Cameron – Liebler line classes

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based on joint work with **Ivan Mogilnykh**,
Institute of Mathematics (Novosibirsk, Russia),

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Justus-Liebig-University (Giessen, Germany), and Ghent University (Belgium).

CoCoA15, July 24, 2015

Definition

We consider a set $\mathcal{L}$ of lines of $PG(3,q)$ such that:

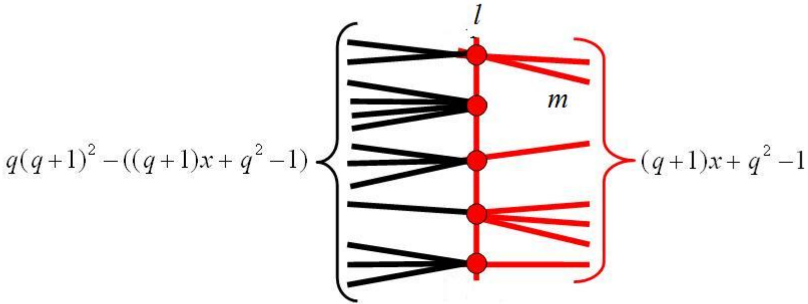
$$\exists \text{ a number } x:$$

$\forall$ line $l \in \mathcal{L}$

$$|\{m \in \mathcal{L} : \ m \text{ meets } l, \ m \neq l\}| = (q+1)x + q^2 - 1$$

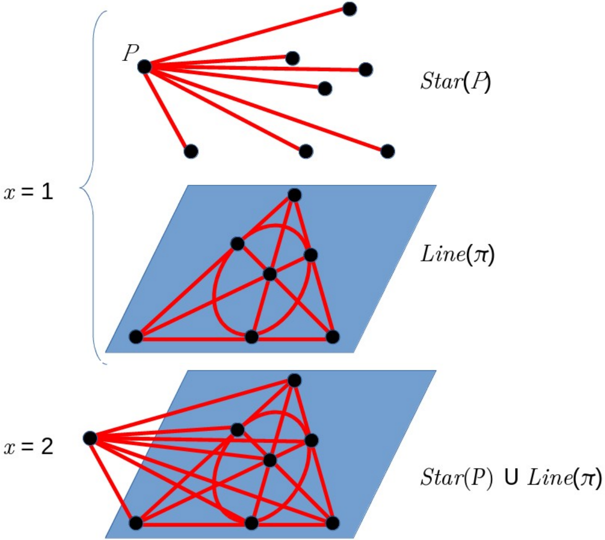
$\forall$ line $k \notin \mathcal{L}$

$$|\{m \in \mathcal{L} : \ m \text{ meets } k\}| = (q+1)x$$



Cameron – Liebler line classes, examples

Any line class that satisfies the property above is called a *Cameron – Liebler* line class, x – its *parameter*.

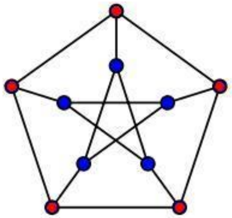


Plan

- Motivation
- Previous results
- New approach and existence condition
- Applications
- Open problems and future directions

Equitable t-partition

- $V(\Gamma) = V_1 \dot\cup V_2 \dot\cup \ldots \dot\cup V_t$,
- every vertex of V_i has exactly p_{ij} neighbours of V_j.



$$\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Motivation

The Grassmann graph $J_q(n,d)$:

- the vertex set: all d-dimensional subspaces of $\mathbb{F}_q^n$,
- U and W are adjacent iff $\dim(U \cap W) = d-1$,
- its diameter equals $\min(d, n-d)$.

In particular, $J_q(4,2)$:

- the vertex set: all lines of $PG(3,q)$,
- two lines are adjacent iff they intersect,
- strongly regular graph.

Cameron-Liebler line classes give rise to:

- Equitable partitions (completely regular codes) of the Grassmann graphs $J_q(4,2)$

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Motivation

- Equitable partitions (completely regular codes) of the Grassmann graphs $J_q(4,2)$
- Conjectures by Cameron and Liebler on tactical decompositions of the point-line design of $PG(n,q)$

Designs

A 2-design with parameters (v, k, λ) is a pair $D = (X, \mathcal{B})$:

- X is a v-set (with elements called points),
- $\mathcal{B}$ is a collection of k-subsets of X (called blocks),
- every 2 distinct points belong to precisely λ blocks.

For a 2-design $D = (X, \mathcal{B})$:

$$|\mathcal{B}| \geqslant |X|.$$

(Fisher's inequality)

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Automorphisms of designs

An automorphism (or a collineation) of D: (γ, δ)

$$\gamma : X \to X,\ \delta : \mathcal{B} \to \mathcal{B} \text{ such that}$$
$$p \in B \iff \gamma(p) \in \delta(B) \text{ for all } p \in X, B \in \mathcal{B}.$$

Consider a group $G \leqslant \mathrm{Aut}(D)$ and its orbits on X and $\mathcal{B}$:

$$\begin{array}{cc} & \mathcal{B} \\ & \circlearrowleft \quad \circlearrowleft \quad \circlearrowleft \\ X \ \begin{array}{c}\circlearrowright \\ \circlearrowright \\ \circlearrowright\end{array} & \left(\begin{array}{c|c|c} \cdots & \cdots & \cdots \\ \hline \cdots & \text{incidence matrix} & \cdots \\ \hline \cdots & \cdots & \cdots \end{array}\right) \end{array}$$

Then

$$\#\{\text{orbits on } \mathcal{B}\} \geqslant \#\{\text{orbits on } X\}.$$

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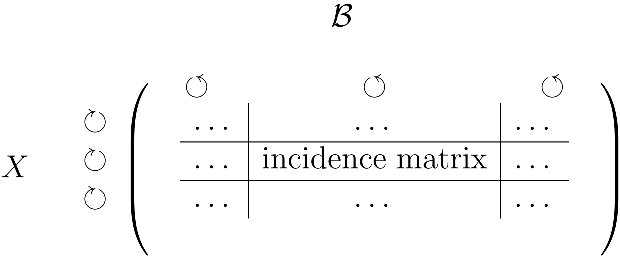

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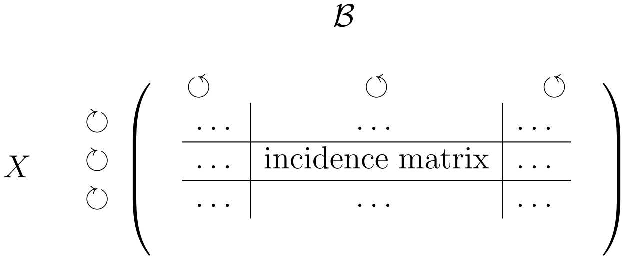

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Designs, tactical decomposition

A tactical decomposition $\mathcal{T}$ of D:

$$X = X_1 \dot\cup \dots \dot\cup X_s,\ \mathcal{B} = \mathcal{L}_1 \dot\cup \dots \dot\cup \mathcal{L}_t$$

such that the incidence matrix $(X_i, \mathcal{L}_j)$ has constant row and column sums for all i, j.

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Projective geometry as a design

Let D be the design on points and lines of $PG(n,q)$ with $\mathrm{Aut}(D) = P\Gamma L(n,q)$ (the point-line design of $PG(n,q)$).

- $n = 2$: D is a symmetric design (projective plane).
  $|X| = |\mathcal{B}|$
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Cameron – Liebler conjecture, 1 (1982)

Which collineation groups (i.e., subgroups of $P\Gamma L(n,q)$) have equally many point orbits and line orbits?

Conjecture on groups (Cameron, Liebler, 1982)

Such a group is:

- line-transitive
  or
- fixes a hyperplane and acts line-transitive on it
  or (dually)
- fixes a point and acts line-transitive on lines through it.

Proven by Bamberg and Penttila (2008).

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What are the symmetric tactical decompositions of $PG(n, q)$?

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A symmetric tactical decomposition of $PG(n, q)$ consists of

- a single point and line class
  or
- two point classes H, $PG(n, q) \setminus H$ and two line classes $line(H)$, $\overline{line(H)}$ for some hyperplane H
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Special line classes

Let $n \geq 3$.

Symmetric t. d. of the point-line design of $PG(n, q)$

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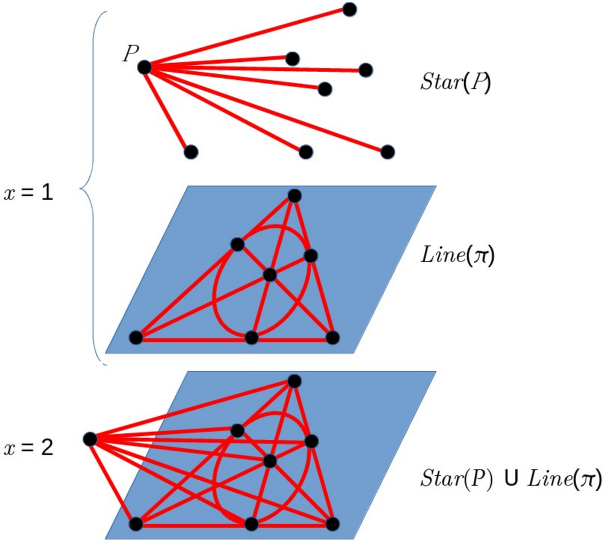
Cameron – Liebler line classes, examples



Cameron – Liebler conjecture, 3 (1982)

A line class $\overline{\mathcal{L}}$ complement to $\mathcal{L}$ is also a Cameron – Liebler line class with $x(\overline{\mathcal{L}}) = q^2 + 1 - x(\mathcal{L})$ $\Rightarrow$ w.l.o.g. $x \leqslant \frac{q^2+1}{2}$

Conjecture on 'special' classes

The only Cameron – Liebler line classes are those shown above (i.e., $x \notin \{3, \ldots, q^2 - 2\}$?).

Counterexample by Drudge (1999).

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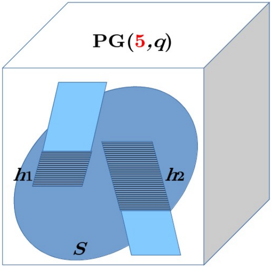
Motivation

- Equitable partitions (completely regular codes) in the Grassmann graphs $J_q(4, 2)$
- Conjectures by Cameron and Liebler on tactical decompositions of the point-line design of $PG(n, q)$
- 2-character sets in $PG(5, q)$

Motivation

A set S of points of $PG(n,q)$ is called a *2-character set* if every hyperplane of $PG(n,q)$ intersects S in either h_1 or h_2 points (intersection numbers).



The Klein correspondence:

lines of $PG(3,q) \longrightarrow$ points of $Q^+(5,q) \subset PG(5,q)$

lines of $\mathcal{L} \longrightarrow$ *tight set* of $Q^+(5,q) \subset PG(5,q)$

tight set of $Q^+(5,q) \longrightarrow$ 2-character set in $PG(5,q)$

Metsch

Properties of a Cameron – Liebler line class $\mathcal{L}$, 1

$\exists$ a number x: for $\forall$ spread S

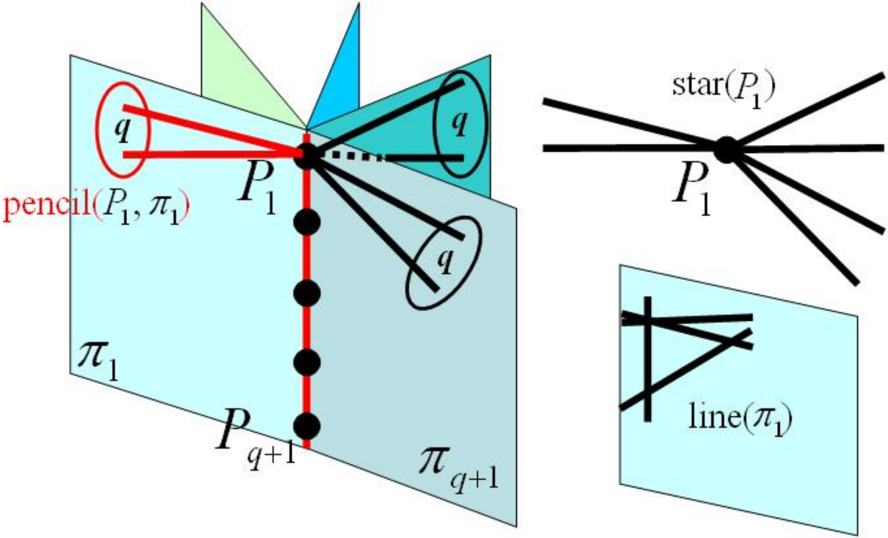
$$|\mathcal{L} \cap S| = x$$

- spread — a line set partitioning the points of $PG(n, q)$

Properties of a Cameron – Liebler line class $\mathcal{L}$, 2

$\exists$ a number x: for $\forall$ point P and $\forall$ plane π with $P \in \pi$:

$$|\text{star}(P) \cap \mathcal{L}| + |\text{line}(\pi) \cap \mathcal{L}| = x + (q+1)|\text{pencil}(P, \pi) \cap \mathcal{L}|$$



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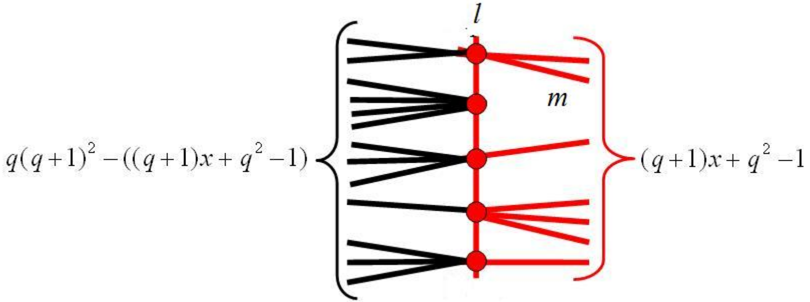
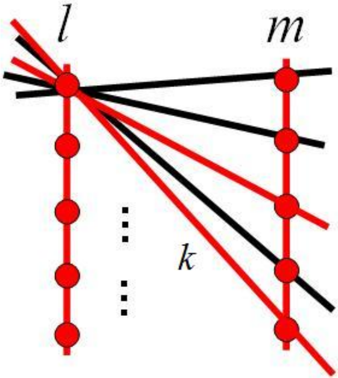


Properties of a Cameron – Liebler line class $\mathcal{L}$, 4

$\exists$ a number x: for $\forall$ skew lines l, m

$$|\{k \in \mathcal{L} : \ k \text{ meets } l \ \& \ m\}| = x + 2q$$



Properties of a Cameron – Liebler line class $\mathcal{L}$, 5

for every regulus $\mathcal{R}$ and its opposite, $\mathcal{R}^{opp}$,

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Properties of a Cameron – Liebler line class

In a summary, if $\mathcal{L}$ is a line class in a symmetric t. d. of D:

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$|\mathcal{L}| = x(q^2+q+1)$ $(\Rightarrow x \leqslant q^2+1)$.

(Cameron, Liebler; Penttila)

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Previous results

- $x \neq 3, 4$ if $q \geqslant 5$.
  (Penttila'91)
- $x \notin \{3, \ldots, \sqrt{q}\}$.
  (Bruen, Drudge'98)
- classification in $PG(3, 3)$ (with one counterexample).
- $x \notin \{3, \ldots, e(q)\}$ where $q + 1 + e(q)$ is the size of the smallest non-trivial blocking set in $PG(2, q)$.
  (Drudge'99)
- a counterexample in $PG(3, q)$ with $x = (q^2 + 1)/2$.
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- $x \neq 4, 5$ and a counterexample with $x = 7$ in $PG(3, 4)$.
  (Govaerts, Penttila'05)
- $x \notin \{3, \ldots, q\}$.
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- a counterexample in $PG(3, q)$ with $x = (q^2 + 1)/2$.

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- $x \neq 4, 5$ and a counterexample with $x = 7$ in $PG(3, 4)$.

  (Govaerts, Penttila'05)

- $x \notin \{3, \dots, q\}$.

  (Metsch'10)

Previous results

- $x \neq 3, 4$ if $q \geqslant 5$.

  (Penttila'91)
- $x \notin \{3, \ldots, \sqrt{q}\}$.

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Previous results

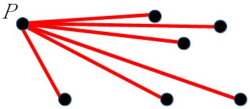
- In 2011 M. Rodgers constructed new Cameron – Liebler line classes for many odd values of q ($q < 200$) satisfying $q \equiv 1 \bmod 4$ and $q \equiv 1 \bmod 3$, having parameter $x = \frac{1}{2}(q^2 - 1)$.

  These new examples are made up of a union of orbits of a cyclic collineation group having order $q^2 + q + 1$.

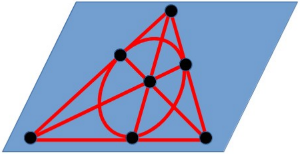
Drudge's approach

The most of the previous results rely on the observation by K. Drudge.

Define a clique of $PG(3,q)$:



$Star(P)$



$Line(\pi)$

A clique $\mathcal{C}$ of $PG(3,q)$ and its lines may be considered as a projective plane $PG(2,q)$ and its points, resp.

A blocking set in $PG(2,q)$ is a set of points that intersects every line but contains no line.

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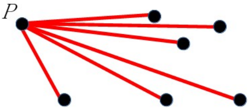

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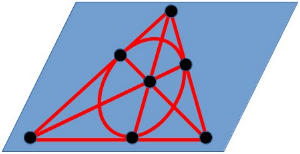

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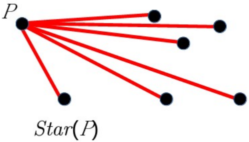

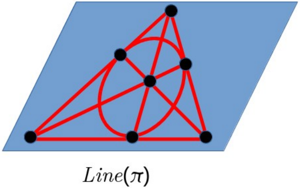

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Drudge's approach

Lemma (Drudge, 1999)

Let $\mathcal{L}$ be a Cameron – Liebler line class with parameter x in $PG(3,q)$, $\mathcal{C}$ be a clique, and assume that there exists no CL line class of parameter $x-1$.

If $x < |\mathcal{C} \cap \mathcal{L}| \leq x+q$ then the lines of $\mathcal{C} \cap \mathcal{L}$ form a blocking set in $\mathcal{C}$.

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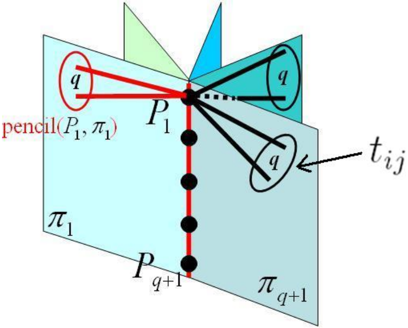
Plan

- Motivation
- Previous results
- New approach and existence condition
- Applications
- Open problems and future directions

Patterns (G. & Mogilnykh, 2012)

Let l be a line of $PG(3,q)$, $\mathcal{L}$ a Cameron – Liebler line class. Consider all the points P_i, $i = 1, \ldots, q+1$ that are on l, and all the planes π_j, $j = 1, \ldots, q+1$ that contain l.



Define a square matrix T of order $q+1$ whose (i,j)-element is $|\text{pencil}(P_i, \pi_j) \cap \mathcal{L} \setminus \{l\}|$

We will call such matrix pattern w.r.t. l.

Properties of patterns

Let $T := (t_{ij})$ be a pattern w.r.t. a line l, and define

$$\chi := \begin{cases} 0 \text{ if } l \notin \mathcal{L}, \\ 1 \text{ if } l \in \mathcal{L}, \end{cases}$$

Then the following hold:

- $t_{ij} \in \mathbb{N}$, $0 \leq t_{ij} \leq q$ for all $i, j \in \{1, \ldots, q+1\}$;
- $\sum_{i,j=1}^{q+1} t_{ij} = x(q+1) + \chi(q^2 - 1)$;
- $\sum_{j=1}^{q+1} t_{kj} + \sum_{i=1}^{q+1} t_{il} = x + (q+1)(t_{kl} + \chi), \forall k, l$;
- $\sum_{i,j=1}^{q+1} t_{ij}^2 = (x - \chi)^2 + q(x - \chi) + \chi q^2(q+1).$

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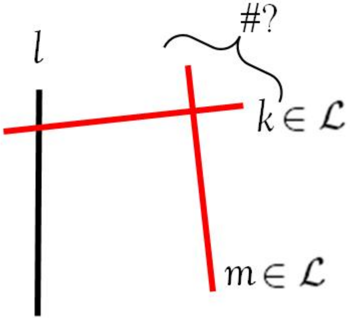
Properties of patterns

$$\sum_{i,j=1}^{q+1} t_{ij}^2 = (x-\chi)^2 + q(x-\chi) + \chi q^2(q+1)$$

follows from the two-side counting of $(k,m) \in \mathcal{L} \times \mathcal{L}$ such that $l \sim k$, $k \sim m$ and $l \not\sim m$.



A new existence condition

As a corollary, we see that if there exists a Cameron – Liebler line class with parameter x, then for all $\chi \in \{0, 1\}$, there should exist $(q+1) \times (q+1)$-matrices T such that:

- $t_{ij} \in \mathbb{N}$, $0 \leq t_{ij} \leq q$ for all $i, j \in \{1, \ldots, q+1\}$;
- $\sum_{i,j=1}^{q+1} t_{ij} = x(q+1) + \chi(q^2 - 1)$;
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Excluded pairs (q, x)

(for which the set of patterns is empty)

q	x	total
4	3,4,8	3 of 8
5	3,4,7,11	4 of 13
7	3,4,5,6,7,11,12,14,15,19,20,22,23	13 of 25
8	3,4,5,6,8,12,14,15,17,21,23,24,26,30,32	15 of 32
9	3,4,5,7,8,9,11,13,14,15,18,19,23,24,27,28,29,	
	31,33,34,35,38,39	23 of 42
11	3,...,9,11,12,14,15,19,20,22,23,27,28,30,31,	
	35,36,38,39,43,44,46,47,51,52,54,55,59,60	35 of 61

Guess (G., Mogilnykh, 2012)

The new existence condition eliminates about a half of possible values of x.

Plan

- Motivation
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(1) Improved bound for x

Klaus Metsch (2013) used the properties of patterns in order to improve his previous bound:

Theorem (Metsch, 2010)

There do not exist Cameron – Liebler line classes in $PG(3, q)$ with parameter x satisfying $2 < x \leq q$.

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(2) Modular equality

Later, we showed that the properties of patterns yield the following modular equation.

Theorem (G., Metsch, 2014)

Suppose $\mathcal{L}$ is a Cameron – Liebler line class of parameter x. Then, for every plane and every point of $PG(3, q)$, one has

$$\binom{x}{2} + \ell(\ell - x) \equiv 0 \bmod (q + 1) \tag{1}$$

where ℓ is the number of lines of $\mathcal{L}$ in the plane respectively through the point.

Corollary

Suppose $PG(3, q)$ has a Cameron – Liebler line class with parameter x. Then (1) has a solution for some ℓ in the set $\{0, 1, \ldots, q\}$.

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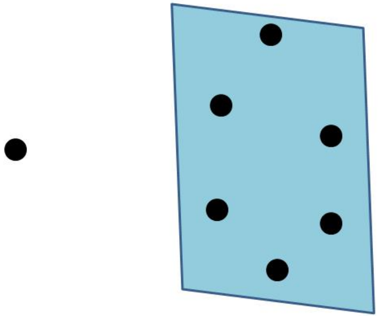
Suppose $PG(3,q)$ has a Cameron – Liebler line class with parameter x. Then (1) has a solution for some ℓ in the set $\{0, 1, \ldots, q\}$.

(3) Cameron – Liebler line classes in $PG(3, 4)$

$x \in \{0!, 1!, 2!, \not{3}, \not{4}, \not{5}, 6?, 7!?, 8?\}$ (as $(q^2 + 1)/2 = 8.5$)

(Govaerts, Penttila'05)

The Govaerts – Penttila class for $x = 7$, $q = 4$.



hyperoval in $PG(2, q)$ – a set of $q + 2$ points, no 3 of which collinear

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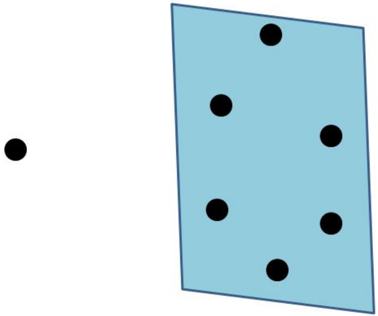

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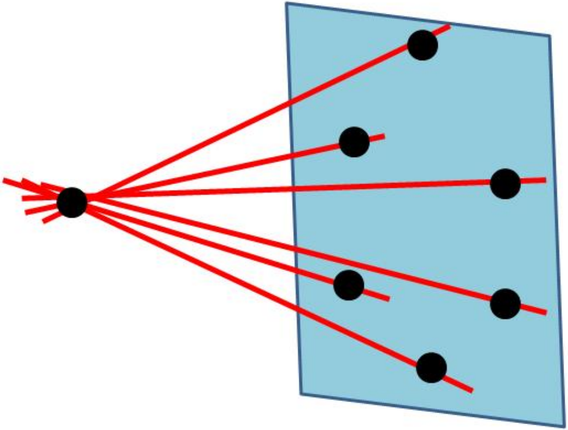

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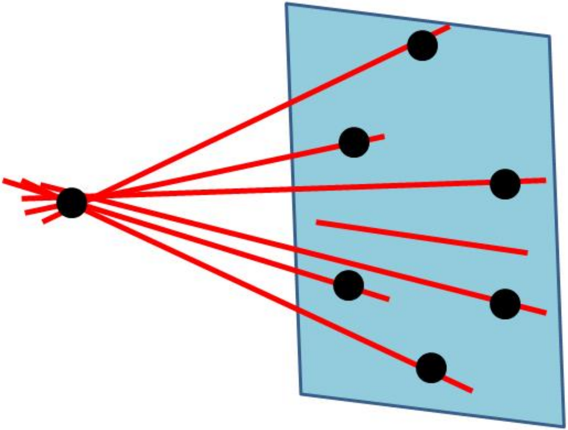

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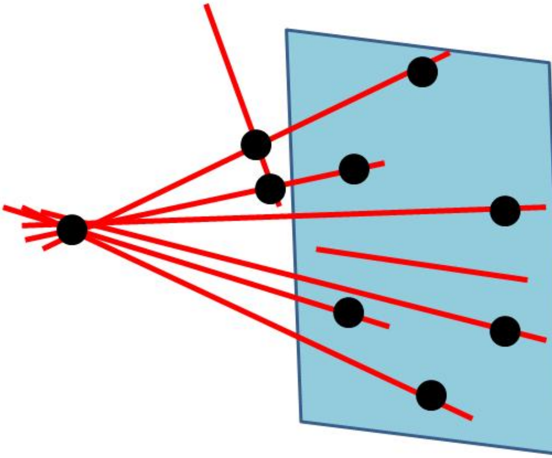

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For $q = 4$ and $x \in \{4,5,6,8\}$ it turns out that there are no matrices admissible w.r.t. our new condition.

Let $x = 7$. We have only the following admissible patterns:

w.r.t. $l \in \mathcal{L}$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 4 & 4 & 2 & 3 & 2 \\ 4 & 4 & 2 & 3 & 2 \\ 3 & 3 & 1 & 2 & 1 \\ 2 & 2 & 0 & 1 & 0 \\ 2 & 2 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 & 4 \end{pmatrix}$$

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(3) Cameron–Liebler line classes in $PG(n, 4)$

Theorem (G., Mogilnykh, 2013)

- A Cameron-Liebler line class with parameter x exists in $PG(3, 4)$ if and only if $x \in \{0!, 1!, 2!, \not{3}, \not{4}, \not{5}, \not{6}, 7!, \not{8}\}$
- the only Cameron-Liebler line classes in $PG(n, 4)$, $n > 3$, are trivial.

(4) Cameron–Liebler line classes in $PG(3,5)$

Theorem (G., Metsch, 2014)

A Cameron-Liebler line class with parameter x exists in $PG(3,5)$ if and only if $x \in \{0!, 1!, 2!, 10!, 12(?!), 13\}$

In particular, we found a new Cameron-Liebler line class with $x = 10$, and proved its uniqueness.

Its construction relies on one of two complete 20-caps found by Abatangelo, Korchmaros, Larato (1996).

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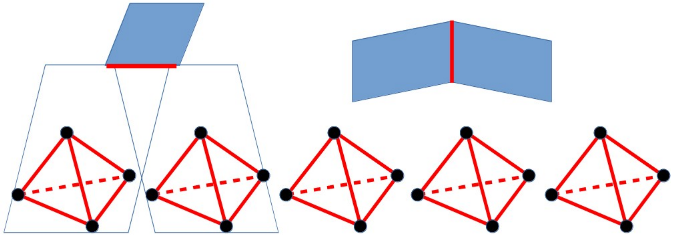
(4) Cameron–Liebler line classes in $PG(3, 5)$

A *cap* – a set of points, no 3 of which are collinear.



It consists of:

- the intersection lines of planes missing the cap K,
- the lines that are edges of the tetrahedra,
- the lines that lie in a plane missing K and two planes meeting K in three points.

(5) New infinite family

- $PG(3,q)$, $x = (q^2+1)/2$,

  Bruen and Drudge, 1998.

- $PG(3,4)$, $x = 7$,

  Govaerts and Penttila, 2004.

- $PG(3,q)$, $q < 200$ odd, $q \equiv 1 \bmod 4$ or $q \equiv 1 \bmod 3$, $x = \frac{1}{2}(q^2-1)$,

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  Momihara, Feng, Xiang, 2014.

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Problem

(1) Show uniqueness of a class with $x = 12$ in $PG(3,5)$

(2) Find all Cameron-Liebler line classes in $PG(n,5)$, $n > 3$.

There are some line classes (among those found by Rodgers), which also seem to be members of an infinite family, however, a general construction for them is not known yet.

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There are some line classes (among those found by Rodgers), which also seem to be members of an infinite family, however, a general construction for them is not known yet.

(2) The bilinear forms graph $Bil_q(2 \times 2)$

- the graph defined on the set $\mathrm{Mat}_{2\times 2}(\mathbb{F}_q)$ with two matrices A, B adjacent iff $\mathrm{rank}(A - B) = 1$.
- the graph defined on the set of lines of $PG(3, q)$ that are skew to a given line, with two lines adjacent iff they intersect.
- It can be viewed as a subgraph of the Grassmann graph $J_q(4, 2)$ induced by the second neighbourhood of a given vertex.

Equitable partition of:

$$J_q(4,2) \quad \longrightarrow Bil_q(2 \times 2)$$
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Frédéric Vanhove (September, 2013) gave the following example:

$$Z_0 := \{A \in \mathrm{Mat}_{2\times 2}(\mathbb{F}_q) : \ \mathrm{trace}(A) = 0\}$$

is a completely regular code in $Bil_q(2 \times 2)$.

Thus, the partition into sets

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gives an equitable 2-partition of $Bil_q(2 \times 2)$.

It is easy to see that $|Z_0| = q^3$ and Z_0 cannot be embedded into a Cameron-Liebler line class.

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(3) Tallini-Scaffati sets

A set S of points of $PG(d,q)$ is called *(m,n)-set* w.r.t. lines if every line of $PG(d,q)$ intersects S in m or n points.

- assume that S is not trivial (up to complement: empty set, a point, a hyperplane),
- there are examples in projective planes ($d = 2$).

  Full classification in $PG(2,9)$ by Royle and Penttila (1995).
- if $d \geq 3$ then q is an odd square, but no such sets are known,
- all m-secants to S form a Cameron-Liebler line class,
- these Cameron-Liebler line classes survive even with our new existence condition!

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(3) Tallini-Scaffati sets

The smallest possible case: $PG(3,9)$ with $(m,n) = (2,5)$.

Intersection of S with any plane = $(2,5)$-set in $PG(2,9)$

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