), we have an exact sequence in \( \mathcal{C}(R) \)

\[
\cdots \rightarrow V_t \otimes_k R \overset{\theta_t}{\rightarrow} V_{t-1} \otimes_k R \overset{\theta_{t-1}}{\rightarrow} \cdots
\]

such that \( \ker \theta_t \subseteq \mathfrak{m}(V_t \otimes_k R) \). Since the elements of \( \mathfrak{m} \) have strictly positive degree, for every \( t \geq 1 \), if \( V_t \neq 0 \),

\[
\iota(\ker \theta_t) \geq \iota(\mathfrak{m}(V_t \otimes_k R)) > \iota(V_t \otimes_k R) = \iota(V_t).
\]

However we also have the short exact sequence

\[ 0 \rightarrow \ker \theta_t \rightarrow V_t \otimes_k R \overset{\theta_t}{\rightarrow} \ker \theta_{t-1} \rightarrow 0, \]

so that

\[
\iota(V_t) = \iota(\ker \theta_t) = \min\{\iota(\ker \theta_t), \iota(\ker \theta_{t-1})\},
\]

forcing

\[
\iota(V_t) = \iota(\ker \theta_{t-1}),
\]

for every \( t \) such that \( V_t \neq 0 \).

Therefore if \( V_t \neq 0 \),

\[
\iota(V_t) = \iota(\ker \theta_{t-1}) > \iota(V_{t-1} \otimes_k R) = \iota(V_{t-1}) \geq i - 1 + \iota(M),
\]

with the last inequality true by induction. Noting the middle strict inequality if \( V_t \neq 0 \), we get \( \iota(V_t) \geq i + \iota(M) \). Of course if \( V_t = 0 \), this last inequality is still true.

This has consequences for the Poincare series of \( \text{Tor}_i^R(M,N) \), generalizing Corollary 6.14.

**Corollary 6.28.** (Lemma 7 ii) and iii) of [1]) If \( M, N \in \mathcal{C}(R) \), then

a. \( P_{\text{Tor}_i^R(M,N)}(t) = \sum_{j \geq i + \iota(M) + \iota(N)} v\dim_k(\text{Tor}_j^R(M,N))_j t^j \).

b. The alternating sum

\[
\chi(M,N)(t) = \sum_{i \geq 0} (-1)^i P_{\text{Tor}_i^R(M,N)}(t)
\]

is a well-defined Laurent series with integer coefficients and

\[
P_R(t) \chi(M,N)(t) = P_M(t) P_N(t).
\]

*Proof.* As in lemma 6.15, let \( V \) be a graded minimal resolution of \( M \), after tensoring with \( N \) we get the graded complex in \( \mathcal{C}(R) \)

\[
V \otimes_k N.
\]

Thus for each “graded” degree \( q \in \mathbb{Z} \), we have a complex

\[
W(q)
\]

of vector spaces over \( k \), where if \( j \geq 0 \) (\( j \) is the “complex-index”)

\[
W(q)_j = \sum_{a+b=q} V_{j,a} \otimes_k N_b,
\]

and the differential \( W(q)_j \rightarrow W(q)_{j-1} \) is induced by

\[
\theta_j \otimes 1 : V_{j,a} \otimes_k N_b \rightarrow V_{j-1,a} \otimes N_b.
\]

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The $i$th homology of this complex $W(q)$ is by definition

$$\text{Tor}^R_i(M, N)_q.$$ 

Now we know that, whenever it makes sense, the alternating sum of the vector space dimensions of the components of a complex equals the alternating sum of the vector space dimensions of the homology groups of that complex, so that (if it makes sense)

$$\sum_{j \geq 0} (-1)^j \text{vdim}_k(W(q)_j) = \sum_{j \geq 0} (-1)^j \text{vdim}_k(\text{Tor}^R_j(M, N)_q).$$

Now, clearly $\text{vdim}_k(W(q)_j) = \sum_{a+b=q} \text{vdim}_k(V_{j,a})\text{vdim}_k(N_b)$.

Therefore, if everything makes sense,

$$\sum_{i \geq 0} \left(-1\right)^i P_{\text{Tor}^R_i(M, N)}(t) = \sum_{i \geq 0} \left(-1\right)^i \left(\sum_{q \in \mathbb{Z}} \text{vdim}_k(\text{Tor}^R_i(M, N)_q) t^q\right) = \sum_q \left(\sum_{i \geq 0} (-1)^i \text{vdim}_k(\text{Tor}^R_i(M, N)_q) t^q\right) q$$

$$= \sum_q \left(\sum_{i \geq 0} (-1)^i \sum_{a+b=q} \text{vdim}_k(V_{i,a})\text{vdim}_k(N_b)\right) t^q$$

$$= \sum_q \left(\sum_{i \geq 0} \sum_{a+b=q} (-1)^i \text{vdim}_k(V_{i,a})\text{vdim}_k(N_b)\right) t^q$$

$$= \left(\sum_r \sum_{i \geq 0} (-1)^i \text{vdim}_k(V_{i,r}) t^r\right) \left(\sum_s \text{vdim}_k(N_s) t^s\right).$$

However, $V \otimes_k R$ is a free resolution of $M$, so that

$$\text{vdim}_k(M_a) = \sum_i (-1)^i \text{vdim}_k(V_i \otimes_k R)_a = \sum_i (-1)^i \sum_{c+d=a} \text{vdim}_k(V_{i,c})\text{vdim}_k(R_d)$$

yielding

$$P_M(t) = \left(\sum_r \sum_i (-1)^i \text{vdim}_k(V_{i,r}) t^r\right) P_R(t),$$

so that

$$P_R(t) \chi(M, N)(t) = P_M(t) P_N(t),$$

if everything makes sense. But everything can be made sense of because of lemma 6.15 and its proof, part a of the proposition and the fact that each $V_i$ is a finite-dimensional graded vector space over $k$, so that $V_{i,a} = 0$ for $|a| > 0$.

Now that we’ve proved this lemma and its corollary, we can make the following:

**Definition 6.29.** If $M, N \in C(R)$, the Euler-Poincare series of $M, N$ is defined to be

$$\chi_{\text{Euler-Poincare}}(M, N)(t) = \sum_{i \geq 0} (-1)^i P_{\text{Tor}^R_i(M, N)}(t).$$

### 6.2 Koszul complexes in $C(R)$

These are defined as in the nongraded case; but we review here very briefly and set notation. Throughout this section suppose that $M$ is in $C(R)$.

Suppose that $x \in \mathfrak{m}$ is a homogeneous element of degree $d(x) > 0$. 
**Definition 6.30.** \(K(x, R)\) is the complex of free modules in \(\mathcal{C}(R)\), with exactly two nonzero terms:

\[
K_1(x, R) = R(-d(x)) \xrightarrow{\partial} R = K_0(x, R),
\]

where \(\partial_z(z) = xz\), for every homogeneous \(z \in K_1(x, R)\).

Thus if \(x_1, \ldots, x_u\) is a sequence of homogeneous elements in \(\mathfrak{m}\), then we may recursively define

**Definition 6.31.** \(K(x_1, \ldots, x_u, R) \cong K(x_1, \ldots, x_{u-1}, R) \otimes_R K(x_u, R)\).

One can then show (by induction; or see later remarks in this section) that \(K(x_1, \ldots, x_u, R)\) is a complex of free modules in \(\mathcal{C}(R)\). Thus, for any \(M\) in \(\mathcal{C}(R)\), by considering \(M\) as a complex in \(\mathcal{C}(R)\) with only one possibly nonzero term \(M\) of complex-index zero, we may also define the complex in \(\mathcal{C}(R)\)

**Definition 6.32.** If \(x_1, \ldots, x_u \in \mathfrak{m}\) are homogeneous, and \(M\) is in \(\mathcal{C}(R)\),

\[
K^R(\bar{x}, M) \cong K(\bar{x}, R) \otimes_R M.
\]

In this section, and later, we often abbreviate notation, denoting the sequence \(x_1, \ldots, x_u\) by the symbol \(\bar{x}\) and/or deleting the superscript \(R\). We also have

**Definition 6.33.** If \(x_1, \ldots, x_u \in \mathfrak{m}\) are homogeneous, and \(M\) is in \(\mathcal{C}(R)\),

\[
H^R_i(\bar{x}, M) = H_i(K^R(\bar{x}, M)),
\]

for every \(i \geq 0\).

For example, if \(u = 1\),

\[
H^R_i(x, M) = \begin{cases} 
0 & i \geq 2 \\
(0 : x)_M & i = 1 \\
M/xM & i = 0.
\end{cases}
\]

We will need the following

**Definition 6.34.** Suppose that \(x_1, \ldots, x_u \in \mathfrak{m}\) is a sequence of homogeneous elements in \(R\) and \(M\) is in \(\mathcal{C}(R)\). Then, this sequence is a \(*M\)-sequence if and only if \(x_1\) is not a zero-divisor on \(M\), and for each \(i > 1\), \(x_i\) is not a zero-divisor on \(M/(x_1, \ldots, x_{i-1})M\). Here, \((x_1, \ldots, x_j)\) denotes the ideal in \(R\) generated by \(x_1, \ldots, x_j\), for each \(j\).

We collect some facts about Koszul complexes. The proof of the next theorem is given exactly by the cited proof in the ungraded case.

**Theorem 6.35.** ([9], Corollary, IV-3; Proposition 2, IV-4) Suppose that \(M\) is in \(\mathcal{C}(R)\). If \(x_1, \ldots, x_u \in \mathfrak{m}\) is a sequence of homogeneous elements that form a \(*M\)-sequence, then \(K^R(\bar{x}, M)\) is an acyclic complex in \(\mathcal{C}(R)\) and \(H^R_0(\bar{x}, M) = M/(x_1, \ldots, x_u)M\).

The first part of the following theorem is a corollary to the theorem immediately above. The remaining parts follow from the definitions.

**Theorem 6.36.** If \(x_1, \ldots, x_u \in \mathfrak{m}\) is a sequence of homogeneous elements that form a \(*R\)-sequence, then

i) \(K(\bar{x}, R)\) is an acyclic complex of free modules in \(\mathcal{C}(R)\), and \(H^R_0(\bar{x}, R) \cong R/(x_1, \ldots, x_u)\).

ii) If, in addition, \(x_1, \ldots, x_u\) generate the superfluous ideal \(\mathfrak{m}\) of \(R\), then \(K(\bar{x}, R)\) is a free resolution of \(k\) in \(\mathcal{C}(R)\) and so, for every \(i\), \(H^R_i(\bar{x}, M)\) is isomorphic in \(\mathcal{C}(R)\) to \(\text{Tor}^R_i(k, M) = \text{Tor}^R_i(M, k)\).

Now, if \(V\) is any graded vector space over \(k\), the exterior products \(\wedge^p(V)\), for \(p \geq 0\), are defined as usual; we note that these vector spaces inherit a grading from that of \(V\).

**Example 6.37.** The exterior algebra formulation of the Koszul complex: Let \(M\) be in \(\mathcal{C}(R)\) and suppose that \(x_1, \ldots, x_u \in \mathfrak{m}\). Then,
i) For every $p \geq 0$, $K^R(\bar{x}, M)_p$ can be identified with
\[ \wedge^p(k(-d(x_1)) \oplus \cdots \oplus k(-d(x_u))) \otimes_k M, \]
as an object in $C(R)$.

ii) Using the identification of i), the differentials of $K^R(\bar{x}, M)$ may be identified as the map defined by
\[ \partial((e_1 \wedge \cdots \wedge e_p) \otimes m) = \sum_{k=1}^{p} (-1)^{k+1}(e_1 \wedge \cdots \wedge e_k \wedge \cdots \wedge e_p) \otimes x_k m. \]

Here, $e_i$ is a basis element of degree $d(x_i)$ in $k(-d(x_1)) \oplus \cdots \oplus k(-d(x_u))$, for $1 \leq i \leq u$.

Example 6.38. Hilbert’s syzygy theorem states that for a finitely generated graded module $M$ over a polynomial ring $S = k[x_1, \ldots, x_n]$, a free resolution of length at most $n$ exists.

Certainly $\bar{x}$ forms a $^*S$-sequence and we’ve shown that when this is the case, $K^S(\bar{x}, S)$ is a free and acyclic complex which forms a free resolution of $k$.

Thus, $\text{Tor}^S_k(\bar{x}, M) \cong \text{Tor}^S_k(M, k) = H^S_k(\bar{x}, M)$. We have also shown that $M$ has a free resolution of the form $\cdots \rightarrow \text{Tor}^S_2(M, k) \otimes_k R \rightarrow \cdots \rightarrow \text{Tor}^S_1(M, k) \otimes_k R \rightarrow 0$, but each term $\text{Tor}^S_i(M, k) = H^S_i(\bar{x}, M)$ and since the exterior algebra is 0 past the dimension of $n$, so too is this resolution.

Example 6.39. In any case, a direct computation reveals
\[ H^R_i(x_1, \ldots, x_u, M) = \begin{cases} 0 & i > u \\ (0 : x_1, \ldots, x_u)_M & i = u \\ M/(x_1, \ldots, x_u)M & i = 0. \end{cases} \]

Lemma 6.40. For a fixed sequence $x_1, \ldots, x_u$, there are functors

- $M \mapsto K^R(x_1, \ldots, x_u, M)$, from $C(R)$ to the category of complexes in $C(R)$, and
- for every $i \geq 0$, $M \mapsto H^R_i(x_1, \ldots, x_u, M)$, from $C(R)$ to itself.

In addition, the first functor is an exact functor, and thus, given any short exact sequence in $C(R)$
\[ 0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0, \]

there is a long exact sequence in $C(R)$
\[ 0 \rightarrow H^R_u(\bar{x}, M') \rightarrow H^R_u(\bar{x}, M) \rightarrow H^R_u(\bar{x}, M'') \rightarrow H^R_{u-1}(\bar{x}, M') \rightarrow \cdots \]
\[ \cdots \rightarrow H^R_1(\bar{x}, M'') \rightarrow H^R_0(\bar{x}, M') \rightarrow H^R_0(\bar{x}, M) \rightarrow H^R_0(\bar{x}, M'') \rightarrow 0. \]

The second statement in the lemma above can be made stronger if $\mathcal{I}$ is a graded ideal of definition for $M$, using the corollary to:

Proposition 6.41. ([9], Proposition 4, IV-7) Suppose that $M$ is in $C(R)$, and that $\mathcal{I}$ is the ideal generated by the homogeneous elements $x_1, \ldots, x_u \in \mathfrak{m}$. Then
\[ \text{Ann}_R(H^R_i(x_1, \ldots, x_u, M)) \supseteq \mathcal{I} + \text{Ann}_R(M), \]

for every $i \geq 0$.

The following two corollaries allow us to compute the Euler characteristics on homology for Koszul complexes defined by graded ideals of definition in $C(R)$. This will serve as one of our main computational tools in chapter 7 on multiplicities.

Corollary 6.42. Suppose that $M$ is in $C(R)$, and that $\mathcal{I}$ is an ideal of definition of $M$, generated by the homogeneous elements $x_1, \ldots, x_u \in \mathfrak{m}$. Then, $H^R_j(\bar{x}, M)$ is in $C(k)$, for every $j \geq 0$. 

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Proof. We know that \( H^R_j(\bar{x}, M) \) is a subquotient \( \ker \partial_j/\text{im} \partial_{j+1} \cong Z_j/B_j \) of \( V_j \otimes_k M \), for a graded vector space \( V_j \) in \( C(k) \). Also the proposition above gives \( \Im H^R_j(\bar{x}, M) = 0 \). Therefore,

\[
H^R_j(\bar{x}, M) = H^R_j(\bar{x}, M)/\Im H^R_j(\bar{x}, M) = (Z_j/B_j)/\Im(Z_j/B_j) \cong Z_j/(\Im Z_j + B_j),
\]

and

\[
Z_j/\Im Z_j \to Z_j/(\Im Z_j + B_j)
\]
is surjective.

For \( r \in R, v \in V_j \), and \( m \in M \), the action of \( R \) on \( V_j \otimes_k M \) is defined by \( r \cdot (v \otimes_k m) = v \otimes_k rm \). Then, \( \Im (V_j \otimes_k M) \cong V_j \otimes_k \Im M \). Now,

\[
(V_j \otimes_k M)/\Im(V_j \otimes_k M) \cong (V_j \otimes_k M)/(V_j \otimes_k \Im M) \\
\cong V_j \otimes_k M/\Im M,
\]

where the second line follows from lemma 6.3. By hypothesis, \( \Im \) is a graded ideal of definition for \( M \) which implies that \( \text{vdim}_k(M/\Im M) < \infty \) and therefore, \( \Im \) is an ideal of definition for \( V_j \otimes_k M \).

By theorem 2.19 \( \Im \) is a graded ideal of definition for \( Z_j \) since \( Z_j \) is a submodule of \( V_j \otimes_k M \). Thus, \( Z_j/\Im Z_j \) and hence \( H^R_j(\bar{x}, M) \) are finite dimensional vector spaces over \( k \).

Since \( H^R_j(x_1, \ldots, x_u, M) = 0 \) for \( j > u \), this has the additional corollary

**Corollary 6.43.** Suppose that \( M \) is in \( C(R) \), and that \( \Im \) is an ideal of definition of \( M \), generated by the homogeneous elements \( x_1, \ldots, x_u \in m \). Then, there is a positive integer \( T \) such that

\[
H^R_j(\bar{x}, M)_t = 0
\]

for every \( j \) and every \( t \geq T \).

### 6.3 A Filtration on the Koszul Complex

Suppose that \( M \) is in \( C(R) \), \( \Im \) is an ideal of definition for \( M \) and \( x_1, \ldots, x_u \) are homogeneous elements that generate \( \Im \) as an ideal in \( R \).

Recall the definition of the associated graded module, \( \text{gr}_\Im(M) = \oplus_{n \geq 0} \Im^n M/\Im^{n+1} M \). We saw in chapter 4 that this is a finitely generated graded module over the associated graded ring \( H(\bar{x}) = \oplus_{n \geq 0} \text{m}(\bar{x})^n/\text{m}(\bar{x})^{n+1} \) (where \( H_0(\bar{x}) = k \)). Again, \( \text{m}(\bar{x}) \) is the superfluous \( k \)-ideal of the graded \( k \)-subring \( k/\bar{x} \).

Specifically, \( \text{gr}_\Im(M) \) is finitely generated by a \( k \)-basis for \( M/\Im M \) as an \( H(\bar{x}) \)-module. Now, \( H(\bar{x}) \) is generated as a graded \( k \)-algebra by any spanning set of the vector space \( \text{m}(\bar{x})/\text{m}(\bar{x})^2 \). In particular, if \( \xi_1, \ldots, \xi_u \) are the images of \( x_1, \ldots, x_u \) in \( \text{m}(\bar{x})/\text{m}(\bar{x})^2 \), then \( \xi_1, \ldots, \xi_u \) generate the superfluous ideal

\[
\text{m}(H) = \oplus_{i \geq 1} \text{m}(\bar{x})^i/\text{m}(\bar{x})^{i+1}
\]
of \( H(\bar{x}) \).

Passing in effect to the category \( C(H(\bar{x})) \), we may form the Koszul complex \( \text{K}^H(\bar{x})(\bar{\xi}, \text{gr}_\Im(M)) \). We denote the homology groups of this complex by \( H^H_j(\bar{x})(\bar{\xi}, \text{gr}_\Im(M)) \).

In this context, the analogues to proposition 6.41 and its corollaries 6.42 and 6.43 are as follows.

**Proposition 6.44.** Suppose that \( M \) is in \( C(R) \), and that \( \Im \) is the ideal generated by the homogeneous elements \( x_1, \ldots, x_u \in m \). If \( \Im \) is an ideal of definition for \( M \)

i.) \( \text{Ann}_{H(\bar{x})}(H^H_j(\bar{\xi}, \text{gr}_\Im(M)) \subseteq \text{m}(H), \) for every \( j \geq 0 \).

ii.) \( H^H_j(\bar{\xi}, \text{gr}_\Im(M)) \) is in \( C(k) \), for every \( j \geq 0 \).

iii.) There exists an \( I \) such that \( H^H_j(\bar{\xi}, \text{gr}_\Im(M))_i = 0, \) for every \( j \), and every \( i > I \).
Definition 6.45. For each \(n \geq 0\), we define the complex \(F^nK\) by defining the \(j\)th \(R\)-submodule of the complex by \((F^nK)_j \cong K^n_j(\vec{x}, \mathcal{I}^{n-j}M)\), for every \(i, j\).

We use the convention that \(\mathcal{I}^pM = M\) when \(p \leq 0\). In particular we have that \((F^iK)_j = K^R(\vec{x}, M)_j\) for \(j \geq i\).

We know that \(F^nK_j\) is an \(R\)-submodule of \(K^R(\vec{x}, M)\) for all \(n\) and \(j\), and it is not hard to check that the differentials for the complex \(K^R(\vec{x}, M)\) restrict to differentials on the complex \(F^nK:\) We have \(F^nK_j = K^n_j(\vec{x}, \mathcal{I}^{n-j}M)\). If we let \(y \in \mathcal{I}^{n-j}M\), using the exterior algebra formulation of the Koszul complex we have \(\partial(e_i \wedge \cdots \wedge e_j \otimes_k y) = \sum_{k=1}^j (e_i \wedge \cdots \wedge \hat{e}_k \wedge \cdots \wedge e_j) \otimes_k x_{ik} y\). Of course, \(x_{ik} \in \mathcal{I}\) so \(x_{ik}y \in \mathcal{I}^{n-j+1}\) for each \(k\) and \(i\). Since \(F^nK_{j-1} = K^n_{j-1}(\vec{x}, \mathcal{I}^{n-j+1}M)\) we see that \(\partial: F^nK_j \rightarrow F^nK_{j-1}\).

For every \(j \geq 0\) we get a filtration of the \(j\)th component of the Koszul complex \(K^n_j(\vec{x}, M)\) as follows:

\[
\cdots \subseteq F^{n+1}K_j \subseteq \cdots \subseteq F^1K_j \subseteq F^0K_j = K^n_j(\vec{x}, M).
\]

Proposition 6.46. Let \(M\) and \(\mathcal{I}\) be as above, then for any \(n\) and \(j\) greater than or equal to 0 we have

i) \((F^nK)_j \cong \mathcal{I}^{n-j}K^n_j(\vec{x}, M)\) as \(R\)-modules.

ii) For every \(j \geq 0\), the filtration of the Koszul complex above is an \(\mathcal{I}\)-bonne filtration.

Proof: i)

\[
F^nK_j \cong K^n_j(\vec{x}, \mathcal{I}^{n-j}M) = K^n_j(\vec{x}, R) \otimes \mathcal{I}^{n-j}M \\
\cong \mathcal{I}^{n-j}K^n_j(\vec{x}, M) \\
\cong \mathcal{I}^{n-j}K^n_j(\vec{x}, M)
\]

ii) Using part i, we have

\[
\mathcal{I} \cdot F^nK_j \cong \mathcal{I}K^n_j(\vec{x}, \mathcal{I}^{n-j}M) \\
\cong \mathcal{I}\mathcal{I}^{n-j}K^n_j(\vec{x}, M) \\
\cong \mathcal{I}^{n+1-j}K^n_j(\vec{x}, M) \\
\cong K^n_{j+1}(\vec{x}, \mathcal{I}^{n+1-j}M) \\
\cong F^{n+1}K_j
\]

\(\square\)

Definition 6.47. Define \(\text{gr}K_j = \bigoplus_i F^iK_j/F^{i+1}K_j\). For each \(j\), \(\text{gr}K_j\) is a finitely generated graded \(R\)-module where the grading is defined \(\text{gr}K_{j,k} = F^K_{j,k}/F^{k+1}_{j,k}\). (Note that \(\text{gr}K_j\) also inherits a grading from \(C(R)\), but for now we may ignore this grading altogether.)

Proposition 6.48. For each \(j\), \(\text{gr}K_j \cong K_R^n(\vec{x}, \mathcal{I}^{k-j}M/\mathcal{I}^{k-j+1}M)\) are isomorphic \(R\)-modules. It follows that for every \(k\), \(\text{gr}K_{j,k} \cong K_j(\vec{x}, \mathcal{I}^{k-j}M/\mathcal{I}^{k-j+1}M)\).
Proof. For any fixed $j$ we have\[\begin{align*}
gr K_j &\cong \oplus_i F^i K_j / F^{i+1} K_j \\
&= \oplus_i (K_j(\bar{x}, R) \otimes_R \mathcal{J}^{i-j} M) / (K_j(\bar{x}, R) \otimes_R \mathcal{J}^{i-j+1} M) \\
&\cong \oplus_i K_j(\bar{x}, R) \otimes_R \mathcal{J}^{i-j} M /
\mathcal{J}^{i-j+1} M \\
&\cong K_j(\bar{x}, R) \otimes_R \oplus_{i} \mathcal{J}^{i-j} M \\
&= K_j(\bar{x}, R) \otimes_R \mathcal{J}^{i-j} M \\
& \cong \mathcal{J}^j M(\bar{x}, \mathfrak{g}_3 M). \\
\end{align*}\]

Using the computation above, we see that\[\begin{align*}
gr K_{j,k} &\cong K_j(\bar{x}, R) \otimes_R (\mathcal{J}^{k-j} M / \mathcal{J}^{k-j+1} M) \\
& \cong K_j(\bar{x}, \mathcal{J}^{k-j} M / \mathcal{J}^{k-j+1} M). \\
\end{align*}\]

\[\square\]

**Definition 6.49.** We can define a graded complex \(\mathfrak{gr} K^R(\bar{x}, M)\), by taking the \(j\)th submodule to be \(gr K_j\), and defining the differentials as follows.

By the lemma above, \(gr K_{j,k} \cong K_j(\bar{x}, \mathcal{J}^{k-j} M / \mathcal{J}^{k-j+1} M)\). We use the exterior algebra formulation of the Koszul complex: \(gr K_{j,k} \cong \Lambda^j(\mathcal{V}) \otimes_k (\mathcal{J}^{k-j} M / \mathcal{J}^{k-j+1} M)\). To compute the differentials, let \(t \in \mathcal{J}^{k-j} M\), we have\[\begin{align*}
\partial (e_{i_1} \wedge \cdots \wedge e_{i_m} \otimes_k (t + \mathcal{J}^{k-j+1} M)) &= \Sigma_{m=1}^i (-1)^{m+1} (e_{i_1} \wedge \cdots \wedge e_{i_m} \wedge \cdots e_{i_j}) \otimes_k (tx_j + \mathcal{J}^{k-j+2} M) \\
\end{align*}\]
The right hand side is an element of \(\Lambda^{j-1}(\mathcal{V}) \otimes_k (\mathcal{J}^{k-j+1} M / \mathcal{J}^{k-j+2} M) \cong gr K_{j-1,k}\), so we have \(\partial : gr K_{j,k} \to gr K_{j-1,k}\) and \(\partial \circ \partial = 0\).

We denote the homology of \(\mathfrak{gr} K^R(\bar{x}, M)\) by \(H_* (\mathfrak{gr} K^R)\) or \(H_* (\oplus_i F^i K / F^{i+1} K)\).

**Lemma 6.50.** The complex \(\mathfrak{gr} K^R(\bar{x}, M) \cong \oplus_i F^i K / F^{i+1} K\) is isomorphic to the complex \(K^H(\bar{\xi}, \mathfrak{g}_3 M)\).

**Proof.** Let’s use the exterior algebra formulation of the Koszul complex. Suppose that \(V\) is the finite dimensional graded vector space over \(k\) satisfying \(K^R(\bar{x}, \mathfrak{g}_3 M) \cong \Lambda^j(\mathcal{V}) \otimes_k \mathfrak{g}_3 M\). Suppose that \(\{e_1, \ldots, e_n\}\) forms a homogeneous basis for \(V\). Then, let \(A_j\) be the finite, ordered multi-index set such that \(\{e_\alpha : \alpha \in A_j\}\) is a graded \(k\)-basis for \(\Lambda^j(V)\).

We have isomorphisms of graded vector spaces over \(k\),
\[\begin{align*}
gr K_j &\cong \mathcal{J}^j M(\bar{x}, \mathfrak{g}_3 M) \\
&\cong \Lambda^j(V) \otimes_k \mathfrak{g}_3 M \\
&\cong \oplus_\alpha \mathfrak{g}_3 M \cdot e_\alpha. \\
\end{align*}\]

On the other hand, let’s consider the complex \(K^H(\bar{\xi}, \mathfrak{g}_3 M)\). Using the exterior algebra formulation, we have \(K^H(\bar{\xi}, \mathfrak{g}_3 M) \cong \Lambda^j(\mathcal{V}_\xi) \otimes_k \mathfrak{g}_3 M\), where \(V_\xi\) is the corresponding \(k\)-vector space to \(\xi\). However, \(\bar{x}\) and \(\xi\) are in one to one correspondence, so we have an isomorphism of free \(k\)-modules \(V\) and \(V_\xi\). Then we can use the same elements \(e_\alpha\), to serve as the “place holders” for basis elements of \(\Lambda^j(V_\xi)\), then we see that
\[\begin{align*}
K^H(\bar{\xi}, \mathfrak{g}_3 M) &\cong \Lambda^j(V_\xi) \otimes_k \mathfrak{g}_3 M \\
&\cong \oplus_\alpha \mathfrak{g}_3 M \cdot e_\alpha. \\
\end{align*}\]
Thus, the $j^{th}$ component of each complex are isomorphic as $k$-vector spaces.

To see that the differentials of the complexes agree, observe that for $\partial : \text{gr} K_{j,k} \rightarrow \text{gr} K_{j-1,k}$, we saw that for $t \in \mathcal{I}^{j-1} M$,

$$\partial (e_1 \wedge \cdots \wedge e_i \otimes (t + \mathcal{I}^{k-j+1} M)) \equiv \sum_{m=1}^{i+1} (-1)^{m+1} (e_1 \wedge \cdots \wedge \hat{e}_m \wedge \cdots e_i) \otimes (t x^{km} + \mathcal{I}^{k-j+2} M).$$

Of course, $\xi_j \cdot (t + \mathcal{I}^{k-j+1} M) \equiv (x_{jk} t + \mathcal{I}^{k-j+2} M)$, so it follows that $\partial$ for $K^H(\xi, \text{gr}_2 M)$ agrees with $\partial$ for $\text{gr} K^R(\bar{x}, M)$.

\begin{corollary}
\begin{proof}
By basic homological algebra, we have that

$$H_j(\text{gr} K^R) \cong H_j(\text{gr} K^F) \cong \oplus H_j(\text{gr}_2 M).$$

Using the isomorphism from the previous lemma, for each $j$ we have that $H_j(\text{gr}_2 M) \cong H_j(F^j K/F^{j+1} K)$. From proposition 6.44, we have that for every $j$, $H_j^R(\bar{x}, \text{gr}_2 M)$ is a finite dimensional $k$-vector space, and for all $j > u$, $H_j^H(\bar{x}, \text{gr}_2 M) = 0$. Using the isomorphism above, we see that for each $j$ there exists an integer $m(j) > 0$ such that $H_j(F^j K/F^{j+1} K) = 0$ whenever $j > m(j)$.

Now by taking $m = \max\{m(1), \ldots, m(u)\}$, then whenever $i > m \geq m(j)$, $H_j(F^i K/F^{i+1} K) = 0$ for all $j$. The second statement follows from induction on $p$, using the short exact sequence of complexes

$$0 \rightarrow F^{i+p-1} K/F^{i+p} K \rightarrow F^i K/F^{i+p-1} K \rightarrow \cdots \rightarrow 0.$$ 

We have seen before that for any $j$, $(F^s K/F^t K)_j \cong K_j^R(\bar{x}, \mathcal{I}^{s-j} M/\mathcal{I}^{t-j} M)$. Moreover, when $p \geq 1$, $K_{j}^R(\bar{x}, \mathcal{I}^{j-p} M/\mathcal{I}^{j+p} M)$ is an $R$-submodule of $K_{j}^R(\bar{x}, \mathcal{I}^j M/\mathcal{I}^{j+p} M)$ for every $j$. Now take the first map in the sequence above to be inclusion and the second to be projection and we see that for each $j$, we get a short exact sequence of graded $R$-modules.

Using the long exact sequence on homology from lemma 6.40, we get the following exact sequences for $i > m$:

$$0 \rightarrow H_{i}^R(F^{i+p-1} K/F^{i+p} K) \rightarrow H_{i}^R(F^i K/F^{i+p-1} K) \rightarrow H_{i+1}^R(F^{i+p-1} K/F^{i+p} K) \rightarrow \cdots$$

Above we proved that $H^R(F^i K/F^{i+1} K) = 0$ for every $j$ and $i > m$, we proceed by induction on $p$, using this as the base case $p = 1$. For the inductive hypothesis, let $n > 1$ and suppose that for every $i > m$, every $p < n$, and every $j$, $H^R(F^i K/F^{i+n} K) = 0$. We need to show that $H^R(F^i K/F^{i+n} K) = 0$ for all $i > m$ and every $j$.

However, the induction hypothesis implies that both $H_j(F^{i+n-1} K/F^{i+n} K)$ and $H_j(F^{i+n-1} K/F^{i+n-1} K)$ equal $0$ for all $j$. Thus, using exactness, $H^R(F^i K/F^{i+n} K) = 0$.

\end{proof}
\end{corollary}

\begin{lemma}
\begin{proof}
The previous corollary gave that $H_j(F^i K/F^{i+p} K) = 0$ for every $j$ and every $p \geq 0$, whenever $i > m$. Now, $F^{i+p} K_j \cong K_j(\bar{x}, \mathcal{I}^{i+p-j} M) \cong \mathcal{P}^i K_j(\bar{x}, \mathcal{I}^{i-j} M) \cong \mathcal{P}^i F^i K_j$, thus $H_j(F^i K/F^{i+p} K) = 0$ for every $j$, every $i > m$, and every $p \geq 0$.

Define $Z^i_j = \ker(\partial : F^i K_j \rightarrow F^i K_{j-1})$ and $B^i_j = \text{Im}(\partial : F^i K_{j+1} \rightarrow F^i K_j)$, we have that $B^i_j \subseteq Z^i_j \subseteq F^i K_j$. Let’s fix $i > m$ and $j \geq 0$, and let $z \in Z^i_j$. Then, the coset $z + \mathcal{P}^i F^i K_j$ is a cycle in $(F^i K/F^{i+p} K)$.

Therefore, there exists a $w \in F^i K_{j+1}$ such that $\partial(w + \mathcal{P}^i F^i K_{j+1}) = z + \mathcal{P}^i F^i K_j$. In other words, $z \in B^i_j + \mathcal{P}^i F^i K_j$. Since this is true for every $p \geq 0$, $z \in \cap_p (B^i_j + \mathcal{P}^i F^i K_j)$.
We now show that $\cap_p (B_j^i + 3^p F^i K_j) = B_j^i + \cap_p 3^p F^i K_j$. This is a general fact for modules in $C(R)$, so we prove it in generality.

Let $X, Y \in C(R)$, with $X$ a submodule of $Y$, and $J$ a graded ideal in $R$. It is immediate that $X + \cap_p 3^p Y \subseteq \cap_p (X + 3^p Y)$. Let $z \in \cap_p (X + 3^p Y)$, since both $X$ and $Y$ are graded, and $J$ is a graded ideal, it follows that $\cap_p (X + 3^p Y)$ is graded, without loss of generality we suppose that $z$ is a homogeneous element of degree $d$. For every $p$, $z \in X + 3^p Y$, and suppose that $z$ may be written as $x_p + \sum_i \alpha_{i,p} y_{p,i}$, where $x_p \in X$, $\alpha_{i,p} \in 3^p$, $y_{p,i} \in Y$.

It follows that for each $p$, if $\deg(x_p) = d$ then $\deg(\alpha_{i,p}) + \deg(y_{p,i}) = d$. Now, $\deg(y_{p,i}) \geq \iota(Y)$ and $\deg(\alpha_{i,p}) \geq p$, but the degrees sum to $d$ for every $p$ so there exists an $N > 0$ such that for $p > N$, $\deg(y_{p,i}) < \iota(Y)$. However, this occurs if and only if for all $i$, $y_{p,i} = 0$ if and only if $z = x_p$. This shows that $\cap_p (X + 3^p Y) = X + \cap_p (3^p Y) = X$.

Returning to the original argument, we have that $z \in Z_j^i$, and also $z \in \cap_p (B_j^i + 3^p F^i K_j) = B_j^i + \cap_p (3^p F^i K_j) = B_j^i$. Of course, by definition $H_j^R(F^i K) \cong Z_j^i / B_j^i$, so we have shown that $H_j^R(F^i K) = 0$ whenever $i > m$.

7 Multiplicities

7.1 Koszul multiplicities

For $M, N \in C(R)$, we’ve already introduced “Euler-Poincare series” $\chi(M, N)(t)$ in section 6.1.

In this section, we suppose that $x_1, \ldots, x_u$ is a sequence of homogeneous elements in the superfluous ideal $m$ of $R$, that $J$ is the ideal generated by $x_1, \ldots, x_u$ in $R$, and that $K^R(\bar{x}, M)$ is the associated Koszul complex, as in section 6.2. Again, using previous established notation, we let $k(\bar{x})$ denote the subring of $R$ generated by $x_1, \ldots, x_u$ and $m(\bar{x})$ denote the superfluous ideal in that ring.

**Definition 7.1.** Suppose that $J$ is a graded ideal of definition for $M$, so that $H_j^R(\bar{x}, M)$ is a finite dimensional graded vector space over $k$, for every $j$. Let

- $h_j^R(\bar{x}, M) \cong \dim_k(H_j^R(\bar{x}, M)))$, for every $j$—this is the total dimension of the graded vector space $H_j^R(\bar{x}, M)$.

- The Euler characteristic associated to the Koszul complex is defined to be

$$\chi(\bar{x}, M) = \sum_{j=0}^{u} (-1)^j h_j^R(\bar{x}, M).$$

We will call this integer the **Koszul multiplicity** of $M$ corresponding to $\bar{x}$.

If $n$ is an integer, then let’s observe that for the graded $R$ module $M(n)$, the associated Koszul complex $K^R(\bar{x}, M(n))$ has exactly the same “complex” grading as $K^R(\bar{x}, M)$; also, if $J$ is an ideal of definition for $M$, it is also an ideal of definition for $M(n)$.

Note also that for every $j$, as ungraded vector spaces over $k$,

$$H_j^R(\bar{x}, M) = H_j^R(\bar{x}, M(n)),$$

and so

$$h_j^R(\bar{x}, M) = h_j^R(\bar{x}, M(n))$$

and

$$\chi(\bar{x}, M) = \chi(\bar{x}, M(n)).$$

**Example 7.2.** Let $M \in C(R)$ and $x_1, \ldots, x_u$ be a *M-sequence, then $K^R(\bar{x}, M)$ has $H_0^R(\bar{x}, M) = M/(x_1, \ldots, x_u)M$ as its only nonzero homology group (theorem 6.35). If it turns out that $x_1, \ldots, x_u$ is also a *sop, then corollaries 6.42 and 6.43 tell us that the Euler characteristic on homology is defined, and in this case $\chi^R(\bar{x}, M) = \text{vdim}_k(M/(x_1, \ldots, x_u)M)$.

This example leads naturally to a discussion of *depth and Cohen-Macaulay rings, but that is beyond the scope of this paper.
**Theorem 7.3.** Suppose that $0 \to N \to M \to P \to 0$ is an exact sequence in $\mathcal{C}(R)$, and that $\mathcal{J}$ is a graded ideal of definition for $M$ generated by homogeneous elements $x_1, \ldots, x_u$. Then, $\mathcal{J}$ is an ideal of definition for $N$ and $P$, and

$$
\chi^R(\bar{x}, M) = \chi^R(\bar{x}, N) + \chi^R(\bar{x}, P).
$$

**Proof.** From this short exact sequence, we get a short exact sequence of Koszul complexes, and hence a long exact sequence on homology of the Koszul complexes via lemma 6.40. Apply corollaries 6.42 and 6.43, and use that $\chi$ adds over the long exact sequences to conclude the result.

### 7.1.1 The Relationship Between Koszul Multiplicities and Samuel Polynomials

Following the exposition in [9], we prove

**Theorem 7.4.** (Theorem 1, IV.3 A, [9]) Suppose that $M$ is in $\mathcal{C}(R)$, and that $\mathcal{J}$ is an ideal of definition for $M$, generated by the homogeneous elements $x_1, \ldots, x_u \in \mathfrak{m}$. Then,

$$
\chi^R(\bar{x}, M) = \sum_{j=0}^{u} (-1)^j h^R_j(\bar{x}, M) = \Delta^u p(\mathcal{J}, M, n),
$$

where $p(\mathcal{J}, M, n)$ is the Samuel function $p(\mathcal{J}, M, n) = \dim_k (M/\mathcal{J}^n M)$, for $n \gg 0$.

**Proof.** By lemma 6.52, there exists an $m \geq 0$ such that for all $i > m$ and all $j$, $H_j(\bar{x}, M) \cong H_j(K^R(\bar{x}, M)/F^i K)$. Therefore, for every $j$, and every $i > m$,

$$
H_j(\bar{x}, M) \cong H_j(K^R(\bar{x}, M)/F^i K).
$$

However note that, for any $i$, using the exterior algebra formulation of the Koszul complex and lemma 6.3 we may compute

$$(F^i K/F^{i+1} K)_j = (\Lambda^j(V) \otimes_k J^{i-j} M)/(\Lambda^j(V) \otimes_k J^{i-j+1} M) = \Lambda^j(V) \otimes_k J^{i-j} M/3^{i-j+1} M,$$

so that, since $\mathcal{J}$ is an ideal of definition for $M$

$$v\dim_k (F^i K/F^{i+1} K)_j < \infty,$$

and, if $j > u$

$$v\dim_k (F^i K/F^{i+1} K)_j = 0.$$

Thus, $\chi(F^i K/F^{i+1} K)$ is defined, for every $i$. In particular,

$$\chi(K^R(\bar{x}, M)/F^i K) = \sum_{j=0}^{u} (-1)^j v\dim_k ((K/F^i K)_j).$$

We compute, for every $i$,

$$v\dim_k ((K^R(\bar{x}, M)/F^i K)_j) = v\dim_k ((\Lambda^j(V) \otimes_k M)/(\Lambda^j(V) \otimes_k J^{i-j} M))$$

$$= v\dim_k (\Lambda^j(V) \otimes_k M/J^{i-j} M)$$

$$= v\dim_k (\Lambda^j(V)) \cdot v\dim_k (M/J^{i-j} M)$$

$$= \binom{u}{j} \cdot p(M, \mathcal{J}, i - j).$$

Now, assume $i \gg 0$, so that at least $i > m$. Then,
\[ \chi^R(\bar{x}, M) = \chi(H_*(K(\bar{x}, M))) = \chi(H_*(K/F^d K)) = \chi(K/F^d K) = \sum_{j=0}^n (-1)^j \binom{n}{j} \cdot p(M, \mathfrak{J}, i - j) = \Delta u p(M, \mathfrak{J}, i - u) \]

The second equality is true since \( i > m \), the third equality is true using Theorem 6.12 from the section on Euler characteristics of complexes, and the last is true using proposition 2.29 since the Samuel function is polynomial-like.

Finally, since \( i \) was taken to be sufficiently large, and \( u \) is a fixed integer, we rewrite the last line taking a dummy variable \( n \) to be sufficiently large, and get

\[ \chi^R(\bar{x}, M) = \Delta u p(M, \mathfrak{J}, n). \]

\[ \Box \]

7.2 Comparing Samuel Multiplicities to Koszul Multiplicities

In this section, \( M \) is in \( \mathcal{C}(R) \) and \( \mathfrak{J} \) is a graded ideal of definition for \( M \).

Now, we have seen that \(*\dim_R(M) = \dim_R(M) = d_1(M) = d(M) = s(M) = s_1(M)\), let us call this integer \( D(M) \) from now on.

We also know that the Samuel function \( p(M, \mathfrak{J}, n) \),

\[ n \mapsto \dim_k(M/\mathfrak{J}^n M) \]

is, for large \( n \), a polynomial in \( n \) of degree equal to \( D(M) \).

Now, using the results of section 2.4 in particular propositions 2.28 and 2.29, for a polynomial-like function \( f(n) \), \( \Delta u f(n) = a_d \cdot \frac{d!}{n^{d-u}} n^{d-u} + \text{L.O.T} \), where \( a_d \) is the leading coefficient of the degree \( d \) polynomial representing \( f(n) \), and \( n \) is sufficiently large. We also saw that \( a_d d! \in \mathbb{Z} \) and if \( f(n) \geq 0 \) for all \( n \) (but not identically 0 for all \( n \)), then \( a_d > 0 \).

We have seen that the Samuel function is polynomial-like, so using the information above, we define

**Definition 7.5.** The **Samuel Multiplicity** for a module \( M \in \mathcal{C}(R) \) with graded ideal of definition \( \mathfrak{J} \) is denoted \( e_R(M, \mathfrak{J}) \), and satisfies the following: For \( n >> 0 \),

\[ p(M, \mathfrak{J}, n) = \frac{e_R(M, \mathfrak{J}) n^{D(M)}}{D(M)!} + \text{terms of lower degree in } n. \]

**Remark 7.6.** By proposition 2.29, we have \( e_R(M, \mathfrak{J}) = \Delta^{D(M)}(p(M, \mathfrak{J}, n)) \), and that \( \Delta u p(M, \mathfrak{J}, n) = 0 \), if \( u > D(M) \). Also, \( e_R(M, \mathfrak{J}) \) is a positive integer if \( M \neq 0 \), using the same proposition.

The main result of this section is the following theorem; it is obtained immediately using our results about \( D(M) \) and its different equivalent definitions, and applying Theorem 7.4, we see

**Theorem 7.7.** Suppose that \( M \) is in \( \mathcal{C}(R) \), and that \( \mathfrak{J} \) is a graded ideal of definition for \( M \), generated by the homogeneous elements \( x_1, \ldots, x_u \in \mathfrak{m} \). Then,

- If \( u > D(M) \), then \( \chi^R(\bar{x}, M) = \sum_{j=0}^u (-1)^j h_j^R(\bar{x}, M) = 0. \)
- If \( u = D(M) \), so that \( x_1, \ldots, x_{D(M)} \) is a * sop for \( M \), then

\[ \chi^R(\bar{x}, M) = \sum_{j=0}^{D(M)} (-1)^j h_j^R(\bar{x}, M) = e_R(M, \mathfrak{J}), \]

so \( \chi^R(\bar{x}, M) \) is a nonnegative integer, and is strictly positive if \( M \neq 0 \).
Theorem 7.4. Proof. The first statement follows from the E Pluribus Unum theorem about dimension, the second from Theorem 7.4.

The last statement follows from the second since \( x_1, \ldots, x_{D(M)} \) is also a *sop for \( M \) as a \( k(\bar{x}) \)-module (from our characterization of graded ideals of definition 4.7) and as we have already noted in corollary 4.15, \( p(M, \mathcal{I}, n) = p(M, \mathfrak{m}(\bar{x}), n) \), for all \( n \geq 0 \).

Putting this all together, we have \( \chi_R^R(\bar{x}, M) = e_r(M, \mathcal{I}) = e_k(\bar{x})(M, \mathfrak{m}(\bar{x})) = \chi^{k(\bar{x})}(\bar{x}, M). \)

7.2.1 Multiplicities and the Euler-Poincare Series

Recall that if \( M, N \in \mathcal{C}(R) \), we have the well-defined Laurent series (corollary 6.28)–the Euler-Poincare series:

\[
\chi_R(M, N)(t) = \sum_{j \geq 0} (-1)^j \tau_{\mathcal{P}r}^j(M, N)(t),
\]

and we know that

\[
P_R(t) \chi_R(M, N)(t) = P_M(t)P_N(t).
\]

Considering again a *sop \( \bar{x} \) for \( M \), we have seen that (corollary 5.10) the graded subring \( k(\bar{x}) \) of \( R \) is a graded polynomial ring over \( k \), and that \( M \in \mathcal{C}(k(\bar{x})) \). Also, whether we consider \( M \in \mathcal{C}(R) \), or \( M \in \mathcal{C}(k(\bar{x})) \), the Poincare series of \( M \) does not change.

The results of chapter 6 (the Hilbert Syzygy Theorem) show that the Koszul complex \( K^{k(\bar{x})}(\bar{x}, k) \) is acyclic, thus is a free, finite resolution of \( k \) as a \( k(\bar{x}) \)-module. In particular, we may tensor this resolution with \( M \) and use it to compute \( \tau_i^{k(\bar{x})}(k, M) = \tau_i^{k(\bar{x})}(M, k) \), showing that

\[
\tau_i^{k(\bar{x})}(M, k) = H_i^{k(\bar{x})}(\bar{x}, M).
\]

This yields

Lemma 7.8. Let \( \bar{x} = x_1, \ldots, x_{D(M)} \) be a *sop for \( M \in \mathcal{C}(R) \), and let \( \mathcal{I} \) be the graded ideal in \( R \) generated by \( \bar{x} \). For every \( i \), \( \tau_i^{k(\bar{x})}(M, k) \in \mathbb{Z}[t, t^{-1}] \), and therefore \( \chi^{k(\bar{x})}(M, k)(t) \in \mathbb{Z}[t, t^{-1}] \). Furthermore,

\[
\chi^{k(\bar{x})}(M, k)(t) = \sum_{j=0}^{D(M)} (-1)^j \tau_{H_i^{k(\bar{x})}}(M, k)(t),
\]

and after evaluating this Laurent polynomial at \( t = 1 \), this means that

\[
\chi^{k(\bar{x})}(M, k)(1) = \chi^{k(\bar{x})}(\bar{x}, M) = \chi^R(\bar{x}, M) = e_R(\mathcal{I}, M).
\]

Proof. Corollary 6.28 shows the first part of the statement, and since the resolution \( K^{k(\bar{x})}(\bar{x}, k) \) is in any case zero for complex degree larger than \( D(M) \), \( \tau_i^{k(\bar{x})}(M, k) \) is also zero for \( i > D(M) \), so

\[
\chi^{k(\bar{x})}(M, k)(t) = \sum_{j=0}^{D(M)} (-1)^j \tau_{H_i^{k(\bar{x})}}(M, k)(t) = \sum_{j=0}^{D(M)} (-1)^j \tau_{H_i^{k(\bar{x})}}(M, k)(t),
\]

being a finite sum of Laurent polynomials is a Laurent polynomial. Also, \( \tau_{H_i^{k(\bar{x})}}(\bar{x}, M)(1) = h_j^{k(\bar{x})}(\bar{x}, M) \), by definition. The rest of the equalities follow from Theorem 7.7.

Thus, summarizing, we have

Theorem 7.9. Suppose that \( M \in \mathcal{C}(R) \) and \( x_1, \ldots, x_{D(M)} \) is a system of parameters for \( M \) as an \( R \)-module generating the ideal \( \mathcal{I} \) of \( R \). Then, the following multiplicities make sense and are equal:

- the Koszul multiplicity \( \chi^R(\bar{x}, M) \);
\begin{itemize}
  \item the Samuel multiplicity $e_R(M, \mathcal{I})$, 
  \item the Samuel multiplicity $e_k(\mathcal{I})(M, m(\mathcal{I}))$, 
  \item the Koszul multiplicity $\chi^k(\mathcal{I})(\mathcal{I}, M)$, 
  \item the value of the Laurent polynomial $\chi_k(\mathcal{I})(M, k)(t)$ at $t = 1$.
\end{itemize}

Also, the common number defined by any of the above is a nonnegative integer, and if $M \neq 0$, this number is a positive integer.

### 7.3 Maiorana’s $C$-multiplicity

Here as usual, for $M \in C(R)$, $D(M)$ is the integer defined in our E Pluribus Unum dimension theorem.

Given $M \in C(R)$, the Hilbert-Serre Theorem says that $t = 1$ is a pole for the Poincaré series $P_M(t)$, and our E Pluribus Unum theorem for dimension says that the order of the pole at $t = 1$ is exactly $D(M)$. Thus, we may make the following definition of a rational number $C(M)$ (adapting Maiorana’s notation [7]):

**Definition 7.10.** If $M \neq 0$, $C(M) = \lim_{t \to 1}(1 - t)^{D(M)}P_M(t)$, where $D(M) = \dim_R(M)$. If $M = 0$, $C(M) = 0$.

**Example 7.11.** We saw in section 3.3 that for $S = k[x_1, \ldots, x_n]$ where $\deg(x_i) = d_i$, if we call $R = S/(f)$ where $f \in S_d$, we have $P_R = \left( \frac{1}{1 - t} \right)^{n-1} \left[ \frac{1}{1 + t + \cdots + t^{d-1}} \right]$. Then,

$$C(R) = \lim_{t \to 1}(1 - t)^{n-1}P_R(t) = \frac{d}{d_1 \cdots d_n}.$$

Except for the fact that it is not always an integer, the Maiorana multiplicity behaves “like a multiplicity should”:

**Lemma 7.12.** Suppose that $0 \to N \to M \to P \to 0$ is an exact sequence in $C(R)$. Then,

- $D(M) = \max\{D(N), D(P)\}$.
- If $D(N) < D(M)$, then $C(M) = C(P)$.
- If $D(P) < D(M)$, then $C(M) = C(N)$.
- If $D(P) = D(N) = D(M)$, then $C(M) = C(N) + C(P)$.

**Proof.** Since vector space dimension is additive over short exact sequences, we have that $P_M(t) = P_N(t) + P_P(t)$. Now, $C(N)$ and $C(P)$ are each greater than or equal to 0, so the order of the pole at $t = 1$ of $P_M(t)$ is the max of the order of the poles for $P_N(t)$ and $P_P(t)$, and the first result follows.

Now, suppose that $D(N) < D(M)$. By the preceding argument, we must have that $D(P) = D(M)$. Now,

$$\lim_{t \to 1}(1 - t)^{D(M)}P_M(t) = \lim_{t \to 1}(1 - t)^{D(M)}P_N(t) + \lim_{t \to 1}(1 - t)^{D(M)}P_P(t) = \lim_{t \to 1}(1 - t)^{D(M)}P_P(t).$$

The last equals sign follows since the order of the pole at $t = 1$ for $P_N(t)$ is strictly less than $D(M)$, and the result follows from this computation. The last two bullet points may be proved in a similar fashion.

**Lemma 7.13.** If $M \in C(R)$, then $C(M(n)) = C(M)$, for any integer $n$. 

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Proof. We have seen that \( P_M(t) = t^n P_{M(n)}(t) \) and that \( D(M) = D(M(n)) \), thus

\[
C(M) = \lim_{t \to 1} (1 - t)^{D(M)} P_M(t) \\
= \lim_{t \to 1} (1 - t)^{D(M(n))} t^n P_{M(n)}(t) \\
= \lim_{t \to 1} (1 - t)^{D(M(n))} P_{M(n)}(t) \cdot \lim_{t \to 1} t^n \\
= \lim_{t \to 1} (1 - t)^{D(M(n))} P_{M(n)}(t) \\
= \lim_{t \to 1} \left( \prod_{i=1}^{D(M)} (1 - t^{d_i}) \right) P_{M(n)}(t) \\
= \prod_{i=1}^{D(M)} (1 - t^{d_i}) \cdot C(M(n)).
\]

We want to compare the Maiorana multiplicity \( C(M) \) to our previously studied multiplicities.

Letting \( \bar{x} \) be a *sop for \( M \in C(R) \), we’ve seen that \( k(\bar{x}) \) is a graded polynomial ring, and as calculated in the examples of Poincare series at the end of section 3.3, we must have

\[
P_{k(\bar{x})}(t) = \frac{1}{\prod_{i=1}^{D(M)} (1 - t^{d_i})},
\]

where \( d_i \) is the degree of the homogeneous element \( x_i \).

Now, corollary 6.28 applied to \( M \) and \( k \) as modules in \( C(k(\bar{x})) \) - recall that \( P_M(t) \) is the same in \( C(R) \) and \( C(k(\bar{x})) \) - gives that

\[
P_k(t) P_M(t) = P_{k(\bar{x})}(t) \chi_R(M, k)(t).
\]

Since \( P_k(t) = 1 \), we have

**Theorem 7.14.** If \( M \in C(R) \) and \( \bar{x} \) is a *sop for \( M \), then

\[
P_M(t) = \frac{\chi_{k(\bar{x})}(M, k)(t)}{\prod_{i=1}^{D(M)} (1 - t^{d_i})},
\]

with \( \chi_{k(\bar{x})}(M, k)(t) \in \mathbb{Z}[t, t^{-1}] \).

Using the result above,

\[
(1 - t)^{D(M)} P_M(t) = \frac{\chi_{k(\bar{x})}(M, k)(t)}{\prod_{i=1}^{D(M)} (1 + t + \ldots + t^{d_i-1})},
\]

and Theorem 7.9 gives that \( \chi_{k(\bar{x})}(M, k)(1) \) is on our list of the various equivalent definitions of multiplicity. Finally, just note that \( \prod_{i=1}^{D(M)} (1 + t + \ldots + t^{d_i-1}) \) evaluated at \( t = 1 \) gives \( \prod_{i=1}^{D(M)} d_i \). In conclusion, we have our main theorem

**Theorem 7.15. (Main Theorem)**

If \( M \) is in \( C(R) \), and \( x_1, \ldots, x_{D(M)} \) of degrees \( d_1, \ldots, d_{D(M)} \) form a *sop for \( M \), which generate the ideal \( \mathcal{I} \) of \( R \), then

\[
C(M) = \frac{e_R(M, \mathcal{I})}{d_1 \cdots d_{D(M)}} = \frac{\chi_R(\bar{x}, M)}{d_1 \cdots d_{D(M)}},
\]

and \( C(M) > 0 \), if \( M \neq 0 \). Thus, the ratio

\[
\frac{e_R(M, \mathcal{I})}{d_1 \cdots d_{D(M)}}
\]

is independent of the choice of system of parameters \( x_1, \ldots, x_{D(M)} \) for \( M \), while \( e_R(M, \mathcal{I}) \) depends only on the product of the degrees of the elements in the *sop.
Example 7.16. Let $R = k[x,y]/(y^2 - x^3)$ be a module over itself where $\deg(x) = 2$ and $\deg(y) = 3$. Then, $x \in R$ forms a $^*$ sop for $R$, since $R$ is finitely generated as a $k(x)$-module by 1, $y$. Similarly, $y$ is also a $^*$ sop for $R$, generated by 1, $x, x^2$ as a $k(y)$-module. Certainly a length 1 $^*$ sop is the least possible length, so by our main dimension theorem, $D(R) = 1$.

By example 7.11, we have that $C(R) = \frac{6}{2\cdot3} = 1$.

Now we apply the main theorem,

$$e(R, x) = \frac{e(R, y)}{2} = \frac{C(R)}{3} = 1,$$

so $e(R, x) = 2$ and $e(R, y) = 3$.

Example 7.17. Let $S = k[x_0, \ldots, x_n]$ be the standard graded polynomial ring with $f \in S_d$ and $R = S/(f)$. We saw in example 5.8 that a $^*$ sop composed of degree 1 elements exists, and by example 7.11 $C(R) = d$. So, if we pick a $^*$ sop composed of degree 1 elements and take $\mathcal{I}$ to be the ideal generated by these elements, we have by the main theorem,

$$C(R) = \frac{e(R, \mathcal{I})}{1\cdot1} = e(R, \mathcal{I}) = d.$$

8 Future Research

Many examples demonstrating the theory of this paper are furnished by the area of mathematics known as group cohomology. For example, if $p$ is a prime number, and $k = \mathbb{Z}/p$ is a field of characteristic $p$, let’s consider a finite $p$-group $G$. The cohomology ring of $G$ with coefficients in $k$, $H^*(G,k)$, is such that the even degree submodules of $H^*(G,k)$ form a finitely generated $k$-algebra, called $H^{ev}(G,k)$, with $H^*(G,k)$ a finitely generated graded module over $H^{ev}$. Thus, all of the theory developed in this paper applies to this particular case.

In the 2014 paper A Degree Formula for Equivariant Cohomology [6], R. Lynn outlines this theory and proves a result about the Maiorana C-multiplicity of the graded module $H^*(G,k)$. She proves that $C(H^*(G,k))$ can be computed as a sum over all classes $A$ of maximal rank elementary abelian $p$-subgroups of $G$, i.e. $C(H^*(G,k)) = \Sigma_A \frac{|H^*(C_G(A),k)|}{|W_G(A)|}$, where $C_G(A)$ is the centralizer and $W_G(A) \cong N_G(A)/C_G(A)$ (the Weyl group of $A$).

I plan to make this paper more robust by adding examples from the cohomology of groups, including verification of R.Lynn’s result. These examples will be computed using already existent computations of group cohomology rings, made public by D. Green[2000] [4] in the book Grobner Bases and the Computation of Group Cohomology. For future research I plan to use his computations and the already developed theory to develop and test my own conjectures.
References


