Note on the Proper Linear Spaces on 18 Points

A. Betten[1] and D. Betten[2]

[1] Fakultät für Mathematik und Physik
Universität Bayreuth
95440 Bayreuth
Anton.Betten@uni-bayreuth.de
[2] Mathematisches Seminar der Universität Kiel
Ludewig-Meyn-Str. 4
24098 Kiel
betten@math.uni-kiel.de

Abstract In [4] we constructed and enumerated all proper linear spaces on 17 points using the so-called TDO-method. This method is also strong enough for the construction and enumeration of all proper linear spaces on 18 points. In the present note we list the results. We get 2 412 890 proper linear spaces on 18 points.

AMS subject classification: 05B25, 05B30, 51E99

1 Introduction

A linear space on a set of v points is a set of subsets called lines such that each line contains at least two points and any two points are contained in exactly one line (cf. [1]). A linear space is called proper if all lines have at least three and at most $v-1$ points. All proper linear spaces on at most 17 points have been classified, [9], [12], [4], see also [11]. In the present note, we extend this classification to the collection of proper linear spaces on 18 points. The number of linear spaces on v points, subsequently denoted by PLIN(v) grows very fast with v. The numbers PLIN(v) for $2 \le v \le 18$ are

v	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
PLIN(v)	0	0	0	0	0	1	0	1	1	1	3	7	1	119	398	161 925	2 412 890

In our computer programs for the construction and analysis of finite geometries we make extensive use of the well known notion of a *tactical decomposition* (TD) of an incidence matrix, see [10]. An important tactical decomposition, which we call TDA, is induced by the orbits of the automorphism group, [5]. Another tactical decomposition is defined by a successive ordering process which we call TDO. It is canonical and may be calculated very quickly, [8], [2], [3]. Each tactical decomposition may be described by its *TD-scheme*, and these schemes are good invariants for the structure of geometries. In order to construct geometries we go in the opposite direction: we start with a given initial parameter set and refine these parameters step by step. See for instance [3], where we proceeded up to parameter depth 2.

Continuing this refining process we will eventually reach a TDO-scheme (or a set of TDO-schemes). ¿From these schemes we then generate the related geometries. Here, a TDO-scheme may or may not be realizable, and if it is, it may produce several geometries.

Using this method, we constructed in [4] all proper linear spaces on 17 points. By the same method it is even possible to construct also the proper linear spaces on 18 points. In the present note we list the results in this case and point out some special spaces and situations.

2 Table of Results

In this section, we present the classification of proper linear spaces on 18 points. We refer to [4] for all definitions and notions used in the following.

Table 1 displays the numbers of proper linear spaces for the various line cases, i. e. distributions of block lengths. For instance $(4^{18}\,3^{15})$ means that we look for a linear space which has 18 blocks of length 4 and 15 blocks of length 3. Similar as in [4] we compute for each line case the set of possible tactical decompositions which may arise as TDO of linear spaces of that type. We took only those line cases into the table where the parameter calculation produced at least one TDO-scheme. In the first three columns we give the number of the case, the line case and the number of TDO-schemes one gets in this case. Since many line cases do not lead to TDO-schemes we get gaps in the (original) numbering.

We call a TDO-scheme *discrete* if all classes of points and all classes of blocks have only one element. In this case the TDO-scheme coincides with the incidence matrix of the space and nothing more is to be done. Only those TDO-schemes which are non-discrete have to be handed to the generator program. The number of non-discrete TDO-schemes for each line case is in column 4. For instance the 740 TDO-schemes in line case no. 15 split into 60 non-discrete TDOs and $740 - 60 = 680$ discrete TDOs.

In column 5 of the table we list the number of linear spaces which are constructed from all non-discrete TDO-schemes in the corresponding line case. In column 6 we give the total number (up to isomorphism) of linear spaces for the respective line case, i. e. the sum of the number of discrete TDO-schemes and the number of geometries constructed from all non-discrete TDO-schemes:

$$\text{number of geometries} = \text{column } 6 = \text{column } 3 - \text{column } 4 + \text{column } 5\ .$$

Table1. Proper linear spaces on 18 points by their line type

no.	line case	# TDO	# TDO .n.d.	# GEO .n.d.	# GEO total
6	$(4^5\,3^{41})$	158 419	87	2 488	160 820
7	$(4^6\,3^{39})$	1 139 617	198	2 657	1 142 076
8	$(4^7\,3^{37})$	407 499	249	115 796	523 046
9	$(4^8\,3^{35})$	166 614	73	379	166 920
10	$(4^9\,3^{33})$	170 060	293	2 171	171 938
11	$(4^{10}\,3^{31})$	124 848	75	288	125 061
12	$(4^{11}\,3^{29})$	48 164	236	428	48 356
13	$(4^{12}\,3^{27})$	14 336	148	170	14 358
14	$(4^{13}\,3^{25})$	3 623	165	207	3 665
15	$(4^{14}\,3^{23})$	740	60	66	746
16	$(4^{15}\,3^{21})$	273	97	164	340
17	$(4^{16}\,3^{19})$	18	8	6	16
18	$(4^{17}\,3^{17})$	5	5	2	2
19	$(4^{18}\,3^{15})$	3	3	2	2
21	$(4^{20}\,3^{11})$	2	2	0	0
22	$(4^{21}\,3^{9})$	1	1	0	0
32	$(5^3\,4^5\,3^{31})$	813	27	84	870
33	$(5^3\,4^6\,3^{29})$	1 413	57	52	1 408
34	$(5^3\,4^7\,3^{27})$	1 713	132	194	1 775
35	$(5^3\,4^8\,3^{25})$	118	4	3	117
36	$(5^3\,4^9\,3^{23})$	160	74	70	156
37	$(5^3\,4^{10}\,3^{21})$	35	6	1	30
38	$(5^3\,4^{11}\,3^{19})$	14	10	5	9
39	$(5^3\,4^{12}\,3^{17})$	2	1	0	1
40	$(5^3\,4^{13}\,3^{15})$	6	6	2	2
41	$(5^3\,4^{14}\,3^{13})$	1	1	1	1
42	$(5^3\,4^{15}\,3^{11})$	4	4	1	1
43	$(5^3\,4^{16}\,3^{9})$	1	1	1	1
54	$(5^6\,4^6\,3^{19})$	1	1	0	0
55	$(5^6\,4^7\,3^{17})$	6	6	1	1
57	$(5^6\,4^9\,3^{13})$	6	6	4	4
63	$(5^6\,4^{15}\,3^{1})$	1	1	0	0
73	$(5^9\,4^9\,3^{3})$	1	1	1	1
85	$(6^1\,4^3\,3^{40})$	1	1	469	469
86	$(6^1\,4^4\,3^{38})$	2	2	29 622	29 622
87	$(6^1\,4^5\,3^{36})$	7 506	38	264	7 732
88	$(6^1\,4^6\,3^{34})$	10 112	78	394	10 428

continued on next page

continued from previous page

no.	line case	# TDO	# TDO .n.d.	# GEO .n.d.	# GEO total
89	$(6^1\,4^7\,3^{32})$	351	2	16	365
90	$(6^1\,4^8\,3^{30})$	907	30	184	1 061
91	$(6^1\,4^9\,3^{28})$	330	9	40	361
92	$(6^1\,4^{10}\,3^{26})$	557	39	67	585
93	$(6^1\,4^{11}\,3^{24})$	32	3	33	62
94	$(6^1\,4^{12}\,3^{22})$	57	24	94	127
95	$(6^1\,4^{13}\,3^{20})$	1	1	1	1
96	$(6^1\,4^{14}\,3^{18})$	2	1	1	2
100	$(6^1\,4^{18}\,3^{10})$	1	1	0	0
102	$(6^1\,4^{20}\,3^{6})$	2	2	0	0
110	$(6^1\,5^3\,4^4\,3^{28})$	146	44	115	217
111	$(6^1\,5^3\,4^5\,3^{26})$	8	7	10	11
112	$(6^1\,5^3\,4^6\,3^{24})$	25	13	12	24
114	$(6^1\,5^3\,4^8\,3^{20})$	1	1	0	0
116	$(6^1\,5^3\,4^{10}\,3^{16})$	1	1	0	0
118	$(6^1\,5^3\,4^{12}\,3^{12})$	7	7	6	6
129	$(6^1\,5^6\,4^4\,3^{18})$	5	5	2	2
157	$(6^2\,4^5\,3^{31})$	30	1	0	29
159	$(6^2\,4^7\,3^{27})$	4	3	3	4
163	$(6^2\,4^{11}\,3^{19})$	5	5	0	0
207	$(6^3\,3^{36})$	1	1	12	12
217	$(6^3\,4^{10}\,3^{16})$	1	1	1	1
276	$(7^1\,4^7\,3^{30})$	10	4	17	23
277	$(7^1\,4^8\,3^{28})$	1	1	6	6
278	$(7^1\,4^9\,3^{26})$	13	1	4	16
280	$(7^1\,4^{11}\,3^{22})$	3	3	1	1
total:		2 258 639	2 367	156 618	2 412 890

In Table 2, we list for each line case the distribution of automorphism group orders for the geometries which come from the non-discrete TDO-schemes. Of course the linear spaces which correspond to the discrete TDO-schemes all have a trivial automorphism group.

Table2. The distribution of automorphism group orders

no.	TDO.n.d.:	distr. of aut. group orders
6	2488:	$1^{1778}, 2^{630}, 3^{74}, 6^{6}$
7	2657:	$1^{1479}, 2^{936}, 3^{234}, 6^{8}$
8	115796:	$1^{113673}, 2^{2012}, 3^{75}, 4^{24}, 6^{12}$
9	379:	$1^{167}, 2^{122}, 3^{86}, 6^{2}, 21^{2}$
10	2171:	$1^{904}, 2^{1056}, 3^{113}, 4^{57}, 6^{29}, 12^{6}, 18^{4}, 36^{1}, 54^{1}$
11	288:	$1^{24}, 2^{194}, 3^{66}, 6^{4}$
12	428:	$1^{35}, 2^{348}, 3^{33}, 4^{6}, 6^{4}, 12^{2}$
13	170:	$1^{9}, 2^{151}, 3^{7}, 6^{2}, 18^{1}$
14	207:	$1^{30}, 2^{173}, 3^{3}, 6^{1}$
15	66:	$1^{2}, 2^{46}, 3^{5}, 4^{12}, 6^{1}$
16	164:	$1^{21}, 2^{99}, 3^{21}, 4^{4}, 6^{13}, 8^{2}, 12^{2}, 24^{2}$
17	6:	2^{6}
18	2:	2^{2}
19	2:	$6^{1}, 18^{1}$
32	84:	$1^{6}, 2^{66}, 3^{8}, 6^{4}$
33	52:	$2^{22}, 3^{28}, 6^{2}$
34	194:	$1^{3}, 2^{173}, 3^{6}, 4^{8}, 6^{4}$
35	3:	$2^{1}, 3^{2}$
36	70:	$2^{39}, 3^{6}, 4^{11}, 6^{10}, 12^{2}, 18^{1}, 36^{1}$
37	1:	2^{1}
38	5:	$2^{4}, 6^{1}$
40	2:	$6^{1}, 12^{1}$
41	1:	6^{1}
42	1:	20^{1}
43	1:	2^{1}
55	1:	4^{1}
57	4:	$3^{1}, 6^{1}, 12^{1}, 36^{1}$
73	1:	108^{1}
85	469:	$1^{264}, 2^{122}, 3^{7}, 4^{57}, 6^{4}, 8^{6}, 9^{1}, 12^{3}, 18^{1}, 24^{3}, 72^{1}$
86	29622:	$1^{27971}, 2^{1386}, 3^{37}, 4^{194}, 6^{9}, 8^{23}, 12^{1}, 24^{1}$
87	264:	$1^{48}, 2^{216}$
88	394:	$1^{78}, 2^{266}, 3^{28}, 4^{18}, 8^{4}$
89	16:	$1^{8}, 3^{8}$
90	184:	$1^{58}, 2^{113}, 3^{3}, 4^{3}, 6^{6}, 12^{1}$
91	40:	$1^{4}, 2^{22}, 4^{14}$
92	67:	$1^{3}, 2^{60}, 4^{4}$
93	33:	$2^{6}, 4^{13}, 6^{3}, 8^{9}, 24^{2}$

continued on next page

continued from previous page

no.	TDO.n.d.:	distr. of aut. group orders
94	94:	$1^6, 2^{44}, 3^{13}, 4^6, 6^9, 8^7, 12^2, 18^2, 24^2, 72^3$
95	1:	3^1
96	1:	3^1
110	115:	$1^3, 2^{74}, 3^4, 4^{22}, 6^1, 8^{10}, 12^1$
111	10:	2^{10}
112	12:	$1^1, 2^6, 6^5$
118	6:	$2^1, 6^1, 8^1, 12^1, 24^1, 72^1$
129	2:	8^2
159	3:	2^3
207	12:	$8^1, 16^1, 24^3, 48^1, 72^1, 144^1, 240^1, 432^1, 648^1, 1296^1$
217	1:	20^1
276	17:	$1^6, 2^{10}, 6^1$
277	6:	$1^2, 3^3, 21^1$
278	4:	$1^2, 2^1, 6^1$
280	1:	2^1

3 Some Special Situations and Geometries

3.1 The Largest Case

Case no. 7 yields the largest number of geometries. Here one can see the advantage of the TDO-method. Among the 1 139 617 TDO-schemes only 198 are non-discrete and need further attention. They generate only 2 657 linear spaces. So, the overwhelming part of this case consists of discrete TDO-schemes, and after having constructed the TDO-schemes, nearly all the work is done.

Because of the high number we did not save all geometries. We put into a file those geometries which came from a non-discrete TDO-scheme, the other geometries which correspond to discrete TDO-schemes were only counted. We use the following notation for the geometries saved in [6]: The first number is the number of the line case, the second number denotes the TDO-scheme and the third number is the number of the geometry generated by this TDO-scheme. For instance 10-151-288 is the linear space no. 288 which is generated starting from TDO-scheme no. 151 belonging to line case no. 10. Of course this numbering is not canonical but depends on the algorithm which has been used.

3.2 A Highly Productive TDO-Scheme

Very remarkable is case no. 8, especially the TDO-scheme no. 249 (the last one). This special TDO-scheme generates 114 672 proper linear spaces, i. e. nearly as much as all 249 non-discrete TDO-schemes together. We present this TDO-scheme here. First in the form that it came out from the program calculating the TDO-schemes, and then in the permutated form which we took for the generating process.

Finally we display one of the 114672 geometries which were generated from this specific TDO-scheme (cf. Fig. 1).

TDO-scheme:

	1	6	1	12	6	6	12
3	1	2	0	4	0	0	0
2	0	3	1	0	3	0	0
1	1	0	1	0	0	6	0
12	0	1	0	2	1	1	3

Transposed and permuted TDO-scheme:

	1	12	2	3
6	1	2	0	0
12	0	3	0	0
6	0	2	1	1
6	0	2	1	0
12	0	2	0	1
1	1	0	0	3
1	1	0	2	0

This TDO-scheme is remarkable in another respect: It was the only one where the generation of the geometries proved to be really hard. We had to find a good permutation for the scheme, had to transpose it and we took another generation program (generating by respecting some canonical ordering). In addition, the tests we took in between had to be well chosen.

Remark: The choice of the permutation of the TDO-scheme and the choice of the intermediate tests may affect the computing time enormously. It seems a crucial problem to find good criteria for these choices. Otherwise one has to carry out long experiments to get a suitable conditioning for the computer run.

Special attention was also needed for the following TDO-schemes: 9-1, 10-151, 10-293 and 16-97. Line cases 85 and 86 were done by transposing and using the order preserving program.

There is a second TDO-scheme which produced many linear spaces, namely TDO-scheme 86-1. We display this scheme together with one of its geometries in Fig. 2. We choose the geometry 86-1-24786, which has the largest automorphism group (order 24).

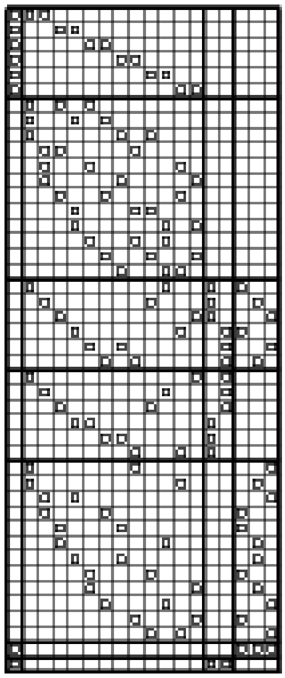


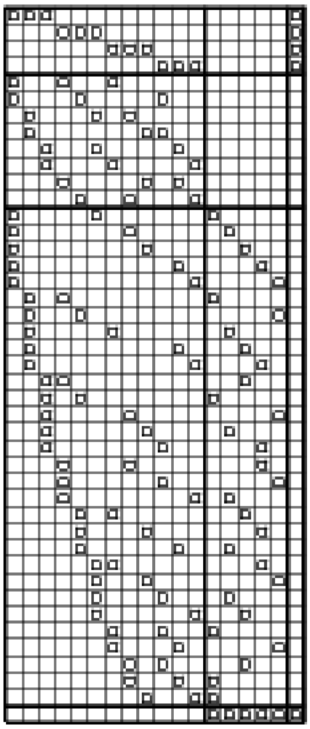
Figure1. One of the 114 672 geometries of TDO-scheme 8-249

3.3 The 12 Latin Squares of Order 6

Line case 207 leads to the well known 12 Latin squares of order 6, see for instance [7] and [11, p. 99]. Compare also [2] where we have explained how a Latin square may be viewed as a linear space: We label the rows and columns and digits by $r_1, \dots, r_6$ and $c_1, \dots c_6$ and $d_1, \dots, d_6$. The Latin Squares of order 6 are the linear spaces of line type $(6^3, 3^{36})$ where the three 6-lines are special: They are $\{r_1, \dots, r_6\}$, $\{c_1, \dots, c_6\}$ and $\{d_1, \dots, d_6\}$. The 36 3-lines correspond to the entries of the Latin Square. Any automorphism either permutes the three 6-lines or not. If it fixes all three, we call the automorphism inner.

These 12 squares have rather large automorphism groups. Using the TDA-schemes of the corresponding linear spaces, we get the orbits on the entries of the Latin Square from the orbits on the 3-blocks in the space.

One Latin Square (cf. Fig. 3, using $d_i = i$ for $i = 1, \dots, 6$) has special properties. The automorphism group has exactly two orbits on the entries. The diagonal elements of the square form one orbit (indicated by small circles), and the other orbit is formed by all off-diagonal elements. The automorphim group is of order 240, generated by the three permutations α, β



TDO-scheme:

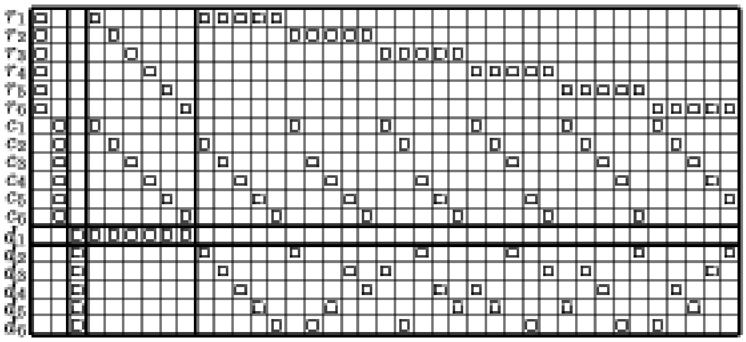
	12	5	1
4	3	0	1
8	3	0	0
30	2	1	0
1	0	5	1

Figure2. A geometry for TDO-scheme 86-1 (with $|Aut| = 24$)

and τ. Here, α and β generate the subgroup of inner automorphisms isomorphic to Sym_5. This group is extended by τ, the mapping which interchanges row i with column i ($i = 1, \ldots, 6$). Recall that there are two non-equivalent subgroups $\mathrm{Sym}_5 \leq \mathrm{Sym}_6$. One of them acts transitively on the six elements and the other one is the stabilizer of Sym_6 on one element. It is remarkable that this special Latin square displays both actions. The group Sym_5 of inner automorphisms acts transitively on the six rows $r_1, r_2, \ldots, r_6$ (and also on the six columns). The other action with a fixed point happens on the six digits: the digit $1 = d_1$ is fixed and Sym_5 acts on the five other digits $d_2, d_3, \ldots, d_6$ in the natural way. Formally, we have the following isomorphism

$$\begin{aligned}\mathrm{Sym}_{\{r_1,\ldots,r_6\}} \simeq \mathrm{Sym}_6 \geq \mathrm{Sym}_5 &\to \mathrm{Sym}_{\{d_2,\ldots,d_6\}},\\ (r_1\, r_4\, r_6\, r_3\, r_2) = \alpha_1 &\mapsto \alpha_3 = (d_2\, d_4\, d_3\, d_5\, d_6),\\ (r_1\, r_5\, r_3\, r_2) = \beta_1 &\mapsto \beta_3 = (d_2\, d_5\, d_4\, d_6)\end{aligned}$$

between these two types of groups Sym_5.



①	2	3	4	5	6
2	①	6	5	3	4
3	6	①	2	4	5
4	5	2	①	6	3
5	3	4	6	①	2
6	4	5	3	2	①

$$\mathrm{Aut} = \langle \alpha, \beta, \tau \rangle,$$
$$\alpha = \alpha_1 \alpha_2 \alpha_3,$$
$$\beta = \beta_1 \beta_2 \beta_3,$$
$$\alpha_1 = (r_1\, r_4\, r_6\, r_3\, r_2) \quad \alpha_2 = (c_1\, c_4\, c_6\, c_3\, c_2) \quad \alpha_3 = (d_2\, d_4\, d_3\, d_5\, d_6),$$
$$\beta_1 = (r_1\, r_5\, r_3\, r_2) \qquad \beta_2 = (c_1\, c_5\, c_3\, c_2) \qquad \beta_3 = (d_2\, d_5\, d_4\, d_6),$$
$$\tau = (r_1\, c_1)(r_2\, c_2)(r_3\, c_3)(r_4\, c_4)(r_5\, c_5)(r_6\, c_6)$$

Figure3. The TDA-decomposition of the special Latin Square

3.4 Some Geometries with a Rather Large Group

Besides the Latin squares there are 26 linear spaces in the list which have automorphism group order ≥ 20. Here are the numbers of these spaces together with the group order (in brackets):

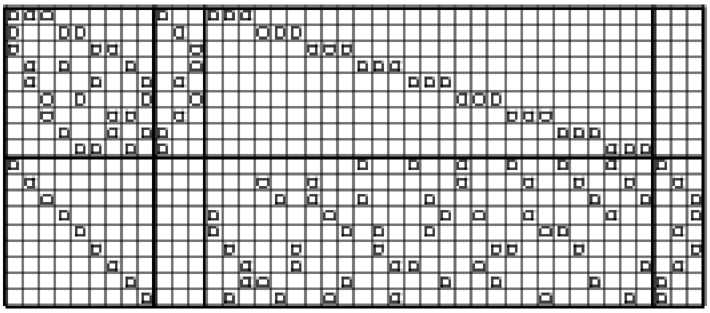
9-1-108(21),	42-4-1(20),	86-1-24786(24),	94-24-14(24),
9-1-5(21),	57-5-1(36),	93-1-14(24),	118-7-1(24),
10-151-263(36),	73-1-1(108),	93-3-12(24),	118-7-2(72),
10-151-288(54),	85-1-372(24),	94-1-8(72),	217-1-1(20),
16-97-7(24),	85-1-464(24),	94-1-13(72),	277-1-6(21).
16-97-8(24),	85-1-468(72),	94-1-14(72),	
36-32-13(36),	85-1-469(24),	94-24-2(24),	

Among these spaces those species might be of special interest, where the automorphism group has only few orbits on points (or on blocks or on flags). There are 4 spaces with exactly two point orbits. Let us illustrate these 4 spaces. We display in each case the TDO-scheme, the TDA-scheme and the incidence matrix with its TDA. In addition, generators for the automorphism group acting on the points $1, \ldots, 18$ are shown (labelled by capital letters $A, B, C, \ldots$). All groups are soluble and a presentation for the group is given. This presentation is adapted to a composition series, i. e. we have

$$1 \trianglelefteq \langle A \rangle \trianglelefteq \langle A, B \rangle \trianglelefteq \langle A, B, C \rangle \trianglelefteq \ldots \trianglelefteq \mathrm{Aut}\mathcal{S}$$

where $\mathcal{S}$ is the linear space in question and successive factors of this series have prime order. The order of the group is thus the product of the indices, i. e. the product of the exponents in the first column of the presentation. We identified the groups in the table of small groups contained in the algebra software package GAP [13] (Version 4). This catalogue is based on the work of H. U. Besche, B. Eick and E. O'Brien. We denote the n-th group of order m in this catalogue by $m\#_{\mathrm{GAP}}n$.

The Linear Space 010-151-288 The automorphism group has order 54 and is isomorphic to $54\#_{\mathrm{GAP}}5$. The TDA-scheme is isomorphic to the TDO-scheme, i. e. the automorphism group induces no refinement.



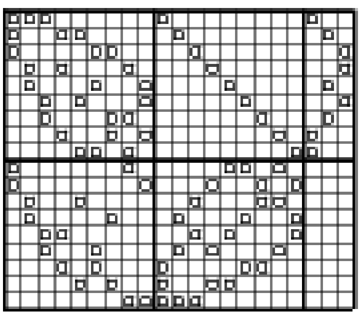
TDA-scheme and the TDO-scheme:

	9	3	27	3
9	3	1	3	0
9	1	0	6	1

$A^3 = id,$
$B^3 = id, A^B = A,$
$C^2 = id, A^C = A^2, B^C = B^2,$
$D^3 = id, A^D = A,\ B^D = AB, C^D = C$

$A = (1\,8\,9)(2\,5\,7)(3\,6\,4)(10\,18\,17)(11\,16\,14)(12\,13\,15)$
$B = (2\,7\,5)(3\,6\,4)(10\,12\,11)(13\,16\,18)(14\,17\,15)$
$C = (4\,6)(5\,7)(8\,9)(11\,12)(13\,14)(15\,16)(17\,18)$
$D = (1\,2\,3)(4\,9\,7)(5\,6\,8)(11\,14\,16)(12\,13\,15)$

The Linear Space 073-1-1 The automorphism group has order 108 (isomorphic to $108\#_{\text{GAP}}17$) and TDO-scheme and TDA-scheme are isomorphic.



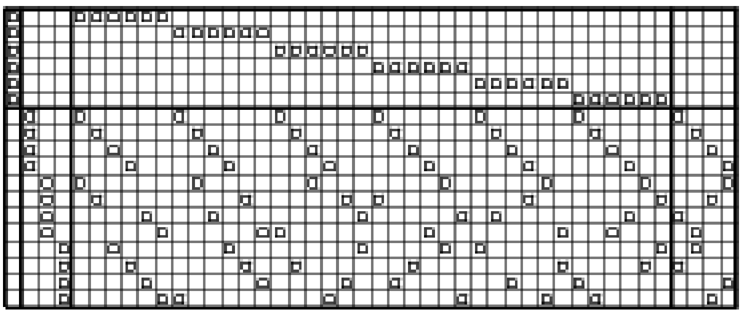
TDA-scheme and the TDO-scheme:

	9	9	3
9	3	1	1
9	2	3	0

$A^3 = id,$
$B^3 = id, A^B = A,$
$C^2 = id, A^C = A^2, B^C = B^2,$
$D^3 = id, A^D = A,\ B^D = AB, C^D = C,$
$E^2 = id, A^E = A,\ B^E = B^2,\ C^E = C, D^E = D^2$

$$A = (1\,8\,9)(2\,5\,7)(3\,6\,4)(10\,11\,18)(12\,13\,17)(14\,16\,15)$$
$$B = (2\,7\,5)(3\,6\,4)(10\,15\,12)(11\,14\,13)(16\,17\,18)$$
$$C = (4\,6)(5\,7)(8\,9)(10\,11)(12\,14)(13\,15)(16\,17)$$
$$D = (1\,2\,3)(4\,9\,7)(5\,6\,8)(12\,17\,13)(14\,16\,15)$$
$$E = (2\,3)(4\,7)(5\,6)(12\,15)(13\,14)(16\,17)$$

The Linear Space 085-1-468 The automorphism group has order 72 (isomorphic to $72\#_{\mathrm{GAP}}44$). The TDO-scheme and the TDA-scheme are isomorphic.



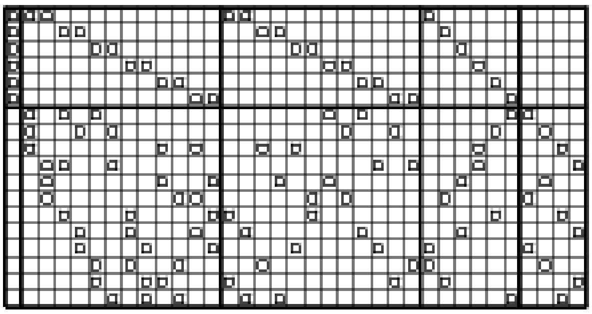
TDA-scheme and the TDO-scheme:

	1	3	36	4
6	1	0	6	0
12	0	1	6	1

$A^2 = id,$
$B^2 = id, A^B = A,$
$C^3 = id, A^C = A, B^C = B,$
$D^2 = id, A^D = A, B^D = B, \ C^D = C^2,$
$E^3 = id, A^E = B, B^E = AB, C^E = C, \ D^D = D$

$$A = (2\,4)(3\,5)(7\,8)(9\,10)(11\,12)(13\,14)(15\,16)(17\,18)$$
$$B = (1\,6)(3\,5)(7\,10)(8\,9)(11\,13)(12\,14)(15\,18)(16\,17)$$
$$C = (7\,13\,16)(8\,14\,15)(9\,12\,18)(10\,11\,17)$$
$$D = (1\,6)(2\,4)(3\,5)(11\,17)(12\,18)(13\,16)(14\,15)$$
$$E = (1\,2\,3)(4\,5\,6)(7\,8\,9)(12\,13\,14)(15\,18\,16)$$

The Linear Space 094-24-2 The automorphism group is isomorphic to Sym_4 of order 24 ($\simeq 24\#_{\mathrm{GAP}}12$). The TDA-scheme is a bit finer than the TDO-scheme. In the natural action on four points, the generators can be identified as $A = (1\,2)(3\,4)$, $B = (1\,3)(2\,4)$, $C = (1\,2\,4)$ and $D = (1\,2)$, for example.



TDO-scheme:

	1	12	18	4
6	1	2	3	0
12	0	3	3	1

TDA-scheme:

	1	12	12	6	4
6	1	2	2	1	0
12	0	3	2	1	1

$A^2 = id,$
$B^2 = id, A^B = A,$
$C^3 = id, A^C = B, B^C = AB,$
$D^2 = id, A^D = A, B^D = AB, C^D = C^2$

$$A = (2\,5)(3\,6)(7\,11)(8\,12)(9\,10)(13\,17)(14\,18)(15\,16)$$
$$B = (1\,4)(3\,6)(7\,14)(8\,13)(9\,16)(10\,15)(11\,18)(12\,17)$$
$$C = (1\,2\,6)(3\,4\,5)(7\,15\,12)(8\,14\,9)(10\,13\,11)(16\,17\,18)$$
$$D = (2\,6)(3\,5)(7\,12)(8\,11)(9\,10)(13\,14)(17\,18)$$

References

1. L. M. Batten and A. Beutelspacher: The theory of finite linear spaces. Cambridge University Press, Cambridge 1993.
2. A. Betten and D. Betten: Regular linear spaces. *Beiträge Algebra Geom.* **38** (1997), 111–124.
3. A. Betten and D. Betten: Linear spaces with at most 12 points. *J. of Combinatorial Designs* **7** (1999), 119–145.
4. A. Betten and D. Betten: The Proper Linear Spaces on 17 Points. *Discrete Applied Mathematics* **95** (1999), 83–108.
5. A. Betten and D. Betten: Tactical decompositions and some configurations v_4. *J. of Geom.* **66** (1999), 27–41.
6. A. Betten and D. Betten: Addendum to *Note on the Proper Linear Spaces on 18 Points* `http://www.mathe2.uni-bayreuth.de/betten/PUB/pub_proper18.html`
7. D. Betten: Die 12 lateinischen Quadrate der Ordnung 12. *Mitt. Math. Sem. Giessen* **163** (1984), 181–188.
8. D. Betten and M. Braun: A tactical decomposition for incidence structures. *Ann. Disc. Math.* **52** (1992), 37–43.
9. A. E. Brouwer: The linear spaces on 15 points. *Ars Combin.* **12** (1981), 3–35.
10. P. Dembowski: *Finite geometries.* Classics in Mathematics. Springer-Verlag, Berlin, 1997. Reprint of the 1968 original.
11. C. Colbourn, J. Dinitz: CRC Handbook of Combinatorial Designs, CRC press, Boca Raton, New York, London, Tokyo, 1996.
12. G. Heathcote: Linear spaces on 16 points. *J. Combin. Des.* **1** (1993), 359–378.
13. The GAP Group: *GAP — Groups, Algorithms, and Programming*, Version 4.1; Aachen, St. Andrews, 1999. (http://www-gap.dcs.st-and.ac.uk/~gap)