Applying block intersection polynomials to study graphs and designs

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The main equations

- Consider the system of equations:

$$\sum_{i=0}^{s} \binom{i}{j} n_i = \binom{s}{j} \lambda_j \quad (j = 0, \ldots, t) \tag{1}$$

where s, t are given non-negative integers, with $s \geq t$, the λ_j are given rational numbers (or symbolic expressions), and we are interested in solution vectors $[n_0, \ldots, n_s]$ of non-negative integers (or symbolic expressions for these solutions), or want to show that no such solutions exist.

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- Systems of equations of this form arise in the study of block designs, especially the study of t-designs, and in the study of graphs with certain regularity properties.

- The block intersection polynomial is a tool to give useful theoretical, symbolic, or exact numerical information about the solutions to the system (1) when t is even and non-negative integers $m_0, \ldots, m_s$ are specified for which $m_i \leq n_i$ must hold.

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- Exact linear or integer programming methods may also be used to study specific instances of the system (1), subject to $m_i \leq n_i$ or other linear inequalities.

The main definition

Definition

The *block intersection polynomial*

$$B(x,[m_0,\ldots,m_s],[\lambda_0,\ldots,\lambda_t])$$

is defined to be

$$\sum_{j=0}^{t}\binom{t}{j}P(-x,t-j)[P(s,j)\lambda_j-\sum_{i=j}^{s}P(i,j)m_i],$$

where for k a non-negative integer,

$$P(x,k):=x(x-1)\cdots(x-k+1).$$

The main theorem

Theorem (P.J. Cameron and S.)

Suppose $[n_0, \ldots, n_s]$ *is an real-vector solution to the system of equations* (1), *where* s, t *are non-negative integers, with* $s \geq t$, $\lambda_0, \ldots, \lambda_t$ *and* $m_0, \ldots, m_s$ *are real numbers, with* $m_i \leq n_i$ *for all* i, *and let*

$$B(x) := B(x, [m_0, \ldots, m_s], [\lambda_0, \ldots, \lambda_t]).$$

Then:

1. $B(x) = \sum_{i=0}^{s} P(i - x, t)(n_i - m_i)$;

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Then:

1. $B(x) = \sum_{i=0}^{s} P(i-x, t)(n_i - m_i)$;
2. *if* t *is even then* $B(m) \geq 0$ *for every integer* m;
3. *if* t *is even and* m *is an integer then* $B(m) = 0$ *if and only if* $m_i = n_i$ *for all* $i \notin \{m, m+1, \ldots, m+t-1\}$, *in which case* $[n_0, \ldots, n_s]$ *is uniquely determined by* $[m_0, \ldots, m_s]$ *and* $[\lambda_0, \ldots, \lambda_t]$.

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- These polynomials are implemented in my DESIGN package for GAP. They are used to provide an upper bound on the number of times a block can be repeated in a t-(v, k, λ) design (given only t, v, k, λ), and to provide a sometimes better bound for this for a resolvable t-(v, k, λ) design with t even.

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- Block intersection polynomials are also used to provide constraints in the DESIGN package function for finding and classifying block designs with user-specified properties.

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- My aim in this talk is to give a simplified introduction to block intersection polynomials, focussing on applications to cliques in edge-regular graphs, in the hope that you will become interested to apply these polynomials in your research.
- All graphs in this talk are finite, undirected, and have no loops or multiple edges.

The main way the main equations arise

- Let Γ be a graph, and let S and Q be given vertex-subsets of Γ, with $s := |S|$. We shall be interested in the number n_i of vertices in Q adjacent to exactly i vertices in S $(i = 0, \ldots, s)$.

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- For $T \subseteq S$, define λ_T to be the number of vertices in Q adjacent to every vertex in T, and for $0 \le j \le s$, define

$$\lambda_j := \binom{s}{j}^{-1} \sum_{T \subseteq S, |T| = j} \lambda_T .$$

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- In many, but not all, applications, λ_T is constant over the j-subsets T of S, in which case, λ_j is simply this constant.

By counting in two ways the number of ordered pairs (T, q) where T is a j-subset of S and q is a vertex in Q adjacent to every vertex in T, we obtain:

$$\sum_{i=0}^{s} \binom{i}{j} n_i = \binom{s}{j} \lambda_j,$$

where n_i is the number of vertices in Q adjacent to exactly i vertices in S.

Example

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Now suppose that S an s-clique of Γ (i.e. an s-set of pairwise adjacent vertices), with $s \geq 2$, and let $Q := V(\Gamma) \setminus S$. Then

$$\lambda_0 = |Q| = v - s, \quad \lambda_1 = k - s + 1, \quad \lambda_2 = \lambda - s + 2,$$

and for $j = 0, 1, 2$ we have:

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Let Γ be the incidence graph of a t-(v, k, λ) design, let S be a subset of the set of point-vertices of Γ, with $s := |S| \geq t$, and let Q be the set of all block-vertices of Γ.

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Then for $0 \leq j \leq t$,

$$\lambda_j = \lambda \binom{v-j}{t-j} / \binom{k-j}{t-j},$$

and

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Note that if S is the point-set of a block of multiplicity at least m, then $n_s \geq m$.

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A *regular clique*, or more specifically, an *m-regular clique* in a graph Γ is a non-empty clique S such that every vertex of Γ not in S is adjacent to exactly m vertices of S, for some constant $m > 0$.

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A *quasiregular clique*, or more specifically, an *m-quasiregular clique* in a graph Γ is a clique S of size at least 2, such that every vertex of Γ not in S is adjacent to exactly m or $m+1$ vertices of S, for some constant $m \geq 0$.

Applying the previous theorem of Cameron and S., we obtain:

Theorem

Let Γ be an edge-regular graph with parameters (v, k, λ), let S be an s-clique of Γ, with $s \geq 2$, and let

$$B(x) := B(x, [0^{s+1}], [v - s, k - s + 1, \lambda - s + 2])$$

$$= x(x + 1)(v - s) - 2xs(k - s + 1) + s(s - 1)(\lambda - s + 2).$$

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3. *if m is a positive integer then $B(m - 1) = B(m) = 0$ if and only if S is m-regular.*

Example

A.A. Makhnev (2011) used block intersection polynomials to study cliques in certain highly regular graphs. In this work, he observed that when

$$v = K((K-1)(R-1)+\alpha)/\alpha,\ k = (K-1)R,\ \lambda = K-2+(R-1)(\alpha-1),$$

for some integers $R, K > 1$ and $\alpha > 0$, we have

$$B(x, [0^{K+1}], [v-K, k-K+1, \lambda-K+2])$$
$$= [\alpha^{-1}K(K-1)(R-1)](x-(\alpha-1))(x-\alpha),$$

to show that in any edge-regular graph having the same (v, k, λ) as a pseudo-geometric strongly regular graph, each K-clique is α-regular.

Generalisation of a result of Neumaier

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1. $\omega(\Delta) \leq s$*, so in particular,* $\omega(\Gamma) = s$*;*
2. *all quasiregular cliques in Δ are m-quasiregular cliques;*
3. *the quasiregular cliques in Δ are precisely the cliques of size s (although Δ may have no cliques of size s).*

Bounding the clique number of an edge-regular graph

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- However, $B(3, [0^9], 65 - 8, 32 - 7, 15 - 6) = -12 < 0$, and so no edge-regular graph with parameters $(65, 32, 15)$ can have a clique of size 8.

On 7-cliques in an SRG$(65, 32, 15, 16)$

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One could split this into subcases depending on the number n_0 of vertices in Γ adjacent to no vertex in S. To eliminate $n_0 \geq 3$, we calculate the block intersection polynomial

$$B(x) := B(x, [3, 0^7], [58, 26, 10]) = 55x^2 - 309x + 420.$$

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Then $B(3) = 0$, and so, if there are two distinct vertices a, b of Γ adjacent to no vertex in some 7-clique S, then every vertex of Γ not in $S \cup \{a, b\}$ is adjacent to just 3 or 4 vertices of S (with exactly $B(4)/2 = 42$ vertices adjacent to exactly 3 vertices of S).

For further results, details, proofs, applications, and implementations, see:

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