Graphs with second largest eigenvalue at most 1

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Graphs with second largest eigenvalue at most -1

Which graphs have second largest eigenvalue at most -1?

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Which graphs have second largest eigenvalue at most -1?

- Let Γ be an n-vertex graph.
- Eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$.

Graphs with second largest eigenvalue at most -1

Let Γ be a connected graph on $n \geqslant 2$ vertices with second largest eigenvalue at most -1.

- The 2-vertex disconnected graph has spectrum $\{[0]^2\}$.
- Interlacing: $\lambda_i \geqslant \mu_i$ for $i \in \{1, \dots, m\}$
  $\implies$ every pair of vertices in Γ must be adjacent.
- Hence Γ must be complete.

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Theorem (Smith 1970)

Let Γ be a connected graph with second largest eigenvalue at most 0. Then Γ is complete multipartite.

Graphs with small second largest eigenvalue

Let $S(b)$ denote the set of connected graphs with second largest eigenvalue at most b.

- Cao and Yuan 1993: $S(1/3)$.
- Petrović 1993: $S(\sqrt{2}-1)$.
- Cvetković and Simić 1995: $S((\sqrt{5}-1)/2)$.

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- Li and Yang 2011: ~~Quad~~Tricyclic graphs.

Plan

Classify graphs Γ with second largest eigenvalue at most 1 such that Γ has precisely three distinct eigenvalues.

- Graphs with three eigenvalues 101.
- Main theorem.
- A structural tool for the proof.
- Idea for the finite search.
- Closing remarks.

Graphs with three eigenvalues

Let Γ be a connected graph (V, E) with eigenvalues $\theta_0 > \theta_1 > \theta_2$. Then

$$(A - \theta_1 I)(A - \theta_2 I) = \alpha\alpha^\top.$$

Graphs with three eigenvalues

Let Γ be a connected graph (V, E) with eigenvalues $\theta_0 > \theta_1 > \theta_2$. Then

$$A^2 = (\theta_1 + \theta_2)A - \theta_1\theta_2 I + \alpha\alpha^\top, \qquad A\alpha = \theta_0\alpha.$$

$$d_x = -\theta_1\theta_2 + \alpha_x^2,$$

$$\nu_{x,y} = (\theta_1 + \theta_2)A_{x,y} + \alpha_x\alpha_y.$$

- Diameter of Γ is 2.
- $\theta_1 \geqslant 0$ and $\theta_2 \leqslant -\sqrt{2}$.

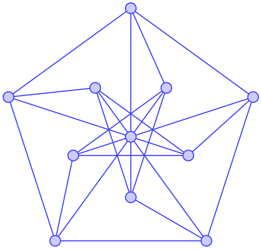
Regular graphs

- Regular graphs with three eigenvalues.
  Strongly regular graphs
- Regular graphs with second largest eigenvalue 1.
  Complement of graphs with smallest eigenvalue -2.
- Regular graphs with three eigenvalues and second largest eigenvalue 1.
  Complement of strongly regular graphs with smallest eigenvalue -2.
- Seidel 1968: classified strongly regular graphs with smallest eigenvalue -2.

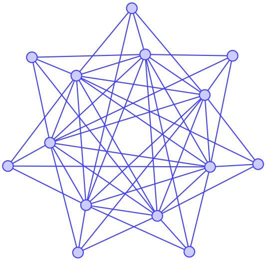
Nonregular graphs

Theorem

Let Γ be a connected nonregular graph with three distinct eigenvalues $\theta_0 > \theta_1 > \theta_2$ and $\theta_1 = 1$. Then $\theta_2 = -2$, and Γ is the Petersen cone or the Van Dam-Fano graph.



Petersen cone



Van Dam-Fano graph

Main theorem

Theorem

Let Γ be a connected graph with three distinct eigenvalues and second largest eigenvalue at most 1. Then Γ is one of the following graphs.

(a) *A complete bipartite graph;*

(b) *The Petersen cone;*

(c) *The Van Dam-Fano graph;*

(d) *A complete multipartite regular graph;*

(e) *The complement of a Seidel SRG.*

Structure of the proof

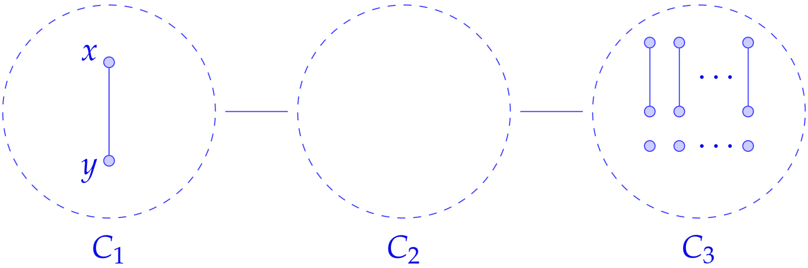
Goal: find connected 3-eigenvalue graphs Γ with $\theta_1 \leqslant 1$.

- Reduce to the case where Γ has second largest eigenvalue *precisely* 1. $\implies$ all eigenvalues are integers.
- Reduce to the case where Γ has at least three distinct valencies.
  - Regular case follows from Seidel (1968).
  - Biregular case [Cheng, Gavrilyuk, GG, Koolen (2015+)].
- Reduce to the case where Γ is not a cone.
- Reduce to the case where the smallest eigenvalue of Γ is at least -29.

A structural lemma

Lemma

Let Γ be a connected graph with second largest eigenvalue 1. For $x \sim y$, let π be a vertex partition with cells $C_1 = \{x, y\}$, $C_2 = \{z \in V(\Gamma) \backslash C_1 \mid z \sim x \text{ or } z \sim y\}$, and $C_3 = \{z \in V(\Gamma) \mid z \not\sim x \text{ and } z \not\sim y\}$. Then the induced subgraph on C_3 has maximum degree 1.



A bound for n

Lemma

Let Γ be a connected n-vertex graph with three eigenvalues and second largest eigenvalue 1. Let m denote the multiplicity of the smallest eigenvalue of Γ. Suppose $x \sim y$. Then $n \leqslant d_x + d_y - \nu_{x,y} + 2m$.

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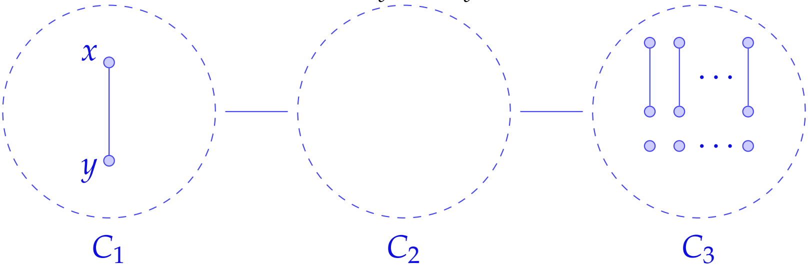
Lemma

Let Γ be a connected n-vertex graph with three eigenvalues and second largest eigenvalue 1. Let m denote the multiplicity of the smallest eigenvalue of Γ. Suppose $x \sim y$. Then $n \leqslant d_x + d_y - \nu_{x,y} + 2m$.

Proof.

$$|C_3| = n - (d_x + d_y - \nu_{x,y}); \quad |C_3| \leqslant 2m.$$



Finite search

Let Γ be a connected n-vertex graph with eigenvalues $s > 1 > -t$ and suppose $-t$ has multiplicity m. (Γ not a cone.)

- $n \leqslant f(t)$ for some rational function f.
- For each $t \in \{3, \ldots, 29\}$, we can enumerate parameters (n, s, m). Denote their set by $\mathcal{S}(t)$.

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t	$\lvert\mathcal{S}(t)\rvert$	— —	— —	t	$\lvert\mathcal{S}(t)\rvert$	— —	— —	t	$\lvert\mathcal{S}(t)\rvert$	— —	— —
3	128			12	497			21	189		
4	196			13	455			22	163		
5	277			14	409			23	143		
6	375			15	377			24	118		
7	492			16	340			25	95		
8	610			17	311			26	76		
9	748			18	273			27	61		
10	898			19	248			28	43		
11	546			20	220			29	27		

Finite search

- $n \leqslant f(t)$ for some rational function f.

- For each $t \in \{3, \ldots, 29\}$, we can enumerate parameters (n, s, m). Denote their set by $\mathcal{S}(t)$.

- For each $S \in \mathcal{S}(t)$, we can enumerate valencies $(k_1, \ldots, k_r)$. Denote by $\mathcal{K}(t)$.

t	$\lvert\mathcal{S}(t)\rvert$	$\lvert\mathcal{K}(t)\rvert$	— —	t	$\lvert\mathcal{S}(t)\rvert$	$\lvert\mathcal{K}(t)\rvert$	— —	t	$\lvert\mathcal{S}(t)\rvert$	$\lvert\mathcal{K}(t)\rvert$	— —
3	128	58		12	497	287		21	189	137	
4	196	116		13	455	237		22	163	137	
5	277	113		14	409	245		23	143	120	
6	375	173		15	377	214		24	118	104	
7	492	159		16	340	220		25	95	92	
8	610	225		17	311	184		26	76	71	
9	748	233		18	273	190		27	61	59	
10	898	297		19	248	162		28	43	43	
11	546	272		20	220	172		29	27	27	

Finite search

- For each $t \in \{3, \ldots, 29\}$, we can enumerate parameters (n, s, m). Denote their set by $\mathcal{S}(t)$.
- For each $S \in \mathcal{S}(t)$, we can enumerate valencies $(k_1, \ldots, k_r)$. Denote by $\mathcal{K}(t)$.
- For each $S \in \mathcal{S}(t)$ and $K \in \mathcal{K}(t)$, we can enumerate valency multiplicities $(n_1, \ldots, n_r)$. Denote by $\mathcal{M}(t)$.

t	$\lvert\mathcal{S}(t)\rvert$	$\lvert\mathcal{K}(t)\rvert$	$\lvert\mathcal{M}(t)\rvert$	t	$\lvert\mathcal{S}(t)\rvert$	$\lvert\mathcal{K}(t)\rvert$	$\lvert\mathcal{M}(t)\rvert$	t	$\lvert\mathcal{S}(t)\rvert$	$\lvert\mathcal{K}(t)\rvert$	$\lvert\mathcal{M}(t)\rvert$
3	128	58	0	12	497	287	0	21	189	137	0
4	196	116	1	13	455	237	0	22	163	137	0
5	277	113	2	14	409	245	0	23	143	120	0
6	375	173	0	15	377	214	0	24	118	104	0
7	492	159	1	16	340	220	0	25	95	92	0
8	610	225	0	17	311	184	0	26	76	71	0
9	748	233	0	18	273	190	0	27	61	59	0
10	898	297	0	19	248	162	0	28	43	43	0
11	546	272	0	20	220	172	0	29	27	27	0

Survivors

t	(n, s, m)	$(k_1, \dots, k_r)$	$(n_1, \dots, n_r)$
4	$(31, 15, 9)$	$(5, 8, 13, 20)$	$(5, 10, 5, 11)$
5	$(36, 19, 9)$	$(7, 13, 23)$	$(6, 12, 18)$
5	$(45, 28, 12)$	$(6, 9, 21, 30)$	$(6, 3, 3, 33)$
7	$(45, 20, 8)$	$(11, 16, 23, 32)$	$(6, 27, 6, 6)$

- Use ad-hoc methods to show nonexistence of graphs corresponding to each of the parameters in the table.

Closing remarks

- D. de Caen: must graphs with three eigenvalues have at most three valencies?
- Regular: Strongly regular graphs.
- Bi-regular: Infinitely many examples.
- Tri-regular: Finitely many known examples.
- At least four valencies: No known examples.