Dr. A. Betten Spring 2009

MATH 369 Linear Algebra

Assignment # 1

Problem # 1

A father and his two sons are together 100 years old. The father is twice as old as his older son and 30 years older than his younger son. How old is each person?

Problem # 2

Determine a, b and c such that the parabola $y = ax^2 + bx + c$ passes through the points A, B and C.
a) $A(2, 1)$, $B(-3, 1)$, $C(1, 0)$.
b) $A(2, -5)$, $B(3, -10)$, $C(4, -19)$.

Problem # 3

Find the integer coefficients a, b, c, d so that the following Chemical reaction formula makes sense:

$$a\mathrm{C_2H_6} + b\mathrm{O_2} \quad \rightarrow \quad c\mathrm{CO_2} + d\mathrm{H_2O}.$$

Problem # 4

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$. Find A^{-1}.

Problem # 5

Solve each of the following systems:

$$a)\quad \begin{aligned} x_1 + x_2 - 2x_3 + 4x_4 &= 5 \\ 2x_1 + 2x_2 - 3x_3 + x_4 &= 3 \\ 3x_1 + 3x_2 - 4x_3 - 2x_4 &= 1 \end{aligned} \qquad\qquad b)\quad \begin{aligned} x_1 + x_2 - 2x_3 + 3x_4 &= 4 \\ 2x_1 + 3x_2 + 3x_3 - x_4 &= 3 \\ 5x_1 + 7x_2 + 4x_3 + x_4 &= 5 \end{aligned}$$

Problem # 6

Write each of the following matrices as a product of elementary matrices:

$$\text{a)} \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} \quad \text{b)} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \text{c)} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Problem # 7

Page 29: 20c). Show your work for full credit.

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Assignment # 2

Problem # 8

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$. Find A^{-1}.

Problem # 9

Write each of the following matrices as a product of elementary matrices:

$$\text{a)} \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} \quad \text{b)} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \text{c)} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Problem # 10

Page 27: 9). Show your work for full credit.

Problem # 11

Solve each of the following systems. Write down the solution set in the form (special solution plus basic solutions).

a)

$$\begin{aligned} 2x_1-3x_2-6x_3-5x_4+2x_5&=7 \\ x_3+3x_4-7x_5&=6 \\ x_4-2x_5&=1 \end{aligned}$$

b)

$$\begin{aligned} 2x-6y+7z&=1 \\ 4y+3z&=8 \\ +2z&=4 \end{aligned}$$

c)

$$\begin{aligned} x+2y-3z&=2 \\ 2x+3y+\ z&=4 \\ 3x+4y+5z&=8 \end{aligned}$$

Problem # 12

Compute the partial fraction decomposition of the function

$$f = \frac{3x^2 + 7x - 2}{(x+1)(x^2-4)}.$$

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Assignment # 3

Problem # 13
Solve the following system. Write down the solution set in the form (distinguished solution plus basic solution):

$$\begin{aligned} x_1+3x_2-\ \,2x_3+\ \,5x_4&=4\\ 2x_1+8x_2-\quad x_3+\ \,9x_4&=9\\ 3x_1+5x_2-12x_3+17x_4&=7 \end{aligned}$$

Problem # 14
Let $A = \begin{bmatrix} 1 & 2 & -4 \\ -1 & -1 & 5 \\ 2 & 7 & -3 \end{bmatrix}$. Find A^{-1}.

Problem # 15
Let $A = \begin{bmatrix} 1 & -3 & 5 \\ 2 & -4 & 7 \\ -1 & -2 & 1 \end{bmatrix}$. Find the LU decomposition of A.

Problem # 16
The determinant of a 3×3 matrix

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}$$

is

$$a_{1,1}a_{2,2}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2} - a_{1,1}a_{3,2}a_{2,3} - a_{2,1}a_{1,2}a_{3,3} - a_{3,1}a_{2,2}a_{1,3}.$$

compute the determinant of the matrix in Problem 15. Then compute the product of the diagonal elements in the LU-decomposition (i.e., both L and U). Do you notice anything special?

Problem # 17
The vector is in the set. What value of the parameters produces that vector?

a) $\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$, $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} i + \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} j \mid i, j \in \mathbb{R} \right\}$

b) $\begin{pmatrix} 0 \\ -4 \\ 2 \end{pmatrix}$, $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} m + \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} n \mid m, n \in \mathbb{R} \right\}$

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Assignment # 4

Problem # 18

Is W a subspace of V or not?

V	W	Yes	No
$\mathbb{R}^3$	all $(a,b,c),\ a \geq 0$		
$\mathbb{R}^3$	all $(a,b,c),\ a = -c+b$		
$\mathbb{R}^3$	all $(a,b,c),\ a^2+b^2+c^2 \leq 1$		
real polynomials	polynomials with integer coefficients		
real polynomials	real polynomials, coeff. of x^2 is 1		
real polynomials	odd polynomials		
sequences in $\mathbb{R}$	polynomials with coefficients in $\mathbb{R}$		
2×2 matrices	2×2 diagonal matrices		
3×3 matrices	3×3 upper triangular matrices		
2×2 matrices	2×2 symmetric matrices		
2×2 matrices	singular 2×2 matrices		
2×2 matrices	nonsingular 2×2 matrices		
all $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}, a,b \in \mathbb{R}$	all $\begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}, b \in \mathbb{R}$		
all $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, a,b,c,d \in \mathbb{R}$	$ad-bc=1$		
all $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, a,b,c,d \in \mathbb{R}$	$a=d$ and $b=-c$		
$\mathbb{R}$, scalars $= \mathbb{Q}$	all $a+b\sqrt{2}, a,b \in \mathbb{Q}$		
$\mathbb{R}$, scalars $= \mathbb{Q}$	all $a-\sqrt{2}, a \in \mathbb{Q}$		
Functions from $\mathbb{R}$ to $\mathbb{R}$	all $a\sin(x)+b\cos(x),\ a,b \in \mathbb{R}$		

Problem # 19

Section 3.2 (pg 133) 11

Problem # 20

Linearly dependent or independent?

Set of vectors	Dependent	Independent
$(1,2),\ (3,-5)$		
$(0,0,0)$		
$2t^2+4t-3,\ 4t^2+8t-6$ (polynomials in t)		
$(1,1,2),\ (2,3,1),\ (4,5,5)$		
Section 3.3 (pg 144) 2a		
Section 3.3 (pg 144) 2b		

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Assignment # 5

Problem # 21
Let V be the vector space of real $n \times n$ matrices. Let

$$U = \{A \in V \mid A^\top = A\}, \quad \text{and} \quad W = \{A \in V \mid A^\top = -A\}.$$

a) Show that $V = U + W$.
b) Show that $V = U \oplus W$.
Hint: For a given matrix A, consider $(A + A^\top)/2$ and $(A - A^\top)/2$.

Problem # 22
Let V be the set

$$\Big\{ p(x) = \sum_{i=0}^{4} a_i x^i \in P_5, \text{ with } p(1) = 0 \Big\}.$$

a) Show that V is a subspace of P_5.
b) Find a basis for V.

Problem # 23
Let

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ -1 \\ -1 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} -1 \\ 8 \\ -5 \\ -6 \\ 1 \end{pmatrix}, \mathbf{x}_3 = \begin{pmatrix} 3 \\ -4 \\ 3 \\ 5 \\ -3 \end{pmatrix}, \mathbf{x}_4 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 2 \\ -2 \end{pmatrix}, \mathbf{x}_5 = \begin{pmatrix} 0 \\ 5 \\ -3 \\ -4 \\ 0 \end{pmatrix}.$$

Find a basis for $\text{Span}\,(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5)$. Once you have found a basis $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k\}$, form a matrix whose rows are the $\mathbf{b}_i$ and compute the row-reduced echelon form.

Problem # 24
Let

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ -2 \\ 1 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} 0 \\ 3 \\ 3 \\ 3 \\ 0 \end{pmatrix}, \mathbf{x}_3 = \begin{pmatrix} 1 \\ -3 \\ 1 \\ -2 \\ 4 \end{pmatrix}.$$

Find a matrix A such that $N(A) = \text{Span}\,(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$. Once you have found such a matrix A, compute its row-reduced echelon form.

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Assignment # 6

Problem # 25
Find a basis for the solution set of the system

$$\begin{array}{rrrcl} x_1 & -4x_2 & +3x_3 - 4_x & = & 0 \\ 2x_1 & -8x_2 & +6x_3 - 2x_4 & = & 0 \end{array}$$

Problem # 26
Find a basis and determine the dimension for each of the following:
a) 2×2 matrices,
b) symmetric 2×2 matrices,
c) symmetric 3×3 matrices,
d) $\mathbb{C}$ as vector space over $\mathbb{R}$.

Problem # 27
Find a basis for each of the following subspaces of P_4, the space of cubic polynomials:
a) $\{p(x) \in P_4 \mid p(7) = 0\}$
b) $\{p(x) \in P_4 \mid p(7) = p(5) = 0\}$
c) $\{p(x) \in P_4 \mid p(7) = p(5) = p(3) = 0\}$
d) $\{p(x) \in P_4 \mid p(7) = p(5) = p(3) = p(1) = 0\}$

Problem # 28
Find a vector $\vec{v}$ that completes the basis for the given space:
a) $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \vec{v} \right\}$ for $\mathbb{R}^2$.
b) $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \vec{v} \right\}$ for $\mathbb{R}^3$.
c) $\left\{ x, 1 + x^2, \vec{v} \right\}$ for P_3 the set of quadratic polynomials.

Problem # 29
Find a basis for the row-space of the following matrix:

$$\begin{pmatrix} 2 & 0 & 3 & 4 \\ 0 & 1 & 1 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 0 & -4 & -1 \end{pmatrix}$$

Problem # 30
Find the rank of each matrix:

a) $\begin{pmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ 1 & 0 & 3 \end{pmatrix}$ b) $\begin{pmatrix} 1 & -1 & 2 \\ 3 & -3 & 6 \\ -2 & 2 & -4 \end{pmatrix}$ c) $\begin{pmatrix} 1 & 3 & 2 \\ 5 & 1 & 1 \\ 6 & 4 & 3 \end{pmatrix}$

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Assignment # 7

Problem # 31

Let $f : V \to W$ be a linear map. Show that $f(\mathbf{0}) = \mathbf{0}$ ($\mathbf{0}$ the zero vector in the appropriate vector space).

Problem # 32

Let $B = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$ be an ordered basis for $\mathbb{R}^3$ with

$$\vec{v}_1 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix},$$

Consider the map $f : \mathbb{R}^3 \mapsto \mathbb{R}^3$

$$f(\vec{x}) = \begin{pmatrix} a \\ b \\ c \end{pmatrix},$$

where a, b, c are the coordinates of $\vec{x}$ with respect to B, that is, $\vec{x} = a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3$.

a) Compute $f(\begin{pmatrix} -27 \\ 1 \\ 2 \end{pmatrix})$.

b) Show that f is a linear map.

c) If $f(\vec{y}) = \begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix}$, what is $\vec{y}$?

Problem # 33

Let D be the differential operator.

a) Compute the matrix of $D : \mathcal{P}_6 \to \mathcal{P}_6$ with respect to the ordered basis $B = (1, x, x^2, x^3, x^4, x^5)$.

b) Compute the matrix of $D : V \to V$ with respect to the ordered basis $B = (e^x, e^{-x})$. Here, $V = \text{Span}\,(B)$.

c) Compute the matrix of $D : V \to V$ with respect to the ordered basis $B = (e^x + e^{-x}, e^x - e^{-x})$.

d) Compute the matrix of $D : V \to V$ with respect to the ordered basis $B = (\sin(x), \cos(x), \sin(2x), \cos(2x), \sin(3x), \cos(3x))$. Here, $V = \text{Span}\,(B)$.

Problem # 34

Let $\vec{v} \in \mathbb{R}^n$ be a non-zero vector.

a) Show that the map

$$f : \mathbb{R}^n \to \mathbb{R}^n, f(\vec{x}) = \vec{x} - 2\frac{\vec{x} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\vec{v}$$

is the reflection in the hyperplane whose normal vector is $\vec{v}$. (here, $\vec{x} \cdot \vec{y}$ is the dot product between $\vec{x}$ and $\vec{y}$). Hint: it suffices to show that $f(\vec{v}) = -\vec{v}$ and that $f(\vec{y}) = \vec{y}$ for each vector $\vec{y}$ in the hyperplane orthogonal to $\vec{v}$.

b) Compute $f \circ f$ by evaluating $f(f(\vec{x}))$. What does this tell you?

c) Show that $f(\vec{x}) \cdot f(\vec{y}) = \vec{x} \cdot \vec{y}$. What does this tell you?

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Assignment # 8

Problem # 35
(pg 205. 5.) Let f be the operator on P_3 defined by

$$f(p(x)) = xp'(x) + p''(x).$$

a) Find the matrix A representing f with respect to the basis $(1, x, x^2)$.
b) Find the matrix B representing f with respect to the basis $(1, x, 1 + x^2)$.
c) Find the matrix S such that $B = S^{-1}AS$.
d) If $p(x) = a_0 + a_1x + a_2(1 + x^2)$, calculate $f^n(p(x))$ (here f^n is applying f n times). *Hint:* Note that $A = SBS^{-1}$ and that $(SBS^{-1})^n = SB^nS^{-1}$.

Problem # 36
(pg 205. 6.) Let V be the subspace of $C[a, b]$ spanned by $(1, e^x, e^{-x})$, and let D be the differential operator on V.
a) Find the matrix A representing D with respect to the ordered basis $(1, e^x, e^{-x})$.
b) Find the matrix B representing D with respect to the ordered basis $(1, \cosh x, \sinh x)$.
c) Find the transition matrix S representing the change of coordinates from the ordered basis $(1, e^x, e^{-x})$ to the ordered basis $(1 \cosh x, \sinh x)$. Recall that $\cosh x = \frac{1}{2}(e^x + e^{-x})$, $\sinh = \frac{1}{2}(x^x - e^{-x})$.
d) Verify that $B = S^{-1}AS$.

Problem # 37
Consider the linear maps $L(x, y, z) = (x + 2y, y + 2z, z + 2x)$ and $M(a, b, c) = (a + c, b + a, b + c)$.
a) Compute the matrix of L
b) Compute the matrix of M
c) Describe the effect of the map $M \circ L$ (i.e. $M(L(x, y, z))$)
d) Compute the matrix of $M \circ L$.
e) How are the matrices in a), b) and d) related?

Problem # 38
Compute the eigenvalues and eigenvectors of the following matrix:

$$\begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}$$

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Assignment # 9

Problem # 39
The populations of U.S. cities, suburbs and nonmetro areas in 2007 are described by the following vector $\mathbf{x}_0$ (in units of one million)

$$\mathbf{x}_0 = \begin{bmatrix} 82 \\ 163 \\ 52 \end{bmatrix}.$$

Let $p_{i,j}$ be the probability that a person in area j will move to area i in any given year (city = 1, suburbs = 2, nonmetro = 3). Assume the transition matrix $P = (p_{i,j})$ is

$$P = \begin{bmatrix} .96 & .01 & .015 \\ .03 & .98 & .005 \\ .01 & .01 & .98 \end{bmatrix}.$$

Determine the long term predictions for the populations of these regions, assuming no change in their total population.

Problem # 40
Determine the characteristic polynomial, eigenvalues and the corresponding eigenspaces of the matrix

$$A = \begin{bmatrix} 5 & -7 & 7 \\ 4 & -3 & 4 \\ 4 & -1 & 2 \end{bmatrix}.$$

Then find a matrix S such that $S^{-1}AS$ is a diagonal matrix.

Problem # 41
Compute e^A for

$$A = \begin{bmatrix} 2 & -8 \\ 1 & -4 \end{bmatrix}.$$

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Assignment # 10

Problem # 42

The data below shows the average temperature, in degree celsius, of the earth's surface from 1975 through 2002 (source: Worldwatch Institute). Find the equation of the line that best fits the data points. Plot the data and the line that you found. Show all your work for full credit.

1975	13.94	1982	14.04	1989	14.19	1996	14.23
1976	13.86	1983	14.25	1990	14.37	1997	14.40
1977	14.11	1984	14.07	1991	14.32	1998	14.56
1978	14.02	1985	14.03	1992	14.14	1999	14.32
1979	14.09	1986	14.12	1993	14.14	2000	14.31
1980	14.16	1987	14.27	1994	14.25	2001	14.46
1981	14.22	1988	14.29	1995	14.37	2002	14.52

Problem # 43

Using the inner product on $\mathcal{P}_3$ defined by $\langle p, q\rangle = \int_0^1 p(x)q(x)dx$, find an orthonormal basis for $\mathcal{P}_3$ starting with the given basis B where

$$B = (x-1, x+2, x^2).$$

Problem # 44

Let $V = \mathbb{R}^{2\times 2}$ with inner product

$$\langle A, B\rangle = \operatorname{tr}(B^\top A).$$

Let $W = \{A \in V \mid A^\top = A\}$.

a) Show that

$$W^\perp = \{A \in V \mid A^\top = -A\}$$

b) Show that every A in V can be written as the sum of matrices from W and from $W^\perp$.

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Assignment # 11

Problem # 45

According to Kepler's first law, a comet has an elliptic, parabolic or hyperbolic orbit around the sun (gravitational influences from planets ignored). In suitable polar coordinates, the position (r, ϕ) of a comet satisfies an equation of the form

$$r = \beta + e(r \cos \phi)$$

where β is a constant and e is the eccentricity of the orbit, with $0 < e < 1$ for an ellipse, $e = 1$ for a parabola and $e > 1$ for a hyperbola.

Given the data below, determine the type of orbit, and predict the position of the comet at $\phi = 4.6$ radians.

$$\begin{bmatrix} \phi & 0.88 & 1.10 & 1.42 & 1.77 & 2.14 \\ r & 3.00 & 2.30 & 1.65 & 1.25 & 1.01 \end{bmatrix}$$

Problem # 46

Find the angle θ between:

a) $\vec{u} = (1, 3, -5, 4)^\top$ and $\vec{v} = (2, -43, 4, 1)^\top$ in $\mathbb{R}^4$

b) $A = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ where $\langle A, B \rangle = \text{tr}(B^\top A)$.

Problem # 47

Let

$$f(x) = \begin{cases} x & \text{if } 0 \leq x \leq \pi \\ x + \pi & \text{if } -\pi \leq x < 0 \end{cases}$$

a) Find the Fourier approximation for f of degrees $n = 2, 3, 4$ and 5.

b) Sketch $y = f(x)$ along with the polynomials in a).

Problem # 48

Let

$$A = \begin{pmatrix} -10 & -18 \\ 9 & 17 \end{pmatrix}$$

a) Find S such that $S^{-1}AS$ is diagonal.

b) Find C such that $C^3 = A$.

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Midterm # 1 Preparation

Problem # 1

Are the following vectors dependent or independent?

a)

$$\begin{pmatrix}2\\1\\1\end{pmatrix}, \begin{pmatrix}3\\2\\3\end{pmatrix}, \begin{pmatrix}1\\2\\2\end{pmatrix},$$

3)

$$\begin{pmatrix}1\\-1\\3\end{pmatrix}, \begin{pmatrix}-5\\8\\-7\end{pmatrix}, \begin{pmatrix}-7\\13\\-5\end{pmatrix}.$$

Problem # 2

Compute a basis for the solution space of the following system

$$\begin{array}{cccccccc} 2x_1 & +3x_2 & +x_3 & -x_4 & +4x_5 & = & 2 \\ -3x_1 & +4x_2 & +3x_3 & +x_4 & +2x_5 & = & -1 \end{array}$$

Problem # 3

Express the vector $\vec{v}$ in terms of the basis $\vec{b}_1, \vec{b}_2, \vec{b}_3$ of $\mathbb{R}^3$.

$$\vec{v} = \begin{pmatrix}-27\\1\\2\end{pmatrix}, \vec{b}_1 = \begin{pmatrix}2\\-1\\3\end{pmatrix}, \vec{b}_2 = \begin{pmatrix}-1\\3\\1\end{pmatrix}, \vec{b}_3 = \begin{pmatrix}4\\2\\-1\end{pmatrix},$$

Problem # 4

$$\vec{v}_1 = \begin{pmatrix}3\\-1\\-2\end{pmatrix}, \vec{v}_2 = \begin{pmatrix}-2\\4\\3\end{pmatrix}, \vec{v}_3 = \begin{pmatrix}1\\0\\2\end{pmatrix}, \vec{v}_4 = \begin{pmatrix}3\\-2\\1\end{pmatrix}.$$

Find a basis for $\text{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$.

Problem # 5

Find a matrix A whose nullspace is $\text{Span}(\vec{v}_1, \vec{v}_2)$ where

$$\vec{v}_1 = \begin{pmatrix}1\\-1\\1\end{pmatrix}, \vec{v}_1 = \begin{pmatrix}2\\3\\-1\end{pmatrix}.$$

Problem # 6

Let V be the space of polynomials $a_0 + a_1 t + a_2 t^2 + a_3 t^3$ in P_4 with $a_0 + a_1 + a_2 + a_3 = 0$. Show that V is ia subspace. Let $p(x) = 3 - t + 2t^2 - 4t^3$. Extend p to a basis for V

Problem # 7
Complete the sentence:

A set $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is linearly dependent if...

Problem # 8
Suppose that $V = U + W$ with V a subspace of $\mathbb{R}^5$, $\dim U = 3$, $\dim W = 3$. What are the possibilities for $\dim V$ and $\dim U \cap W$?

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Practice # 1

Problem # 1

Consider the inner product $\langle A, B\rangle = \mathrm{tr}(B^\top A)$ on the space of all 2×2 matrices over $\mathbb{R}$. Compute

$$\mathrm{Span}\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right)^\perp.$$

Problem # 2

If $x^3 = 1$, then

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

equals

(A) $(cx^2+bx+a)\begin{vmatrix} 1 & b & c \\ x & c & a \\ x^2 & a & b \end{vmatrix}$; (B) $(cx^2+bx+a)\begin{vmatrix} x & b & c \\ 1 & c & a \\ x^2 & a & b \end{vmatrix}$

(C) $(cx^2+bx+a)\begin{vmatrix} x^2 & b & c \\ x & c & a \\ 1 & a & b \end{vmatrix}$; (D) $(cx^2+bx+a)\begin{vmatrix} 1 & b & c \\ x^2 & c & a \\ x & a & b \end{vmatrix}$

Problem # 3

Let A be an $n\times n$ matrix with $A^2 - 4A + 5I = 0$. Show that n is even.

Problem # 4

Compute the determinant

$$|A| = \begin{vmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ -1 & 0 & 3 & \dots & n-1 & n \\ -1 & -2 & 0 & \dots & n-1 & n \\ \vdots & & & & & \\ -1 & -2 & -3 & \dots & -(n-1) & 0 \end{vmatrix}$$

Problem # 5

Compute the determinant

$$|A| = \begin{vmatrix} 1 & n & n & \dots & n \\ n & 2 & n & \dots & n \\ n & n & 3 & \dots & n \\ \vdots & & & & \\ n & n & \dots & n & n \end{vmatrix}$$

Problem # 6
Compute the characteristic polynomial of

$$A = \begin{bmatrix} 0 & 0 & 0 & \dots & -a_0 \\ 1 & 0 & 0 & \dots & -a_1 \\ 0 & 1 & 0 & \dots & -a_2 \\ \vdots & & & & \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{bmatrix}$$

Problem # 7
For $n \in \mathbb{N}$, let

$$T_n = \{f \in \mathbb{R}[x] \mid \deg(f) = n \text{ or } f = 0\}$$

and

$$P_n = \{f \in \mathbb{R}[x] \mid \deg(f) \leq n \text{ or } f = 0\}.$$

a) Is T_n a subspace of $\mathbb{R}[x]$ over $\mathbb{R}$?
b) Is P_n a subspace of $\mathbb{R}[x]$ over $\mathbb{R}$?

Problem # 8
Prove that if the rank of the $m \times n$ matrix A is m, then the system $Ax = b$ is consistent for all $b \in \mathbb{R}^m$.

Problem # 9
Let A be an $m \times n$ matrix and B be the transpose of A. Let $P = BA$. Prove that A and P have the same null space and the same rank.

Problem # 10
Find the LR factorization of

$$\begin{bmatrix} 1 & 1 & 3 \\ 2 & 2 & 5 \\ 3 & 1 & 2 \end{bmatrix}$$

Problem # 11
Let A, B, C, D be $n \times n$ matrices such that $CD = DC$ and D is invertible, show that

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(AD - BC)$$

Hint: multiply on the right by $\begin{bmatrix} I & 0 \\ X & I \end{bmatrix}$ for suitable X.

Problem # 12
Let A be an $m \times n$ matrix with $n > m$. Show that $\det A^\top A = 0$. *Hint:* If f and g are linear maps, show that $\dim \operatorname{Im}(g \circ f) \leq \dim \operatorname{Im}(f)$.

Dr. A. Betten Spring 2009

MATH 369 Linear Algebra

Practice # 2

Problem # 1
Prove: If A and B are nonsingular $n \times n$ matrices, then AB is also nonsingular and $(AB)^{-1} = B^{-1}A^{-1}$.

Problem # 2
Let A by an $m \times n$ matrix.

a. Explain why the matrix multiplications $A^T A$ and AA^T are possible.

b. Show that $A^T A$ and AA^T are symmetric

Problem # 3
A matrix A is said to be *skew-symmetric* if $A^T = -A$. Show that if a matrix is skew-symmetric then its diagonal entries must all be 0.

Problem # 4
Given

$$A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

compute A^{-1} and use it to:

a. Find a 2×2 matrix X such that $AX = B$.

b. Find a 2×2 matrix Y such that $YA = B$.

Problem # 5
Let $A = [\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}]$ be a 3×3 matrix blocked off by columns. Suppose that

$$\mathbf{a_1} = 3\mathbf{a_2} - 2\mathbf{a_3}.$$

Will the system $A\mathbf{x} = \mathbf{0}$ have a nontrivial solution? Is A nonsingular? Explain your answers.

Problem # 6
Let U be an $n \times n$ upper triangular matrix with nonzero diagonal entries.

a. Explain why U must be nonsingular.

b. Explain why U^{-1} must be upper triangular.

Problem # 7
Let A be a nonsingular $n \times n$ matrix and let B be an $n \times r$ matrix. Show that the reduced row echelon form of $[A, B]$ is $[I, C]$. What is C?

Problem # 8
Let A and B be $n \times n$ matrices and define $2n \times 2n$ matrices S and M by

$$S = \begin{bmatrix} I & A \\ 0 & I \end{bmatrix}, \qquad M = \begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix}$$

Determine the block form of S^{-1} and use it to compute the block form of the product $S^{-1}MS$.

Problem # 9
Let A be a matrix of the form

$$A = \begin{bmatrix} \alpha & \beta \\ 2\alpha & 2\beta \end{bmatrix}$$

where α and β are fixed scalars not both equal to 0.

a. Explain why the system

$$A\mathbf{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

must be inconsistent(i.e., a solution does not exist).

b. How can one choose a nonzero vector $\mathbf{b}$ so that the system $A\mathbf{x} = \mathbf{b}$ will be consistent? Explain.

Problem # 10
Find all possible choices of c that would make the following matrix singular.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 9 & c \\ 1 & c & 3 \end{bmatrix}$$

Note: You could combine information from reducing the matrix and using determinants.

Problem # 11
Let A and B be 3×3 matrices with $\det(A) = 4$ and $\det(B) = 5$. Find the value of (a)$\det(AB)$, (b)$\det(3A)$, (c)$\det(2AB)$, (d)$\det(A^{-1}B)$.

Problem # 12
Let A be an $n \times n$ matrix. Is it possible for $A^2 + I = 0$ in the case where n is odd? Answer the same question in the case where n is even. Note: This problem is *different.* Use facts about determinants.

Problem # 13
In each of the following answer *true* if the statement is always true and *false* otherwise. In the case of a *true* statement, explain or prove your answer. In the case of a *false* statement, give an example to show that the statement is not always true. In each of the following, assume that all the matrices are $n \times n$.

a. $\det(AB) = \det(BA)$

b. $\det(A + B) = \det(A) + \det(B)$

c. $\det(cA) = c\det(A)$

d. $\det((AB)^T) = \det(A)\det(B)$

e. $\det(A) = \det(B)$ implies $A = B$.

f. $\det(A^k) = \det(A)^k$

g. A triangular matrix is nonsingular if and only if its diagonal entries are all nonzero.

h. If $\mathbf{x}$ is a nonzero vector in R^n and $A\mathbf{x} = \mathbf{0}$, then $\det(A) = 0$.

i. If $A \neq 0_{n\times n}$, but $A^k = 0_{n\times n}$ for some positive integer k, then A must be singular.

Problem # 14
If A is an $n \times n$ matrix and $B = S^{-1}AS$ for some nonsingular matrix S, then $\det(B) = \det(A)$.

Problem # 15
Determine whether the following sets form subspaces of R^3.

a. $\{[x_1, x_2, x_3]^T : x_1 + x_3 = 1\}$

b. $\{[x_1, x_2, x_3]^T : x_1 = x_2 = x_3\}$

c. $\{[x_1, x_2, x_3]^T : x_3 = x_1 + x_2\}$

Problem # 16
Determine the row space, column space, null space, and dimensions of each for

$$\begin{bmatrix} 1 & 3 & -4 \\ 2 & -1 & -1 \\ -1 & -3 & 4 \end{bmatrix}$$

Problem # 17
Determine whether the following are subspaces of P_4(polynomials of degree 4).

a. The set of polynomials in P_4 of even degree

b. The set of all polynomials of degree 3

c. The set of all polynomials $p(x)$ in P_4 such that $p(0) = 0$.

d. The set of all polynomials in P_4 having at least one real root.

Problem # 18
Which of the following are spanning sets of R^3?

a. $\{[1, 0, 0]^T, [0, 1, 1]^T, [1, 0, 1]^T\}$

b. $\{[2, 1, -2]^T, [3, 2, -2]^T, [2, 2, 0]^T\}$

c. $\{[1, 1, 3]^T, [0, 2, 1]^T\}$

Problem # 19
Which of the following are spanning sets for P_3?

a. $\{1, x^2, x^2 - 2\}$

b. $\{2, x + 1, x^2 - 3, 5x^4\}$

Problem # 20
Determine whether the following vectors are linearly independent in R^3.

a. $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$

b. $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$

Problem # 21
Determine whether the following vectors are linearly independent in $R^{2\times 2}$.

a. $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$

Problem # 22
Let $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_k}$ be linearly independent vectors in a vector space V.

a. If we add a vector $\mathbf{x_{k+1}}$ to the collection, will we still have a linearly independent collection of vectors? Explain.

b. If we delete a vector, say $\mathbf{x_k}$, from the collection, will we still have a linearly independent collection of vectors? Explain.

Problem # 23
Given $\mathbf{u_1} = [1,1,1]^T, \mathbf{u_2} = [1,2,2]^T, \mathbf{u_3} = [2,3,4]^T$, find the coordinates in terms of $\mathbf{u_i}, i = 1:3$, of the following vectors.

a) $[3,2,5]^T$

b) $[2,3,2]^T$

Problem # 24
Given

$$A = \begin{bmatrix} 1 & 2 & 2 & 3 & 1 & 4 \\ 2 & 4 & 5 & 5 & 4 & 9 \\ 3 & 6 & 7 & 8 & 5 & 9 \end{bmatrix}$$

a. Compute the RREF form of A.

b. What are the dimensions of the row space, column space, and null space of A? How many free variables are there?

c. Define the general solution of $A\mathbf{x} = \mathbf{0}$.

Problem # 25
Let A be a 6×5 matrix. If $\dim(N(A)) = 2$, what are the dimensions of the row and column spaces of A?

Problem # 26
Prove that a linear system $A\mathbf{x} = \mathbf{b}$ is consistent(i.e., a solution exists) if and only if the rank of $[A, \mathbf{b}]$ equals the rank of A. Note: The rank equals the number of pivots when bringing a matrix to RREF.

Problem # 27
Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 3 \\ 1 & 1 & -1 \end{bmatrix}$$

a. Factor A into a product SDS^{-1}.

b. Use the factorization to compute A^6.

a) Use the factorization to compute A^{-1}.

Problem # 28
Let A be a diagonalizable matrix whose eigenvalues are all either 1 or -1. Show that $A^{-1} = A$.

Problem # 29
Let A be a 4×4 matrix and let λ be an eigenvalue of algebraic multiplicity 3. If $A - \lambda I$ has rank 1, what is the geometric multiplicity of λ?

Problem # 30
Let A be an $n \times n$ matrix with an eigenvalue λ of multiplicity n. Show that A is diagonalizable if and only if $A = \lambda I$.

Problem # 31
A nilpotent matrix $A_{n\times n}$ is such that $A^2 = A$. What are the eigenvaules of A? What is the algebraic multiplicity of each eigenvalue;i.e., if one of the eigenvalues has algebraic multiplicity m, what is the algebraic multiplicity of the other eigenvalues? Hint: Find a polynomial of degree $\leq n$ such that $p(A) = 0_{n\times n}$.

Problem # 32
Suppose the matrix A is diagonalizable (in terms of eigenvectors). How are the eigenvectors corresponding to the eigenvalue $\lambda = 0$ related to the null space of A? How is this related to the number of free variables for A? Explain.

Problem # 33
If S diagonalizes A with a similarity transformation, show that $R = (S^{-1})^T$ diagonalizes A^T.

Problem # 34
Let

$$\mathbf{y_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{y_2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{y_3} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and let T be the linear operator on R^3 defined by

$$T(c_1\mathbf{y_1} + c_2\mathbf{y_2} + c_3\mathbf{y_3}) = (c_1 + c_2 + c_3)\mathbf{y_1} + (2c_1 + c_3)\mathbf{y_2} - (2c_2 + c_3)\mathbf{y_3}.$$

a. Find a matrix representing T with respect to the ordered basis $\{\mathbf{y_1}, \mathbf{y_2}, \mathbf{y_3}\}$.

b. For each of the following, write the vector $\mathbf{x}$ as a linear combination of $\mathbf{y_1}, \mathbf{y_2}, \mathbf{y_3}$ and use the matrix from part(a) to determine $T(\mathbf{x})$. (i)$\mathbf{x} = [7, 5, 2]^T$, (ii) $\mathbf{x} = [3, 2, 1]^T$, (iii) $\mathbf{x} = [1, 2, 3]^T$.

Problem # 35
Let T be the linear transformation mapping P_2, polynomials of degree less than or equal to 2, into R^2 define by

$$T(p(x)) = \begin{bmatrix} \int_0^1 p(x)dx \\ p(0) \end{bmatrix}$$

Find a matrix A such that

$$T(\alpha + \beta x) = A \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

Problem # 36
Let

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 12 \\ 6 \\ 18 \end{bmatrix}.$$

a. Use the Gram-Schmidt process to find an orthonormal basis for the column space of A.

b. Factor A into QR where the columns of Q correspond the orthonormal basis in part (a).

c. Solve $A\mathbf{x} = \mathbf{b}$ using the QR decomposition.

Problem # 37
The vectors $\mathbf{x_1} = \frac{1}{2}[1, 1, 1, -1]^T$ and $\mathbf{x_2} = \frac{1}{6}[1, 1, 3, 5]^T$ form an orthonormal set in R^4. Extend this set to an orthonormal basis for R^4 by finding an orthonormal basis for the nullspace of

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 3 & 5 \end{bmatrix}$$

Hint: First find a basis for the nullspace and then use the Gram-Schmidt process.

Problem # 38
Use the Gram-Schmidt process to find an orthonormal basis for the subspace of R^4 spanned by $\mathbf{x_1} = [2, 0, 0, 2]^T$ and $\mathbf{x_2} = [1, 1, -1, 1]^T$.

Problem # 39
Find the linear least squares solution for the data points $(2, -3), (3, -2.5), (4, -2.25), (5, -2)$.