Automorphism Groups of Algebraic Curves

Gábor Korchmáros

Università degli Studi della Basilicata Italy

Joint work with M. Giulietti

Combinatorics and Computer Algebra, CoCoA 2015,
July 19-25 2015 Forth Collins

Some background from Algebraic Geometry

Some background from Algebraic Geometry

Algebraic curve:= projective (absolutely irreducible, non-singular) algebraic variety of dimension 1 in a projective space $PG(r, \mathbb{K})$;

Some background from Algebraic Geometry

Algebraic curve:= projective (absolutely irreducible, non-singular) algebraic variety of dimension 1 in a projective space $PG(r, \mathbb{K})$;
$\mathbb{K}$:= field of characteristic $p \geq 0$,

Assumption (w.l.g.): $\mathbb{K}$ is algebraically closed field;

Some background from Algebraic Geometry

Algebraic curve:= projective (absolutely irreducible, non-singular) algebraic variety of dimension 1 in a projective space $PG(r, \mathbb{K})$;
$\mathbb{K}$:= field of characteristic $p \geq 0$,

Assumption (w.l.g.): $\mathbb{K}$ is algebraically closed field;

Every algebraic curve $\mathcal{X}$ has a plane model $\mathcal{C}$:

Some background from Algebraic Geometry

Algebraic curve:= projective (absolutely irreducible, non-singular) algebraic variety of dimension 1 in a projective space $PG(r, \mathbb{K})$;
$\mathbb{K}$:= field of characteristic $p \geq 0$,

Assumption (w.l.g.): $\mathbb{K}$ is algebraically closed field;

Every algebraic curve $\mathcal{X}$ has a plane model $\mathcal{C}$:
$\mathcal{C}$:= irreducible (in general singular) curve in $PG(2, \mathbb{K})$;

Some background from Algebraic Geometry

Algebraic curve:= projective (absolutely irreducible, non-singular) algebraic variety of dimension 1 in a projective space $PG(r, \mathbb{K})$;
$\mathbb{K}$:= field of characteristic $p \geq 0$,

Assumption (w.l.g.): $\mathbb{K}$ is algebraically closed field;

Every algebraic curve $\mathcal{X}$ has a plane model $\mathcal{C}$:
$\mathcal{C}$:= irreducible (in general singular) curve in $PG(2, \mathbb{K})$;
$\mathcal{C}$ is a *model* of $\mathcal{X}$ if $\exists$ a map

Some background from Algebraic Geometry

Algebraic curve:= projective (absolutely irreducible, non-singular) algebraic variety of dimension 1 in a projective space $PG(r, \mathbb{K})$; $\mathbb{K}$:= field of characteristic $p \geq 0$,

Assumption (w.l.g.): $\mathbb{K}$ is algebraically closed field;

Every algebraic curve $\mathcal{X}$ has a plane model $\mathcal{C}$:
$\mathcal{C}$:= irreducible (in general singular) curve in $PG(2, \mathbb{K})$;
$\mathcal{C}$ is a *model* of $\mathcal{X}$ if $\exists$ a map

$$\Phi = \begin{cases} PG(r, \mathbb{K}) \to PG(2, \mathbb{K}) \\ (Y_0, Y_1, \ldots, Y_r) \to (X_0, X_1, X_2) \\ \mathcal{X} \to \mathcal{C} \end{cases}$$

Some background from Algebraic Geometry

Algebraic curve:= projective (absolutely irreducible, non-singular) algebraic variety of dimension 1 in a projective space $PG(r, \mathbb{K})$;
$\mathbb{K}$:= field of characteristic $p \geq 0$,

Assumption (w.l.g.): $\mathbb{K}$ is algebraically closed field;

Every algebraic curve $\mathcal{X}$ has a plane model $\mathcal{C}$:
$\mathcal{C}$:= irreducible (in general singular) curve in $PG(2, \mathbb{K})$;
$\mathcal{C}$ is a *model* of $\mathcal{X}$ if $\exists$ a map

$$\Phi = \begin{cases} PG(r, \mathbb{K}) \to PG(2, \mathbb{K}) \\ (Y_0, Y_1, \ldots, Y_r) \to (X_0, X_1, X_2) \\ \mathcal{X} \to \mathcal{C} \end{cases}$$

$X_i = U_i(Y_0, Y_1, \ldots, Y_r)$ homogeneous polynomials of degree $d \geq 1$,

Some background from Algebraic Geometry

Algebraic curve:= projective (absolutely irreducible, non-singular) algebraic variety of dimension 1 in a projective space $PG(r, \mathbb{K})$;
$\mathbb{K}$:= field of characteristic $p \geq 0$,

Assumption (w.l.g.): $\mathbb{K}$ is algebraically closed field;

Every algebraic curve $\mathcal{X}$ has a plane model $\mathcal{C}$:
$\mathcal{C}$:= irreducible (in general singular) curve in $PG(2, \mathbb{K})$;
$\mathcal{C}$ is a *model* of $\mathcal{X}$ if $\exists$ a map

$$\Phi = \begin{cases} PG(r, \mathbb{K}) \to PG(2, \mathbb{K}) \\ (Y_0, Y_1, \ldots, Y_r) \to (X_0, X_1, X_2) \\ \mathcal{X} \to \mathcal{C} \end{cases}$$

$X_i = U_i(Y_0, Y_1, \ldots, Y_r)$ homogeneous polynomials of degree $d \geq 1$,
Φ is injective and surjective (apart from a finite number of points);

Algebraic curve:= projective (absolutely irreducible, non-singular) algebraic variety of dimension 1 in a projective space $PG(r, \mathbb{K})$;
$\mathbb{K}$:= field of characteristic $p \geq 0$,

Assumption (w.l.g.): $\mathbb{K}$ is algebraically closed field;

Every algebraic curve $\mathcal{X}$ has a plane model $\mathcal{C}$:
$\mathcal{C}$:= irreducible (in general singular) curve in $PG(2, \mathbb{K})$;
$\mathcal{C}$ is a *model* of $\mathcal{X}$ if $\exists$ a map

$$\Phi = \begin{cases} PG(r, \mathbb{K}) \to PG(2, \mathbb{K}) \\ (Y_0, Y_1, \ldots, Y_r) \to (X_0, X_1, X_2) \\ \mathcal{X} \to \mathcal{C} \end{cases}$$

$X_i = U_i(Y_0, Y_1, \ldots, Y_r)$ homogeneous polynomials of degree $d \geq 1$,
Φ is injective and surjective (apart from a finite number of points);
Two algebraic curves with the same plane model are *birationally isomorphic*;

Some background from Algebraic Geometry

Algebraic curve:= projective (absolutely irreducible, non-singular) algebraic variety of dimension 1 in a projective space $PG(r, \mathbb{K})$;
$\mathbb{K}$:= field of characteristic $p \geq 0$,

Assumption (w.l.g.): $\mathbb{K}$ is algebraically closed field;

Every algebraic curve $\mathcal{X}$ has a plane model $\mathcal{C}$:
$\mathcal{C}$:= irreducible (in general singular) curve in $PG(2, \mathbb{K})$;
$\mathcal{C}$ is a *model* of $\mathcal{X}$ if $\exists$ a map

$$\Phi = \begin{cases} PG(r, \mathbb{K}) \to PG(2, \mathbb{K}) \\ (Y_0, Y_1, \ldots, Y_r) \to (X_0, X_1, X_2) \\ \mathcal{X} \to \mathcal{C} \end{cases}$$

$X_i = U_i(Y_0, Y_1, \ldots, Y_r)$ homogeneous polynomials of degree $d \geq 1$,
Φ is injective and surjective (apart from a finite number of points);
Two algebraic curves with the same plane model are *birationally isomorphic*;

Concept of automorphism

Concept of automorphism

Let $\mathcal{C}$ be a plane model of $\mathcal{X}$;

Concept of automorphism

Let $\mathcal{C}$ be a plane model of $\mathcal{X}$;
$\mathcal{C}$:= (projective, irreducible) plane curve with homogeneous equation $F(X_0, X_1, X_2) = 0$;

Concept of automorphism

Let $\mathcal{C}$ be a plane model of $\mathcal{X}$;
$\mathcal{C}:=$ (projective, irreducible) plane curve with homogeneous equation $F(X_0, X_1, X_2) = 0$;
homogeneous equation $\longrightarrow$ affine equation;
$F(X_0, X_1, X_2) \rightarrow f(X, Y) = F(1, X, Y)$;

Concept of automorphism

Let $\mathcal{C}$ be a plane model of $\mathcal{X}$;
$\mathcal{C}$:= (projective, irreducible) plane curve with homogeneous equation $F(X_0, X_1, X_2) = 0$;
homogeneous equation $\longrightarrow$ affine equation;
$F(X_0, X_1, X_2) \rightarrow f(X, Y) = F(1, X, Y)$;

Concept of automorphism

Let $\mathcal{C}$ be a plane model of $\mathcal{X}$;
$\mathcal{C}$:= (projective, irreducible) plane curve with homogeneous equation $F(X_0, X_1, X_2) = 0$;
homogeneous equation $\rightarrow$ affine equation;
$F(X_0, X_1, X_2) \rightarrow f(X, Y) = F(1, X, Y)$;
α:=automorphism of $\mathcal{C}$ (and of $\mathcal{X}$);

$$\alpha = \begin{cases} PG(2, \mathbb{K}) \rightarrow PG(2, \mathbb{K}) \\ (X_0, X_1, X_2) \rightarrow (X_0', X_1', X_2') \\ \mathcal{C} \rightarrow \mathcal{C} \end{cases}$$

Concept of automorphism

Let $\mathcal{C}$ be a plane model of $\mathcal{X}$;
$\mathcal{C}$:= (projective, irreducible) plane curve with homogeneous equation $F(X_0, X_1, X_2) = 0$;
homogeneous equation $\rightarrow$ affine equation;
$F(X_0, X_1, X_2) \rightarrow f(X, Y) = F(1, X, Y)$;
α:=automorphism of $\mathcal{C}$ (and of $\mathcal{X}$);

$$\alpha = \begin{cases} PG(2, \mathbb{K}) \rightarrow PG(2, \mathbb{K}) \\ (X_0, X_1, X_2) \rightarrow (X_0', X_1', X_2') \\ \mathcal{C} \rightarrow \mathcal{C} \end{cases}$$

$X_i' = U_i(X_0, X_1, X_2)$ homogeneous polynomials of degree $d \geq 1$,

Concept of automorphism

Let $\mathcal{C}$ be a plane model of $\mathcal{X}$;
$\mathcal{C}$:= (projective, irreducible) plane curve with homogeneous equation $F(X_0, X_1, X_2) = 0$;
homogeneous equation $\rightarrow$ affine equation;
$F(X_0, X_1, X_2) \rightarrow f(X, Y) = F(1, X, Y)$;
α:=automorphism of $\mathcal{C}$ (and of $\mathcal{X}$);

$$\alpha = \begin{cases} PG(2, \mathbb{K}) \rightarrow PG(2, \mathbb{K}) \\ (X_0, X_1, X_2) \rightarrow (X_0', X_1', X_2') \\ \mathcal{C} \rightarrow \mathcal{C} \end{cases}$$

$X_i' = U_i(X_0, X_1, X_2)$ homogeneous polynomials of degree $d \geq 1$,
α is surjective and injective on $\mathcal{C}$ (apart from a finite number of points);

Concept of automorphism

Let $\mathcal{C}$ be a plane model of $\mathcal{X}$;
$\mathcal{C}$:= (projective, irreducible) plane curve with homogeneous equation $F(X_0, X_1, X_2) = 0$;
homogeneous equation $\rightarrow$ affine equation;
$F(X_0, X_1, X_2) \rightarrow f(X, Y) = F(1, X, Y)$;
α:=automorphism of $\mathcal{C}$ (and of $\mathcal{X}$);

$$\alpha = \begin{cases} PG(2, \mathbb{K}) \rightarrow PG(2, \mathbb{K}) \\ (X_0, X_1, X_2) \rightarrow (X_0', X_1', X_2') \\ \mathcal{C} \rightarrow \mathcal{C} \end{cases}$$

$X_i' = U_i(X_0, X_1, X_2)$ homogeneous polynomials of degree $d \geq 1$,
α is surjective and injective on $\mathcal{C}$ (apart from a finite number of points);
$\mathrm{Aut}(\mathcal{X})$:=group of all automorphisms of $\mathcal{X}$;

Concept of automorphism

Let $\mathcal{C}$ be a plane model of $\mathcal{X}$;
$\mathcal{C}$:= (projective, irreducible) plane curve with homogeneous equation $F(X_0, X_1, X_2) = 0$;
homogeneous equation $\rightarrow$ affine equation;
$F(X_0, X_1, X_2) \rightarrow f(X, Y) = F(1, X, Y)$;
α:=automorphism of $\mathcal{C}$ (and of $\mathcal{X}$);

$$\alpha = \begin{cases} PG(2, \mathbb{K}) \rightarrow PG(2, \mathbb{K}) \\ (X_0, X_1, X_2) \rightarrow (X_0', X_1', X_2') \\ \mathcal{C} \rightarrow \mathcal{C} \end{cases}$$

$X_i' = U_i(X_0, X_1, X_2)$ homogeneous polynomials of degree $d \geq 1$,
α is surjective and injective on $\mathcal{C}$ (apart from a finite number of points);
$\mathrm{Aut}(\mathcal{X})$:=group of all automorphisms of $\mathcal{X}$;
$G < \mathrm{Aut}(\mathcal{X})$:= *tame* when $p \nmid |G|$, otherwise *non-tame*.

Genus of a curve

Genus of a curve

Choose $\mathcal{C} = \mathbf{v}(f(X, Y))$ with only ordinary singularities;

Genus of a curve

Choose $\mathcal{C} = \mathbf{v}(f(X, Y))$ with only ordinary singularities;
$P \in \mathcal{C}$ ordinary r-fold point:=exactly r distinct tangents at P;

Choose $\mathcal{C} = \mathbf{v}(f(X, Y))$ with only ordinary singularities;
$P \in \mathcal{C}$ ordinary r-fold point:=exactly r distinct tangents at P;

$P_1, \ldots P_k$ singular points of $\mathcal{C}$, (P_i is r_i-fold);

Genus of a curve

Choose $\mathcal{C} = \mathbf{v}(f(X, Y))$ with only ordinary singularities;
$P \in \mathcal{C}$ ordinary r-fold point:=exactly r distinct tangents at P;

$P_1, \ldots P_k$ singular points of $\mathcal{C}$, (P_i is r_i-fold);
The *genus* g of $\mathcal{X}$ (and $\mathcal{C}$) is

$$g := \frac{1}{2}(n-1)(n-2) - \frac{1}{2}\sum_{i=1}^{k} r_i(r_i - 1);$$

Genus of a curve

Choose $\mathcal{C} = \mathbf{v}(f(X,Y))$ with only ordinary singularities;
$P \in \mathcal{C}$ ordinary r-fold point:=exactly r distinct tangents at P;

$P_1, \ldots P_k$ singular points of $\mathcal{C}$, (P_i is r_i-fold);
The *genus* g of $\mathcal{X}$ (and $\mathcal{C}$) is

$$g := \frac{1}{2}(n-1)(n-2) - \frac{1}{2}\sum_{i=1}^{k} r_i(r_i - 1);$$

Lines (and irreducible conics) have genus 0;

Choose $\mathcal{C} = \mathbf{v}(f(X, Y))$ with only ordinary singularities;
$P \in \mathcal{C}$ ordinary r-fold point:=exactly r distinct tangents at P;

$P_1, \ldots P_k$ singular points of $\mathcal{C}$, (P_i is r_i-fold);
The *genus* g of $\mathcal{X}$ (and $\mathcal{C}$) is

$$g := \frac{1}{2}(n-1)(n-2) - \frac{1}{2}\sum_{i=1}^{k} r_i(r_i - 1);$$

Lines (and irreducible conics) have genus 0;
Elliptic curves (birationally isomorphic to nonsingular plane cubics) have genus 1;

Genus of a curve

Choose $\mathcal{C} = \mathbf{v}(f(X, Y))$ with only ordinary singularities;
$P \in \mathcal{C}$ ordinary r-fold point:=exactly r distinct tangents at P;

$P_1, \ldots P_k$ singular points of $\mathcal{C}$, (P_i is r_i-fold);
The *genus* g of $\mathcal{X}$ (and $\mathcal{C}$) is

$$g := \frac{1}{2}(n-1)(n-2) - \frac{1}{2}\sum_{i=1}^{k} r_i(r_i - 1);$$

Lines (and irreducible conics) have genus 0;
Elliptic curves (birationally isomorphic to nonsingular plane cubics) have genus 1;
$\mathrm{Aut}(\mathcal{X})$ is infinite if and only if $g(\mathcal{X}) \leq 1$;

Genus of a curve

Choose $\mathcal{C} = \mathbf{v}(f(X,Y))$ with only ordinary singularities;
$P \in \mathcal{C}$ ordinary r-fold point:=exactly r distinct tangents at P;

$P_1, \ldots P_k$ singular points of $\mathcal{C}$, (P_i is r_i-fold);
The *genus* g of $\mathcal{X}$ (and $\mathcal{C}$) is

$$g := \frac{1}{2}(n-1)(n-2) - \frac{1}{2}\sum_{i=1}^{k} r_i(r_i-1);$$

Lines (and irreducible conics) have genus 0;
Elliptic curves (birationally isomorphic to nonsingular plane cubics) have genus 1;
$\mathrm{Aut}(\mathcal{X})$ is infinite if and only if $g(\mathcal{X}) \leq 1$;
If $\mathcal{X}$ is a (nonsingular) plane curve with $g(\mathcal{X}) \geq 2$ then $\mathrm{Aut}(\mathcal{X}) < PGL(3,\mathbb{K})$.

p-rank of a curve

p-rank of a curve

Abelian group defined on the points of an elliptic curve;

p-rank of a curve

Abelian group defined on the points of an elliptic curve;

(Not straightforward) generalization from $g = 1$ to $g \geq 2$:

p-rank of a curve

Abelian group defined on the points of an elliptic curve;

(Not straightforward) generalization from $g = 1$ to $g \geq 2$: Jacobian variety $\mathcal{J}(\mathcal{X})$,

p-rank of a curve

Abelian group defined on the points of an elliptic curve;

(Not straightforward) generalization from $g = 1$ to $g \geq 2$:
Jacobian variety $\mathcal{J}(\mathcal{X})$,
equivalently, the zero Picard group Pic_0.

p-rank of a curve

Abelian group defined on the points of an elliptic curve;

(Not straightforward) generalization from $g = 1$ to $g \geq 2$:
Jacobian variety $\mathcal{J}(\mathcal{X})$,
equivalently, the zero Picard group Pic_0.
For any prime ℓ,

$$G_\ell := \{g | g \in \mathrm{Pic}_0, g^\ell = 1\}.$$

p-rank of a curve

Abelian group defined on the points of an elliptic curve;

(Not straightforward) generalization from $g = 1$ to $g \geq 2$:
Jacobian variety $\mathcal{J}(\mathcal{X})$,
equivalently, the zero Picard group Pic_0.
For any prime ℓ,

$$G_\ell := \{g | g \in \mathrm{Pic}_0, g^\ell = 1\}.$$

If $\ell \neq p$, then $|G_\ell| = \ell^{2g}$.

p-rank of a curve

Abelian group defined on the points of an elliptic curve;

(Not straightforward) generalization from $g = 1$ to $g \geq 2$:
Jacobian variety $\mathcal{J}(\mathcal{X})$,
equivalently, the zero Picard group Pic_0.
For any prime ℓ,

$$G_\ell := \{g | g \in \mathrm{Pic}_0, g^\ell = 1\}.$$

If $\ell \neq p$, then $|G_\ell| = \ell^{2g}$.
If $\ell = p$, then $|G_\ell| \leq \ell^g$.

p-rank of a curve

Abelian group defined on the points of an elliptic curve;

(Not straightforward) generalization from $g = 1$ to $g \geq 2$:
Jacobian variety $\mathcal{J}(\mathcal{X})$,
equivalently, the zero Picard group Pic_0.
For any prime ℓ,

$$G_\ell := \{g | g \in \mathrm{Pic}_0, g^\ell = 1\}.$$

If $\ell \neq p$, then $|G_\ell| = \ell^{2g}$.
If $\ell = p$, then $|G_\ell| \leq \ell^g$.

$|G_p| = p^\gamma$ and γ is the *p-rank* of $\mathcal{X}$ (also called the Hasse-Witt invariant);

Abelian group defined on the points of an elliptic curve;

(Not straightforward) generalization from $g = 1$ to $g \geq 2$:
Jacobian variety $\mathcal{J}(\mathcal{X})$,
equivalently, the zero Picard group Pic_0.
For any prime ℓ,

$$G_\ell := \{g | g \in \mathrm{Pic}_0, g^\ell = 1\}.$$

If $\ell \neq p$, then $|G_\ell| = \ell^{2g}$.
If $\ell = p$, then $|G_\ell| \leq \ell^{g}$.

$|G_p| = p^\gamma$ and γ is the *p-rank* of $\mathcal{X}$ (also called the Hasse-Witt invariant);

$$0 \leq \gamma(\mathcal{X}) \leq g(\mathcal{X}).$$

Tools from Algebraic Geometry

Tools from Algebraic Geometry

G:= finite automorphism group of an algebraic curve $\mathcal{X}$;

Tools from Algebraic Geometry

G:= finite automorphism group of an algebraic curve $\mathcal{X}$;
G acts faithfully on $\mathcal{X}$.

Tools from Algebraic Geometry

G:= finite automorphism group of an algebraic curve $\mathcal{X}$;
G acts faithfully on $\mathcal{X}$.

- G has a finite number of short orbits: $\theta_1, \ldots, \theta_k$;

Tools from Algebraic Geometry

G:= finite automorphism group of an algebraic curve $\mathcal{X}$;
G acts faithfully on $\mathcal{X}$.

- G has a finite number of short orbits: $\theta_1, \ldots, \theta_k$;
- The stabilizer G_P of $P \in \mathcal{X}$ is $S \rtimes H$ with $|S| = p^h$, $p \nmid |H|$ and H cyclic;

Tools from Algebraic Geometry

G:= finite automorphism group of an algebraic curve $\mathcal{X}$;
G acts faithfully on $\mathcal{X}$.

- G has a finite number of short orbits: $\theta_1, \ldots, \theta_k$;
- The stabilizer G_P of $P \in \mathcal{X}$ is $S \rtimes H$ with $|S| = p^h$, $p \nmid |H|$ and H cyclic;

$\exists$ a curve $\mathcal{Y}$ whose points are the G-orbits of $\mathcal{X}$;

Tools from Algebraic Geometry

G:= finite automorphism group of an algebraic curve $\mathcal{X}$;
G acts faithfully on $\mathcal{X}$.

- G has a finite number of short orbits: $\theta_1, \ldots, \theta_k$;
- The stabilizer G_P of $P \in \mathcal{X}$ is $S \rtimes H$ with $|S| = p^h$, $p \nmid |H|$ and H cyclic;

$\exists$ a curve $\mathcal{Y}$ whose points are the G-orbits of $\mathcal{X}$;
$\mathcal{Y}$:= *quotient curve of* $\mathcal{X}$ by G;

Tools from Algebraic Geometry

G:= finite automorphism group of an algebraic curve $\mathcal{X}$;
G acts faithfully on $\mathcal{X}$.

- G has a finite number of short orbits: $\theta_1, \ldots, \theta_k$;
- The stabilizer G_P of $P \in \mathcal{X}$ is $S \rtimes H$ with $|S| = p^h$, $p \nmid |H|$ and H cyclic;

$\exists$ a curve $\mathcal{Y}$ whose points are the G-orbits of $\mathcal{X}$;
$\mathcal{Y}$:= *quotient curve of* $\mathcal{X}$ by G; $N_{\mathrm{Aut}\mathcal{X}}(G)/G \le \mathrm{Aut}(\mathcal{Y})$;

Tools from Algebraic Geometry

G:= finite automorphism group of an algebraic curve $\mathcal{X}$;
G acts faithfully on $\mathcal{X}$.

- G has a finite number of short orbits: $\theta_1, \ldots, \theta_k$;
- The stabilizer G_P of $P \in \mathcal{X}$ is $S \rtimes H$ with $|S| = p^h$, $p \nmid |H|$ and H cyclic;

$\exists$ a curve $\mathcal{Y}$ whose points are the G-orbits of $\mathcal{X}$;
$\mathcal{Y}$:= *quotient curve of* $\mathcal{X}$ by G; $N_{\mathrm{Aut}\mathcal{X}}(G)/G \leq \mathrm{Aut}(\mathcal{Y})$;
If $k = 0$ then $\mathcal{X}|\mathcal{Y}$ is *unramified Galois extension*;

Tools from Algebraic Geometry

G:= finite automorphism group of an algebraic curve $\mathcal{X}$;
G acts faithfully on $\mathcal{X}$.

- G has a finite number of short orbits: $\theta_1, \ldots, \theta_k$;
- The stabilizer G_P of $P \in \mathcal{X}$ is $S \rtimes H$ with $|S| = p^h$, $p \nmid |H|$ and H cyclic;

$\exists$ a curve $\mathcal{Y}$ whose points are the G-orbits of $\mathcal{X}$;
$\mathcal{Y}$:= *quotient curve of* $\mathcal{X}$ by G; $N_{\mathrm{Aut}\mathcal{X}}(G)/G \leq \mathrm{Aut}(\mathcal{Y})$;
If $k = 0$ then $\mathcal{X}|\mathcal{Y}$ is *unramified Galois extension*;
Hurwitz-Riemann Formula: If $p \nmid |G|$ then

$$g(\mathcal{X}) - 1 = |G|(g(\mathcal{Y}) - 1) + \frac{1}{2}\sum_{i=1}^{k}(|G| - |\theta_i|)$$

Tools from Algebraic Geometry

G:= finite automorphism group of an algebraic curve $\mathcal{X}$;
G acts faithfully on $\mathcal{X}$.

- G has a finite number of short orbits: $\theta_1, \ldots, \theta_k$;
- The stabilizer G_P of $P \in \mathcal{X}$ is $S \rtimes H$ with $|S| = p^h$, $p \nmid |H|$ and H cyclic;

$\exists$ a curve $\mathcal{Y}$ whose points are the G-orbits of $\mathcal{X}$;
$\mathcal{Y}$:= *quotient curve of* $\mathcal{X}$ by G; $N_{\mathrm{Aut}\mathcal{X}}(G)/G \le \mathrm{Aut}(\mathcal{Y})$;
If $k = 0$ then $\mathcal{X}|\mathcal{Y}$ is *unramified Galois extension*;
Hurwitz-Riemann Formula: If $p \nmid |G|$ then

$$g(\mathcal{X}) - 1 = |G|(g(\mathcal{Y}) - 1) + \tfrac{1}{2}\sum_{i=1}^{k}(|G| - |\theta_i|)$$

Deuring-Shafarveich Formula: If $|G| = p^h$ then

$$\gamma(\mathcal{X}) - 1 = |G|(\gamma(\mathcal{Y}) - 1) + \sum_{i=1}^{k}(|G| - |\theta_i|)$$

Outline

Outline

- Upper bounds on $|\mathrm{Aut}(\mathcal{X})|$ depending on g, a survey.

Outline

- Upper bounds on $|\mathrm{Aut}(\mathcal{X})|$ depending on g, a survey.
- What are the possibilities for $\mathrm{Aut}(\mathcal{X})$ when $\mathcal{X}$ has zero p-rank? A classification in even characteristic

Outline

- Upper bounds on $|\mathrm{Aut}(\mathcal{X})|$ depending on g, a survey.
- What are the possibilities for $\mathrm{Aut}(\mathcal{X})$ when $\mathcal{X}$ has zero p-rank? A classification in even characteristic
- p-subgroups of $\mathrm{Aut}(\mathcal{X})$ of curves with positive p-rank.

Outline

- Upper bounds on $|\mathrm{Aut}(\mathcal{X})|$ depending on g, a survey.
- What are the possibilities for $\mathrm{Aut}(\mathcal{X})$ when $\mathcal{X}$ has zero p-rank? A classification in even characteristic
- p-subgroups of $\mathrm{Aut}(\mathcal{X})$ of curves with positive p-rank.

Remark

In zero characteristic, the general study of $\mathrm{Aut}(\mathcal{X})$ relies on the fundamental group of the curve,

Outline

- Upper bounds on $|\mathrm{Aut}(\mathcal{X})|$ depending on g, a survey.
- What are the possibilities for $\mathrm{Aut}(\mathcal{X})$ when $\mathcal{X}$ has zero p-rank? A classification in even characteristic
- p-subgroups of $\mathrm{Aut}(\mathcal{X})$ of curves with positive p-rank.

Remark

In zero characteristic, the general study of $\mathrm{Aut}(\mathcal{X})$ relies on the fundamental group of the curve, In positive characteristic, this approach only works in some special cases, see R. Pries and K. Stevenson, A survey of Galois theory of curves in characteristic p, Amer. Math. Soc., (2011)

Outline

- Upper bounds on $|\mathrm{Aut}(\mathcal{X})|$ depending on g, a survey.
- What are the possibilities for $\mathrm{Aut}(\mathcal{X})$ when $\mathcal{X}$ has zero p-rank? A classification in even characteristic
- p-subgroups of $\mathrm{Aut}(\mathcal{X})$ of curves with positive p-rank.

Remark

In zero characteristic, the general study of $\mathrm{Aut}(\mathcal{X})$ relies on the fundamental group of the curve, In positive characteristic, this approach only works in some special cases, see R. Pries and K. Stevenson, A survey of Galois theory of curves in characteristic p, Amer. Math. Soc., (2011)
For further developments the potential of Finite Group Theory is needed.

The classical Hurwitz bound

From now on $g \geq 2$ *is assumed*

From now on $g \geq 2$ is assumed

- $\mathrm{Aut}(\mathcal{X})$ is a finite group.

The classical Hurwitz bound

From now on $g \geq 2$ is assumed

- $\mathrm{Aut}(\mathcal{X})$ is a finite group.
- If G is tame then $|G| \leq 84(g-1)$. (Hurwitz bound)

The classical Hurwitz bound

From now on $g \geq 2$ is assumed

- $\mathrm{Aut}(\mathcal{X})$ is a finite group.
- If G is tame then $|G| \leq 84(g-1)$. (Hurwitz bound)
- $|\mathrm{Aut}(\mathcal{X})| < 16g^4$; up to one exception, the Hermitian curve, [Stichtenoth (1973)].

The classical Hurwitz bound

From now on $g \geq 2$ is assumed

- $\mathrm{Aut}(\mathcal{X})$ is a finite group.
- If G is tame then $|G| \leq 84(g-1)$. (Hurwitz bound)
- $|\mathrm{Aut}(\mathcal{X})| < 16g^4$; up to one exception, the Hermitian curve, [Stichtenoth (1973)].
- $|\mathrm{Aut}(\mathcal{X})| < 8g^3$; up to four exceptions. [Henn (1976)]

The classical Hurwitz bound

From now on $g \geq 2$ is assumed

- $\mathrm{Aut}(\mathcal{X})$ is a finite group.
- If G is tame then $|G| \leq 84(g-1)$. (Hurwitz bound)
- $|\mathrm{Aut}(\mathcal{X})| < 16g^4$; up to one exception, the Hermitian curve, [Stichtenoth (1973)].
- $|\mathrm{Aut}(\mathcal{X})| < 8g^3$; up to four exceptions. [Henn (1976)]

Remark

Large automorphism group G of $\mathcal{X}$:=$|G| > g(\mathcal{X})$;

The classical Hurwitz bound

From now on $g \geq 2$ is assumed

- $\mathrm{Aut}(\mathcal{X})$ is a finite group.
- If G is tame then $|G| \leq 84(g-1)$. (Hurwitz bound)
- $|\mathrm{Aut}(\mathcal{X})| < 16g^4$; up to one exception, the Hermitian curve, [Stichtenoth (1973)].
- $|\mathrm{Aut}(\mathcal{X})| < 8g^3$; up to four exceptions. [Henn (1976)]

Remark

Large automorphism group G of $\mathcal{X}$:=$|G| > g(\mathcal{X})$;
Very large automorphism group G of $\mathcal{X}$:=$|G| > cg(\mathcal{X})^2$ with a constant c.

Four infinite families of curves $\mathcal{X}$ with $|\mathrm{Aut}(\mathcal{X})| \geq 8g^3$

Four infinite families of curves $\mathcal{X}$ with $|\mathrm{Aut}(\mathcal{X})| \geq 8g^3$

(I) $\mathbf{v}(Y^2 + Y + X^{2^k+1})$, $p = 2$, a hyperelliptic curve of genus $g = 2^{k-1}$ with $\mathrm{Aut}(\mathcal{X})$ fixing a point of $\mathcal{X}$.
$|\mathrm{Aut}(\mathcal{X})| = 2^{2k+1}(2^k + 1)$.

Four infinite families of curves $\mathcal{X}$ with $|\mathrm{Aut}(\mathcal{X})| \geq 8g^3$

(I) $\mathbf{v}(Y^2 + Y + X^{2^k+1})$, $p = 2$, a hyperelliptic curve of genus $g = 2^{k-1}$ with $\mathrm{Aut}(\mathcal{X})$ fixing a point of $\mathcal{X}$.
$|\mathrm{Aut}(\mathcal{X})| = 2^{2k+1}(2^k + 1)$.

(II) The Roquette curve:
$\mathbf{v}(Y^2 - (X^q - X))$ with $p > 2$, a hyperelliptic curve of genus $g = \frac{1}{2}(q-1)$; $\mathrm{Aut}(\mathcal{X})/M \cong \mathrm{PSL}(2, q)$ or $\mathrm{Aut}(\mathcal{X})/M \cong \mathrm{PGL}(2, q)$, where $q = p^r$ and $|M| = 2$;

Four infinite families of curves $\mathcal{X}$ with $|\mathrm{Aut}(\mathcal{X})| \geq 8g^3$

(I) $\mathbf{v}(Y^2 + Y + X^{2^k+1})$, $p = 2$, a hyperelliptic curve of genus $g = 2^{k-1}$ with $\mathrm{Aut}(\mathcal{X})$ fixing a point of $\mathcal{X}$. $|\mathrm{Aut}(\mathcal{X})| = 2^{2k+1}(2^k + 1)$.

(II) The Roquette curve: $\mathbf{v}(Y^2 - (X^q - X))$ with $p > 2$, a hyperelliptic curve of genus $g = \frac{1}{2}(q-1)$; $\mathrm{Aut}(\mathcal{X})/M \cong \mathrm{PSL}(2,q)$ or $\mathrm{Aut}(\mathcal{X})/M \cong \mathrm{PGL}(2,q)$, where $q = p^r$ and $|M| = 2$;

(III) The Hermitian curve: $\mathbf{v}(Y^n + Y - X^{n+1})$ with $n = p^r$, genus $\frac{1}{2}\,n(n-1)$, $\mathrm{Aut}(\mathcal{X}) \cong \mathrm{PGU}(3,n)$. $|\mathrm{Aut}(\mathcal{X})| = (n^3+1)n^3(n^2-1)$.

Four infinite families of curves $\mathcal{X}$ with $|\mathrm{Aut}(\mathcal{X})| \geq 8g^3$

(I) $\mathbf{v}(Y^2+Y+X^{2^k+1})$, $p=2$, a hyperelliptic curve of genus $g=2^{k-1}$ with $\mathrm{Aut}(\mathcal{X})$ fixing a point of $\mathcal{X}$.
$|\mathrm{Aut}(\mathcal{X})| = 2^{2k+1}(2^k+1)$.

(II) The Roquette curve:
$\mathbf{v}(Y^2-(X^q-X))$ with $p>2$, a hyperelliptic curve of genus $g=\frac{1}{2}(q-1)$; $\mathrm{Aut}(\mathcal{X})/M \cong \mathrm{PSL}(2,q)$ or $\mathrm{Aut}(\mathcal{X})/M \cong \mathrm{PGL}(2,q)$, where $q=p^r$ and $|M|=2$;

(III) The Hermitian curve:
$\mathbf{v}(Y^n+Y-X^{n+1})$ with $n=p^r$, genus $\frac{1}{2}\,n(n-1)$,
$\mathrm{Aut}(\mathcal{X}) \cong \mathrm{PGU}(3,n)$.
$|\mathrm{Aut}(\mathcal{X})| = (n^3+1)n^3(n^2-1)$.

(IV) The DLS curve (Deligne-Lusztig curve of Suzuki type):
$\mathbf{v}(X^{n_0}(X^n+X)+Y^n+Y)$, with $p=2$, $n_0=2^r\geq 2$, $n=2n_0^2$, $g=n_0(n-1)$, $\mathrm{Aut}(\mathcal{X}) \cong \mathrm{Sz}(n)$ where $\mathrm{Sz}(n)$ is the Suzuki group, $|\mathrm{Aut}(\mathcal{X})| = (n^2+1)n^2(n-1)$

Two more infinite families of curves $\mathcal{X}$ with large $\mathrm{Aut}(\mathcal{X})$

(V) The DLR curve (the Deligne-Lusztig curve arising from the Ree group):
$\mathbf{v}(Y^{n^2} - [1 + (X^n - X)^{n-1}]Y^n + (X^n - X)^{n-1}Y - X^n(X^n - X)^{n+3n_0})$, with $p = 3$, $n_0 = 3^r$, $n = 3n_0^2$;
$g = \frac{3}{2}n_0(n-1)(n+n_0+1)$; $\mathrm{Aut}(\mathcal{X}) \cong \mathrm{Ree}(n)$ where $\mathrm{Ree}(n)$ is the Ree group, $|\mathrm{Aut}(\mathcal{X})| = (n^3+1)n^3(n-1)$.

Two more infinite families of curves $\mathcal{X}$ with large $\mathrm{Aut}(\mathcal{X})$

(V) The DLR curve (the Deligne-Lusztig curve arising from the Ree group):
$\mathbf{v}(Y^{n^2} - [1 + (X^n - X)^{n-1}]Y^n + (X^n - X)^{n-1}Y - X^n(X^n - X)^{n+3n_0})$, with $p = 3$, $n_0 = 3^r$, $n = 3n_0^2$;
$g = \frac{3}{2}n_0(n-1)(n+n_0+1)$; $\mathrm{Aut}(\mathcal{X}) \cong \mathrm{Ree}(n)$ where $\mathrm{Ree}(n)$ is the Ree group, $|\mathrm{Aut}(\mathcal{X})| = (n^3+1)n^3(n-1)$.

(VI) The $\mathrm{G.K}$ curve:
$\mathbf{v}(Y^{n^3+1} + (X^n + X)(\sum_{i=0}^{n}(-1)^{i+1}X^{i(n-1)})^{n+1})$, a curve of genus $g = \frac{1}{2}(n^3+1)(n^2-2)+1$ with $\mathrm{Aut}(\mathcal{X})$ containing a subgroup isomorphic to $\mathrm{SU}(3,n)$, $n = p^r$.
$|\mathrm{Aut}(\mathcal{X})| = (n^3+1)n^3(n-1)$.

Remark

All the above examples have zero p-rank.

Problems on curves with large automorphism groups, $\gamma = 0$

Remark

All the above examples have zero p-rank.

- Problem 1: Find a function $f(g)$ such that if $|\mathrm{Aut}(\mathcal{X})| > f(g)$ then $\gamma = 0$.

Problems on curves with large automorphism groups, $\gamma = 0$

Remark

All the above examples have zero p-rank.

- Problem 1: Find a function $f(g)$ such that if $|\mathrm{Aut}(\mathcal{X})| > f(g)$ then $\gamma = 0$.
- Problem 2: Determine the structure of large automorphism groups of curves with $\gamma = 0$.

Remark

All the above examples have zero p-rank.

- Problem 1: Find a function $f(g)$ such that if $|\mathrm{Aut}(\mathcal{X})| > f(g)$ then $\gamma = 0$.
- Problem 2: Determine the structure of large automorphism groups of curves with $\gamma = 0$.
- Problem 3: $\exists$ simple groups, other than those in the examples (II),...(VI), occurring as a section of a large automorphism group of an algebraic curve?

Problems on curves with large automorphism groups, $\gamma = 0$

Remark

All the above examples have zero p-rank.

- Problem 1: Find a function $f(g)$ such that if $|\mathrm{Aut}(\mathcal{X})| > f(g)$ then $\gamma = 0$.
- Problem 2: Determine the structure of large automorphism groups of curves with $\gamma = 0$.
- Problem 3: $\exists$ simple groups, other than those in the examples (II),...(VI), occurring as a section of a large automorphism group of an algebraic curve?

Theorem (Giulietti, K. 2015)

Let $p > 2$. If G is solvable and $|G| > 144g(\mathcal{X})^2$ then $\gamma(\mathcal{X}) = 0$ and G fixes a point.

Problems on zero p-rank curves with very large p-group of automorphisms

- Curves with a large p-group S of automorphisms have p-rank γ equal to zero, (Stichtenoth, 1973, Nakajima, 1987).

Problems on zero p-rank curves with very large p-group of automorphisms

- Curves with a large p-group S of automorphisms have p-rank γ equal to zero, (Stichtenoth, 1973, Nakajima, 1987).
- Problem 4: "*Big action problem*" (Lehr-Matignon): What about zero p-rank curves with very large p-group S of automorphisms fixing a point?

Problems on zero p-rank curves with very large p-group of automorphisms

- Curves with a large p-group S of automorphisms have p-rank γ equal to zero, (Stichtenoth, 1973, Nakajima, 1987).
- Problem 4: "*Big action problem*" (Lehr-Matignon): What about zero p-rank curves with very large p-group S of automorphisms fixing a point? $|S| \geq (4g^2)/(p-1)^2 \Rightarrow \mathcal{X} = \mathbf{v}(Y^q - Y + f(X))$ s. t. $f(X) = XP(X) + cX$, $q = p^h$ and $P(X)$ is an additive polynomial of $\mathbb{K}[X]$, (Lehr-Matignon 2005).

Problems on zero p-rank curves with very large p-group of automorphisms

- Curves with a large p-group S of automorphisms have p-rank γ equal to zero, (Stichtenoth, 1973, Nakajima, 1987).
- Problem 4: "*Big action problem*" (Lehr-Matignon): What about zero p-rank curves with very large p-group S of automorphisms fixing a point? $|S| \geq (4g^2)/(p-1)^2 \Rightarrow \mathcal{X} = \mathbf{v}(Y^q - Y + f(X))$ s. t. $f(X) = XP(X) + cX$, $q = p^h$ and $P(X)$ is an additive polynomial of $\mathbb{K}[X]$, (Lehr-Matignon 2005).
- If $\mathrm{Aut}(\mathcal{X})$ fixes no point and $|S| > pg/(p-1)$ then $\mathcal{X}$ is one of the curves (II) ... (VI). (Giulietti-K. 2010).

Large p-subgroups of automorphisms of zero p-rank curves

Lemma

[Bridge lemma]

Lemma

[Bridge lemma] Let $\mathcal{X}$ be a zero p-rank curve, i.e. $\gamma = 0$. Let $S \le \mathrm{Aut}(\mathcal{X})$ with $|S| = p^h$. Then S fixes a point of P of $\mathcal{X}$, and no non-trivial element in S fixes a point distinct from P.

Lemma

[Bridge lemma] Let $\mathcal{X}$ be a zero p-rank curve, i.e. $\gamma = 0$. Let $S \leq \mathrm{Aut}(\mathcal{X})$ with $|S| = p^h$. Then S fixes a point of P of $\mathcal{X}$, and no non-trivial element in S fixes a point distinct from P.

Definition

A Sylow p-subgroup S_p of a finite group G is a *trivial intersection set* if S_p meets any other Sylow p-subgroup of G trivially.

Lemma

[Bridge lemma] Let $\mathcal{X}$ be a zero p-rank curve, i.e. $\gamma = 0$. Let $S \leq \mathrm{Aut}(\mathcal{X})$ with $|S| = p^h$. Then S fixes a point of P of $\mathcal{X}$, and no non-trivial element in S fixes a point distinct from P.

Definition

A Sylow p-subgroup S_p of a finite group G is a *trivial intersection set* if S_p meets any other Sylow p-subgroup of G trivially. If this is the case, G has the *TI*-condition with respect to the prime p.

Lemma

[Bridge lemma] Let $\mathcal{X}$ be a zero p-rank curve, i.e. $\gamma = 0$. Let $S \leq \mathrm{Aut}(\mathcal{X})$ with $|S| = p^h$. Then S fixes a point of P of $\mathcal{X}$, and no non-trivial element in S fixes a point distinct from P.

Definition

A Sylow p-subgroup S_p of a finite group G is a *trivial intersection set* if S_p meets any other Sylow p-subgroup of G trivially. If this is the case, G has the *TI*-condition with respect to the prime p.

Theorem (Giulietti-K. 2005)

Let $\mathcal{X}$ be a curve with $\gamma = 0$. Then $G < \mathrm{Aut}(\mathcal{X})$ with $p \mid |G|$ satisfies the TI-condition for its p-subgroups of Sylow.

Lemma

[Bridge lemma] Let $\mathcal{X}$ be a zero p-rank curve, i.e. $\gamma = 0$. Let $S \leq \mathrm{Aut}(\mathcal{X})$ with $|S| = p^h$. Then S fixes a point of P of $\mathcal{X}$, and no non-trivial element in S fixes a point distinct from P.

Definition

A Sylow p-subgroup S_p of a finite group G is a *trivial intersection set* if S_p meets any other Sylow p-subgroup of G trivially. If this is the case, G has the *TI*-condition with respect to the prime p.

Theorem (Giulietti-K. 2005)

Let $\mathcal{X}$ be a curve with $\gamma = 0$. Then $G < \mathrm{Aut}(\mathcal{X})$ with $p \mid |G|$ satisfies the TI-condition for its p-subgroups of Sylow.

.

Theorem (Burnside-Gow, 1976)

Finite groups satisfying TI-condition for some prime p

.

Theorem (Burnside-Gow, 1976)

Let G be a finite solvable group satisfying the TI-condition for p. Then a Sylow p-subgroup S_p is either normal or cyclic, or $p = 2$ and S_2 is a generalized quaternion group.

Finite groups satisfying TI-condition for some prime p

.

Theorem (Burnside-Gow, 1976)

Let G be a finite solvable group satisfying the TI-condition for p. Then a Sylow p-subgroup S_p is either normal or cyclic, or $p = 2$ and S_2 is a generalized quaternion group.

Remark

Non-solvable groups satisfying the TI-condition are also exist. The known examples include the simple groups involved in the examples (II) ... (VI).

.

Theorem (Burnside-Gow, 1976)

Let G be a finite solvable group satisfying the TI-condition for p. Then a Sylow p-subgroup S_p is either normal or cyclic, or $p = 2$ and S_2 is a generalized quaternion group.

Remark

Non-solvable groups satisfying the TI-condition are also exist. The known examples include the simple groups involved in the examples (II) . . . (VI).
Their complete classification is not done yet,

.

Theorem (Burnside-Gow, 1976)

Let G be a finite solvable group satisfying the TI-condition for p. Then a Sylow p-subgroup S_p is either normal or cyclic, or $p = 2$ and S_2 is a generalized quaternion group.

Remark

Non-solvable groups satisfying the TI-condition are also exist. The known examples include the simple groups involved in the examples (II) ... (VI).
Their complete classification is not done yet,
Important partial classifications (under further conditions) were given by Hering, Herzog, Aschbacher, and more recently by Guralnick-Pries-Stevenson.

Theorem (Giulietti-K. 2010)

Theorem (Giulietti-K. 2010)

Let $p = 2$ *and* $\mathcal{X}$ *a zero 2-rank algebraic curve of genus* $g \geq 2$*. Let* $G \leq \mathrm{Aut}(\mathcal{X})$ *with* $2 \mid |G|$*.*

Theorem (Giulietti-K. 2010)

Let $p = 2$ *and* $\mathcal{X}$ *a zero 2-rank algebraic curve of genus* $g \geq 2$. *Let* $G \leq \mathrm{Aut}(\mathcal{X})$ *with* $2 \mid |G|$. *Then one of the following cases holds.*

Theorem (Giulietti-K. 2010)

Let $p = 2$ and $\mathcal{X}$ a zero 2-rank algebraic curve of genus $g \geq 2$. Let $G \leq \mathrm{Aut}(\mathcal{X})$ with $2 \mid |G|$. Then one of the following cases holds.

(a) *G fixes no point of $\mathcal{X}$ and the subgroup N of G generated by all its 2-elements is isomorphic to one of the groupsn :* $\mathrm{PSL}(2, n)$, $\mathrm{PSU}(3, n)$, $\mathrm{SU}(3, n)$, $\mathrm{Sz}(n)$ *with $n = 2^r \geq 4$; Here N coincides with the commutator subgroup G' of G.*

Theorem (Giulietti-K. 2010)

Let $p = 2$ and $\mathcal{X}$ a zero 2-rank algebraic curve of genus $g \geq 2$. Let $G \leq \mathrm{Aut}(\mathcal{X})$ with $2 \mid |G|$. Then one of the following cases holds.

(a) *G fixes no point of $\mathcal{X}$ and the subgroup N of G generated by all its 2-elements is isomorphic to one of the groupsn :* $\mathrm{PSL}(2, n)$, $\mathrm{PSU}(3, n)$, $\mathrm{SU}(3, n)$, $\mathrm{Sz}(n)$ *with* $n = 2^r \geq 4$; *Here N coincides with the commutator subgroup G' of G.*

(b) *G fixes no point of $\mathcal{X}$ and it has a non-trivial normal subgroup of odd order. A Sylow 2-subgroup S_2 of G is either a cyclic group or a generalized quaternion group.*

Theorem (Giulietti-K. 2010)

Let $p = 2$ *and* $\mathcal{X}$ *a zero 2-rank algebraic curve of genus* $g \geq 2$. *Let* $G \leq \mathrm{Aut}(\mathcal{X})$ *with* $2 \mid |G|$. *Then one of the following cases holds.*

(a) *G fixes no point of $\mathcal{X}$ and the subgroup N of G generated by all its 2-elements is isomorphic to one of the groupsn* : $\mathrm{PSL}(2, n)$, $\mathrm{PSU}(3, n)$, $\mathrm{SU}(3, n)$, $\mathrm{Sz}(n)$ *with* $n = 2^r \geq 4$; *Here N coincides with the commutator subgroup G' of G.*

(b) *G fixes no point of $\mathcal{X}$ and it has a non-trivial normal subgroup of odd order. A Sylow 2-subgroup S_2 of G is either a cyclic group or a generalized quaternion group.*

Furthermore, either $G = O(G) \rtimes S_2$, *or* $G/O(G) \cong \mathrm{SL}(2, 3)$, *or* $G/O(G) \cong \mathrm{GL}(2, 3)$, *or* $G/O(G) \cong \mathcal{G}_{48}$.

Theorem (Giulietti-K. 2010)

Let $p = 2$ and $\mathcal{X}$ a zero 2-rank algebraic curve of genus $g \geq 2$. Let $G \leq \mathrm{Aut}(\mathcal{X})$ with $2 \mid |G|$. Then one of the following cases holds.

(a) *G fixes no point of $\mathcal{X}$ and the subgroup N of G generated by all its 2-elements is isomorphic to one of the groupsn :* $\mathrm{PSL}(2,n)$, $\mathrm{PSU}(3,n)$, $\mathrm{SU}(3,n)$, $\mathrm{Sz}(n)$ *with $n = 2^r \geq 4$; Here N coincides with the commutator subgroup G' of G.*

(b) *G fixes no point of $\mathcal{X}$ and it has a non-trivial normal subgroup of odd order. A Sylow 2-subgroup S_2 of G is either a cyclic group or a generalized quaternion group.*

Furthermore, either $G = O(G) \rtimes S_2$, or $G/O(G) \cong \mathrm{SL}(2,3)$, or $G/O(G) \cong \mathrm{GL}(2,3)$, or $G/O(G) \cong \mathcal{G}_{48}$.

(c) *G fixes a point of $\mathcal{X}$, and $G = S_2 \rtimes H$, with a subgroup H of odd order.*

Corollary

Let $\mathcal{X}$ be a zero 2-rank curve such that the subgroup G of $\mathrm{Aut}(\mathcal{X})$ fixes no point of $\mathcal{X}$.

Corollary

Let $\mathcal{X}$ be a zero 2-rank curve such that the subgroup G of $\mathrm{Aut}(\mathcal{X})$ fixes no point of $\mathcal{X}$.

- *If G is a solvable, then the Hurwitz bound holds for G; more precisely $|G| \leq 72(g-1)$.*

Corollary

Let $\mathcal{X}$ be a zero 2-rank curve such that the subgroup G of $\mathrm{Aut}(\mathcal{X})$ fixes no point of $\mathcal{X}$.

- *If G is a solvable, then the Hurwitz bound holds for G; more precisely $|G| \leq 72(g-1)$.*
- *If G is not solvable, then G is known and the possible genera of $\mathcal{X}$ are computed from the order of its commutator subgroup G' provided that G is large enough, namely whenever $|G| \geq 24g(g-1)$.*

- Problem 5: Find some more examples of zero 2-rank curves of genus g with $|\mathrm{Aut}(\mathcal{X})| \geq 24g(g-1)$.

- Problem 5: Find some more examples of zero 2-rank curves of genus g with $|\mathrm{Aut}(\mathcal{X})| \geq 24g(g-1)$.
- Problem 6: Characterize such examples using their automorphism groups.

- Problem 5: Find some more examples of zero 2-rank curves of genus g with $|\mathrm{Aut}(\mathcal{X})| \geq 24g(g-1)$.
- Problem 6: Characterize such examples using their automorphism groups.
- Problem 7: How extend the above results to zero p-rank curves for $p > 2$?

- Problem 5: Find some more examples of zero 2-rank curves of genus g with $|\mathrm{Aut}(\mathcal{X})| \geq 24g(g-1)$.
- Problem 6: Characterize such examples using their automorphism groups.
- Problem 7: How extend the above results to zero p-rank curves for $p > 2$?
- For Problem 7, progress made by Guralnick-Malmskog-Pries 2012.

Curves with large p-groups of automorphisms, case $\gamma > 0$

$\mathcal{X}$:=curve with genus g and p-rank $\gamma > 0$.

Curves with large p-groups of automorphisms, case $\gamma > 0$

$\mathcal{X}$:=curve with genus g and p-rank $\gamma > 0$.
S:=p-subgroup of $\mathrm{Aut}(\mathcal{X})$;

Curves with large p-groups of automorphisms, case $\gamma > 0$

$\mathcal{X}$:=curve with genus g and p-rank $\gamma > 0$.
S:=p-subgroup of $\mathrm{Aut}(\mathcal{X})$;
Nakajima's bound (1987):

$\mathcal{X}$:=curve with genus g and p-rank $\gamma > 0$.
S:=p-subgroup of $\mathrm{Aut}(\mathcal{X})$;
Nakajima's bound (1987):

$$|S| \leq \begin{cases} 4(\gamma - 1) \quad \text{for} \quad p = 2,\ \gamma > 1 \\ \frac{p}{p-2}(\gamma - 1) \quad \text{for} \quad p \neq 2,\ \gamma > 1, \\ g - 1 \quad \text{for} \quad \gamma = 1. \end{cases}$$

Curves with large p-groups of automorphisms, case $\gamma > 0$

$\mathcal{X}$:=curve with genus g and p-rank $\gamma > 0$.
S:=p-subgroup of $\mathrm{Aut}(\mathcal{X})$;
Nakajima's bound (1987):

$$|S| \leq \begin{cases} 4(\gamma - 1) & \text{for} \quad p = 2, \ \gamma > 1 \\ \frac{p}{p-2}(\gamma - 1) & \text{for} \quad p \neq 2, \ \gamma > 1, \\ g - 1 & \text{for} \quad \gamma = 1. \end{cases}$$

Problem 9: Determine the possibilities for the structures of S when $\mathcal{X}$ extremal w.r. Nakajima's bound, or $|S|$ is closed to it.

$\mathcal{X}$:=curve with genus g and p-rank $\gamma > 0$.
S:=p-subgroup of $\mathrm{Aut}(\mathcal{X})$;
Nakajima's bound (1987):

$$|S| \leq \begin{cases} 4(\gamma-1) & \text{for} \quad p=2,\ \gamma>1 \\ \frac{p}{p-2}(\gamma-1) & \text{for} \quad p\neq 2,\ \gamma>1, \\ g-1 & \text{for} \quad \gamma=1. \end{cases}$$

Problem 9: Determine the possibilities for the structures of S when $\mathcal{X}$ extremal w.r. Nakajima's bound, or $|S|$ is closed to it.

Hypothesis (I): $|S| > \frac{p^2}{p^2-p-1}(g-1)$ *(and* $|S| \geq 8$*)*,

$\mathcal{X}$:=curve with genus g and p-rank $\gamma > 0$.
S:=p-subgroup of $\mathrm{Aut}(\mathcal{X})$;
Nakajima's bound (1987):

$$|S| \leq \begin{cases} 4(\gamma-1) & \text{for} \quad p=2,\ \gamma>1 \\ \frac{p}{p-2}(\gamma-1) & \text{for} \quad p\neq 2,\ \gamma>1, \\ g-1 & \text{for} \quad \gamma=1. \end{cases}$$

Problem 9: Determine the possibilities for the structures of S when $\mathcal{X}$ extremal w.r. Nakajima's bound, or $|S|$ is closed to it.

Hypothesis (I): $|S| > \frac{p^2}{p^2-p-1}(g-1)$ *(and* $|S| \geq 8$*),*

If S fixes a point then $|S| \leq pg/(p-1)$.

$\mathcal{X}$:=curve with genus g and p-rank $\gamma > 0$.
S:=p-subgroup of $\mathrm{Aut}(\mathcal{X})$;
Nakajima's bound (1987):

$$|S| \leq \begin{cases} 4(\gamma - 1) & \text{for} \quad p = 2,\ \gamma > 1 \\ \frac{p}{p-2}(\gamma - 1) & \text{for} \quad p \neq 2,\ \gamma > 1, \\ g - 1 & \text{for} \quad \gamma = 1. \end{cases}$$

Problem 9: Determine the possibilities for the structures of S when $\mathcal{X}$ extremal w.r. Nakajima's bound, or $|S|$ is closed to it.

Hypothesis (I): $|S| > \frac{p^2}{p^2-p-1}(g-1)$ *(and* $|S| \geq 8$*)*,

If S fixes a point then $|S| \leq pg/(p-1)$.

Hypothesis (II): *S fixes no point on $\mathcal{X}$.*

Case $p > 2$

Case $p > 2$

Theorem

(Giulietti-K. 2014/15) Let $p > 2$. If $|S| > \frac{p^2}{p^2-p-1}(g-1)$ and S fixes no point on $\mathcal{X}$, then one of the following holds

Case $p > 2$

Theorem

(Giulietti-K. 2014/15) Let $p > 2$. *If* $|S| > \frac{p^2}{p^2-p-1}(g-1)$ *and* S *fixes no point on* $\mathcal{X}$, *then one of the following holds*

- *(i)* $|S| = p$, *and* $\mathcal{X}$ *is an ordinary curve of genus* $g = p - 1$.

Case $p > 2$

Theorem

(Giulietti-K. 2014/15) Let $p > 2$. *If* $|S| > \frac{p^2}{p^2-p-1}(g-1)$ *and* S *fixes no point on* $\mathcal{X}$, *then one of the following holds*

- *(i)* $|S| = p$, *and* $\mathcal{X}$ *is an ordinary curve of genus* $g = p - 1$.
- *(ii)* $\mathcal{X}$ *is an ordinary Nakajima extremal curve, and it is an unramified Galois extension of a curve in (i).*
  S *is generated by two elements and the Galois extension is abelian, then* S *has maximal nilpotency class.*

Theorem

(Giulietti-K. 2014/15) Let $p > 2$*. If* $|S| > \frac{p^2}{p^2-p-1}(g-1)$ *and* S *fixes no point on* $\mathcal{X}$*, then one of the following holds*

- *(i)* $|S| = p$*, and* $\mathcal{X}$ *is an ordinary curve of genus* $g = p - 1$*.*
- *(ii)* $\mathcal{X}$ *is an ordinary Nakajima extremal curve, and it is an unramified Galois extension of a curve in (i).*
  S *is generated by two elements and the Galois extension is abelian, then* S *has maximal nilpotency class.*
- *In both cases, either* $\mathrm{Aut}(\mathcal{X}) = S \rtimes D$ *with* D *a subgroup of a dihedral group of order* $2(p-1)$*, or* $p = 3$ *and,* $\exists\ M < S$ *of index* 3*,* $\mathrm{Aut}(\mathcal{X})/M \cong L$ *with* $L < GL(2,3)$*.*

Case $p > 2$, infinite families of examples

Case $p > 2$, infinite families of examples

- $F{:=}\mathbb{K}(x,y),\ x(y^p - y) - x^2 + c = 0$, $c \in \mathbb{K}^*$;
  $g(F) = \gamma(F) = 2$.

Case $p > 2$, infinite families of examples

- $F{:=}\mathbb{K}(x,y),\ \ x(y^p-y)-x^2+c=0,\ c\in\mathbb{K}^*;$
  $g(F)=\gamma(F)=2.$
- $\varphi := (x,y)\mapsto(x,y+1),$

Case $p > 2$, infinite families of examples

- $F{:=}\mathbb{K}(x,y),\ x(y^p-y)-x^2+c=0,\ c\in\mathbb{K}^*;$
  $g(F)=\gamma(F)=2.$
- $\varphi := (x,y)\mapsto(x,y+1), \varphi\in \mathrm{Aut}(F).$

Case $p > 2$, infinite families of examples

- $F{:=}\mathbb{K}(x,y),\ x(y^p-y)-x^2+c=0,\ c\in\mathbb{K}^*;$
  $g(F)=\gamma(F)=2.$
- $\varphi := (x,y)\mapsto(x,y+1), \varphi\in \mathrm{Aut}(F).$
- F_N:=largest unramified abelian extension of F of exponent N with two generators,

Case $p > 2$, infinite families of examples

- $F := \mathbb{K}(x, y)$, $x(y^p - y) - x^2 + c = 0$, $c \in \mathbb{K}^*$; $g(F) = \gamma(F) = 2$.
- $\varphi := (x, y) \mapsto (x, y + 1), \varphi \in \mathrm{Aut}(F)$.
- F_N:=largest unramified abelian extension of F of exponent N with two generators,
  - (i) $F_N|F$ is an unramified Galois extension of degree p^{2N},
  - (ii) F_N is generated by all function fields which are cyclic unramified extensions of F of degree p^N,
  - (iii) $\mathrm{Gal}(F_N|F) = C_{p^N} \times C_{p^N}$ and $u^{p^N} = 1$ for every element $u \in \mathrm{Gal}(F_N|F)$.

Case $p > 2$, infinite families of examples

- $F := \mathbb{K}(x,y)$, $x(y^p - y) - x^2 + c = 0$, $c \in \mathbb{K}^*$; $g(F) = \gamma(F) = 2$.
- $\varphi := (x,y) \mapsto (x, y+1), \varphi \in \mathrm{Aut}(F)$.
- F_N:=largest unramified abelian extension of F of exponent N with two generators,
  - (i) $F_N|F$ is an unramified Galois extension of degree p^{2N},
  - (ii) F_N is generated by all function fields which are cyclic unramified extensions of F of degree p^N,
  - (iii) $\mathrm{Gal}(F_N|F) = C_{p^N} \times C_{p^N}$ and $u^{p^N} = 1$ for every element $u \in \mathrm{Gal}(F_N|F)$.
- M :=Galois closure of $F_N|\mathbb{K}$.

Case $p > 2$, infinite families of examples

- $F := \mathbb{K}(x, y)$, $x(y^p - y) - x^2 + c = 0$, $c \in \mathbb{K}^*$;
  $g(F) = \gamma(F) = 2$.
- $\varphi := (x, y) \mapsto (x, y + 1), \varphi \in \mathrm{Aut}(F)$.
- F_N:=largest unramified abelian extension of F of exponent N with two generators,
  (i) $F_N|F$ is an unramified Galois extension of degree p^{2N},
  (ii) F_N is generated by all function fields which are cyclic unramified extensions of F of degree p^N,
  (iii) $\mathrm{Gal}(F_N|F) = C_{p^N} \times C_{p^N}$ and $u^{p^N} = 1$ for every element $u \in \mathrm{Gal}(F_N|F)$.
- M :=Galois closure of $F_N|\mathbb{K}$.

Lemma

$\mathrm{Gal}(M|\mathbb{K}(x))$ *preserves* F.

Case $p > 2$, infinite families of examples

- $F := \mathbb{K}(x, y)$, $x(y^p - y) - x^2 + c = 0$, $c \in \mathbb{K}^*$; $g(F) = \gamma(F) = 2$.
- $\varphi := (x, y) \mapsto (x, y + 1), \varphi \in \mathrm{Aut}(F)$.
- F_N:=largest unramified abelian extension of F of exponent N with two generators,
  - (i) $F_N|F$ is an unramified Galois extension of degree p^{2N},
  - (ii) F_N is generated by all function fields which are cyclic unramified extensions of F of degree p^N,
  - (iii) $\mathrm{Gal}(F_N|F) = C_{p^N} \times C_{p^N}$ and $u^{p^N} = 1$ for every element $u \in \mathrm{Gal}(F_N|F)$.
- M :=Galois closure of $F_N|\mathbb{K}$.

Lemma

$\mathrm{Gal}(M|\mathbb{K}(x))$ *preserves* F. $\Rightarrow \mathrm{Gal}(M|\mathbb{K}(x)) \leq \mathrm{Aut}(F_N)$.

Case $p > 2$, infinite families of examples

- $F{:=}\mathbb{K}(x,y)$, $x(y^p - y) - x^2 + c = 0$, $c \in \mathbb{K}^*$;
  $g(F) = \gamma(F) = 2$.
- $\varphi := (x,y) \mapsto (x, y+1), \varphi \in \mathrm{Aut}(F)$.
- F_N:=largest unramified abelian extension of F of exponent N with two generators,
  - (i) $F_N|F$ is an unramified Galois extension of degree p^{2N},
  - (ii) F_N is generated by all function fields which are cyclic unramified extensions of F of degree p^N,
  - (iii) $\mathrm{Gal}(F_N|F) = C_{p^N} \times C_{p^N}$ and $u^{p^N} = 1$ for every element $u \in \mathrm{Gal}(F_N|F)$.
- M :=Galois closure of $F_N|\mathbb{K}$.

Lemma

$\mathrm{Gal}(M|\mathbb{K}(x))$ *preserves* F. $\Rightarrow \mathrm{Gal}(M|\mathbb{K}(x)) \le \mathrm{Aut}(F_N)$.

Corollary

F_N is an extremal function field w.r. Nakajima's bound.

Case $p = 3$, Examples for small genera

Case $p = 3$, Examples for small genera

- If $|S| = 3$ then $\mathcal{X} = \mathbf{v}((X(Y^3 - Y) - X^2 + c)$ with $c \in \mathbb{K}^*$.

Case $p = 3$, Examples for small genera

- If $|S| = 3$ then $\mathcal{X} = \mathbf{v}((X(Y^3 - Y) - X^2 + c)$ with $c \in \mathbb{K}^*$.
- If $|S| = 9$ then $S = C_3 \times C_3$ and $\mathcal{X} = \mathbf{v}((X^3 - X)((Y^3 - Y) + c)$ with $c \in \mathbb{K}^*$, $g(\mathcal{X}) = 4$.

Case $p = 3$, Examples for small genera

- If $|S| = 3$ then $\mathcal{X} = \mathbf{v}((X(Y^3 - Y) - X^2 + c)$ with $c \in \mathbb{K}^*$.
- If $|S| = 9$ then $S = C_3 \times C_3$ and $\mathcal{X} = \mathbf{v}((X^3 - X)((Y^3 - Y) + c)$ with $c \in \mathbb{K}^*$, $g(\mathcal{X}) = 4$.
- If $|S| = 27$ then $S = UT(3,3)$ and $\mathcal{X} = \mathbf{v}((X^3 - X)(Y^3 - Y) + c, Z^3 - Z - X^3Y + YX^3)$ with $c \in \mathbb{K}^*$, $g(\mathcal{X}) = 10$.

Case $p = 3$, Examples for small genera

- If $|S| = 3$ then $\mathcal{X} = \mathbf{v}((X(Y^3 - Y) - X^2 + c)$ with $c \in \mathbb{K}^*$.
- If $|S| = 9$ then $S = C_3 \times C_3$ and $\mathcal{X} = \mathbf{v}((X^3 - X)((Y^3 - Y) + c)$ with $c \in \mathbb{K}^*$, $g(\mathcal{X}) = 4$.
- If $|S| = 27$ then $S = UT(3,3)$ and $\mathcal{X} = \mathbf{v}((X^3 - X)(Y^3 - Y) + c, Z^3 - Z - X^3Y + YX^3)$ with $c \in \mathbb{K}^*$, $g(\mathcal{X}) = 10$.
- For $|S| = 81$ an explicit example: $S \cong \mathrm{Syl}_3(\mathrm{Sym}_9)$, $\mathcal{X} = \mathbf{v}((X^3 - X)(Y^3 - Y) + c, U^3 - U - X,$ $(U - Y)(W^3 - W) - 1, (U - (Y + 1))(T^3 - T) - 1)$ with $c \in \mathbb{K}^*$, $g(\mathcal{X}) = 28$.

Case $p = 2$

Case $p = 2$

Theorem (Giulietti-K. 2012)

Case $p = 2$

Theorem (Giulietti-K. 2012)

Let $p = 2$. If $|S| > 2(g-1)$, $|S| \geq 8$ and S fixes no point on $\mathcal{X}$, then one of the following cases occurs

Case $p = 2$

Theorem (Giulietti-K. 2012)

Let $p = 2$. If $|S| > 2(g-1)$, $|S| \geq 8$ and S fixes no point on $\mathcal{X}$, then one of the following cases occurs

- $|S| = 4(g-1)$, $\mathcal{X}$ *is an ordinary bielliptic curve.*

Theorem (Giulietti-K. 2012)

Let $p = 2$. If $|S| > 2(g-1)$, $|S| \geq 8$ and S fixes no point on $\mathcal{X}$, then one of the following cases occurs

- $|S| = 4(g-1)$, $\mathcal{X}$ *is an ordinary bielliptic curve. Either*
  (ia) S *is dihedral, or*

Case $p = 2$

Theorem (Giulietti-K. 2012)

Let $p = 2$. If $|S| > 2(g-1)$, $|S| \geq 8$ and S fixes no point on $\mathcal{X}$, then one of the following cases occurs

- $|S| = 4(g-1)$, $\mathcal{X}$ *is an ordinary bielliptic curve. Either*
  - (ia) S *is dihedral, or*
  - (ib) $S = (E \times \langle u \rangle) \rtimes \langle w \rangle$ *where* E *is cyclic group of order* $g-1$ *and* u *and* w *are involutions.*

Theorem (Giulietti-K. 2012)

Let $p = 2$. If $|S| > 2(g-1)$, $|S| \geq 8$ and S fixes no point on $\mathcal{X}$, then one of the following cases occurs

- $|S| = 4(g-1)$, $\mathcal{X}$ *is an ordinary bielliptic curve. Either*
  - (ia) S *is dihedral, or*
  - (ib) $S = (E \times \langle u \rangle) \rtimes \langle w \rangle$ *where* E *is cyclic group of order* $g-1$ *and* u *and* w *are involutions.*
- $|S| = 2g+2$, *and* $S = A \rtimes B$, A *is an elementary abelian subgroup of index 2 and* $B = 2$;

Case $p = 2$

Theorem (Giulietti-K. 2012)

Let $p = 2$. If $|S| > 2(g-1)$, $|S| \geq 8$ and S fixes no point on $\mathcal{X}$, then one of the following cases occurs

- $|S| = 4(g-1)$, $\mathcal{X}$ *is an ordinary bielliptic curve. Either*
  - (ia) S *is dihedral, or*
  - (ib) $S = (E \times \langle u \rangle) \rtimes \langle w \rangle$ *where E is cyclic group of order $g-1$ and u and w are involutions.*
- $|S| = 2g+2$, *and* $S = A \rtimes B$, *A is an elementary abelian subgroup of index 2 and $B = 2$;*
- *Every central involution of S is inductive.*

Case $p = 2$

Theorem (Giulietti-K. 2012)

Let $p = 2$. If $|S| > 2(g-1)$, $|S| \geq 8$ and S fixes no point on $\mathcal{X}$, then one of the following cases occurs

- $|S| = 4(g-1)$, $\mathcal{X}$ *is an ordinary bielliptic curve. Either*
  - (ia) S *is dihedral, or*
  - (ib) $S = (E \times \langle u \rangle) \rtimes \langle w \rangle$ *where* E *is cyclic group of order* $g-1$ *and* u *and* w *are involutions.*
- $|S| = 2g+2$, *and* $S = A \rtimes B$, A *is an elementary abelian subgroup of index* 2 *and* $B = 2$*;*
- *Every central involution of* S *is inductive.*

Involution $u \in Z(S)$ is inductive:$=$ $S/\langle u \rangle$, viewed as a subgroup of $\mathrm{Aut}(\bar{\mathcal{X}})$ of the quotient curve $\mathcal{X} = \mathcal{X}/\langle u \rangle$ satisfies the hypotheses of the theorem.

Case $p = 2$, examples

Case $p = 2$, examples

- For every 2^h, $\exists$ a curve of type (ia): (extremal curve w.r. Nakajima's bound with dihedral 2-group of automorphisms).

Case $p = 2$, examples

- For every 2^h, $\exists$ a curve of type (ia): (extremal curve w.r. Nakajima's bound with dihedral 2-group of automorphisms).
- $\exists$ a sporadic example of type (ib) with $g = 9$ and $S = D_8 \times C_2$.

Case $p = 2$, examples

- For every 2^h, $\exists$ a curve of type (ia): (extremal curve w.r. Nakajima's bound with dihedral 2-group of automorphisms).
- $\exists$ a sporadic example of type (ib) with $g = 9$ and $S = D_8 \times C_2$.
- For $q = 2^h$, the hyperelliptic curve

$$\mathcal{X} := \mathbf{v}((Y^2 + Y + X)(X^q + X) + \sum_{\alpha \in \mathbb{F}_q} \frac{X^q + X}{X + \alpha})$$

has genus $g = q - 1$ and an elementary abelian automorphism group of order $2q$.

Case $p = 2$, examples

- For every 2^h, $\exists$ a curve of type (ia): (extremal curve w.r. Nakajima's bound with dihedral 2-group of automorphisms).
- $\exists$ a sporadic example of type (ib) with $g = 9$ and $S = D_8 \times C_2$.
- For $q = 2^h$, the hyperelliptic curve

$$\mathcal{X} := \mathbf{v}((Y^2 + Y + X)(X^q + X) + \sum_{\alpha \in \mathbb{F}_q} \frac{X^q + X}{X + \alpha})$$

  has genus $g = q - 1$ and an elementary abelian automorphism group of order $2q$.
- Examples involving inductive involutions are also known.

Case $p = 2$, examples

- For every 2^h, $\exists$ a curve of type (ia): (extremal curve w.r. Nakajima's bound with dihedral 2-group of automorphisms).
- $\exists$ a sporadic example of type (ib) with $g = 9$ and $S = D_8 \times C_2$.
- For $q = 2^h$, the hyperelliptic curve

$$\mathcal{X} := \mathbf{v}((Y^2 + Y + X)(X^q + X) + \sum_{\alpha \in \mathbb{F}_q} \frac{X^q + X}{X + \alpha})$$

  has genus $g = q - 1$ and an elementary abelian automorphism group of order $2q$.
- Examples involving inductive involutions are also known.

Problem 11: Construct infinite family of curves of type (ib).