Symmetric coverings and the Bruck-Ryser-Chowla theorem

Daniel Horsley (Monash University, Australia)

Joint work with

Darryn Bryant, Melinda Buchanan, Barbara Maenhaut and Victor Scharaschkin

and with

Nevena Francetić and Sarada Herke

Part 1:

The Bruck-Ryser-Chowla theorem

Symmetric designs

Symmetric designs

A (v, k, λ)-design is a set of v points and a collection of blocks, each with k points, such that any two points occur together in exactly λ blocks.

Symmetric designs

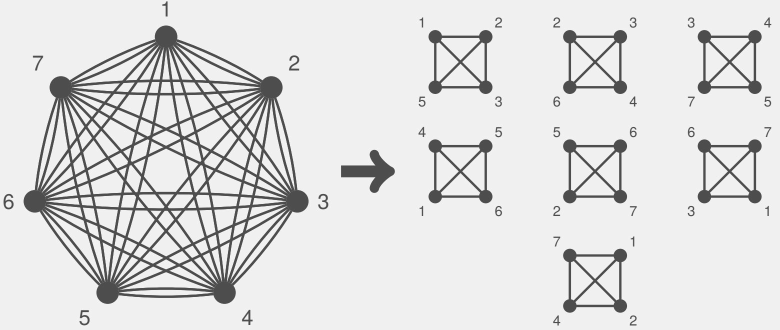
A (v, k, λ)-design is a set of v points and a collection of blocks, each with k points, such that any two points occur together in exactly λ blocks.

A (v, k, λ)-design is symmetric if it has exactly v blocks.

Symmetric designs

A (v, k, λ)-design is a set of v points and a collection of blocks, each with k points, such that any two points occur together in exactly λ blocks.

A (v, k, λ)-design is symmetric if it has exactly v blocks.

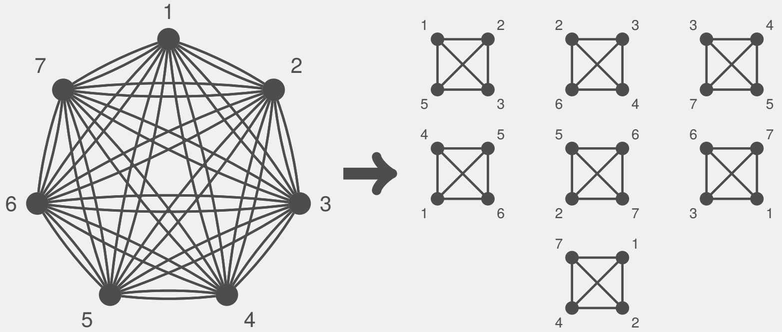


A symmetric $(7, 4, 2)$-design

Symmetric designs

A (v, k, λ)-design is a set of v points and a collection of blocks, each with k points, such that any two points occur together in exactly λ blocks.

A (v, k, λ)-design is symmetric if it has exactly v blocks.



A symmetric $(7, 4, 2)$-design

A symmetric (v, k, λ)-design has $v = \frac{k(k-1)}{\lambda} + 1$.

The BRC theorem

The BRC theorem

Bruck-Ryser-Chowla theorem (1950) If a symmetric (v, k, λ)-design exists then

- if v is even, then $k - \lambda$ is square; and
- if v is odd, then $x^2 = (k - \lambda)y^2 + (-1)^{(v-1)/2}\lambda z^2$ has a solution for integers x, y, z, not all zero.

BRC proof

BRC proof

The incidence matrix M of a symmetric (v, k, λ)-design is a $v \times v$ matrix whose (i, j) entry is 1 if point i is in block j and 0 otherwise.

BRC proof

The incidence matrix M of a symmetric (v, k, λ)-design is a $v \times v$ matrix whose (i, j) entry is 1 if point i is in block j and 0 otherwise.

$$\text{point } x_1 \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ \\ \\ \end{pmatrix}$$

BRC proof

The incidence matrix M of a symmetric (v, k, λ)-design is a $v \times v$ matrix whose (i, j) entry is 1 if point i is in block j and 0 otherwise.

$$\begin{array}{c} \\ \text{point } x_1 \\ \\ \end{array} \overset{\qquad\qquad\qquad\qquad b_1}{\begin{pmatrix} & & & & & & & & & & & & \\ 1 & 0 & 0 & 0 & \color{red}{1} & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ & & & & & & & & & & & & \\ \end{pmatrix}}$$

BRC proof

The incidence matrix M of a symmetric (v, k, λ)-design is a $v \times v$ matrix whose (i, j) entry is 1 if point i is in block j and 0 otherwise.

$$
\begin{array}{r c}
 & \begin{array}{ccccccccccccc} & & & & & & & & & b_2 & & & \end{array} \\
\text{point } x_1 & \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & {\color{red}0} & 0 & 0 & 0 \\ \\ \\ \end{pmatrix}
\end{array}
$$

BRC proof

The incidence matrix M of a symmetric (v, k, λ)-design is a $v \times v$ matrix whose (i, j) entry is 1 if point i is in block j and 0 otherwise.

$$
\text{point } x_1 \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ & & & & & & & & & & & & \\ & & & & & & & & & & & & \end{pmatrix}
$$

BRC proof

The incidence matrix M of a symmetric (v, k, λ)-design is a $v \times v$ matrix whose (i, j) entry is 1 if point i is in block j and 0 otherwise.

$$\text{point } x_1 \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ & & & & & & & & & & & & \\ & & & & & & & & & & & & \end{pmatrix}$$

The dot product of two distinct rows is λ.

BRC proof

The incidence matrix M of a symmetric (v, k, λ)-design is a $v \times v$ matrix whose (i, j) entry is 1 if point i is in block j and 0 otherwise.

$$
\begin{array}{r} \text{point } x_1 \\ \\ \text{point } x_2 \end{array}
\begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & & & \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
$$

The dot product of two distinct rows is λ.

BRC proof

The incidence matrix M of a symmetric (v, k, λ)-design is a $v \times v$ matrix whose (i, j) entry is 1 if point i is in block j and 0 otherwise.

$$\begin{array}{r} \text{point } x_1 \\ \\ \text{point } x_2 \end{array} \begin{pmatrix} 1 & 0 & 0 & 0 & \color{red}{1} & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ \\ 0 & 1 & 0 & 0 & \color{red}{1} & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The dot product of two distinct rows is λ.

BRC proof

The incidence matrix M of a symmetric (v, k, λ)-design is a $v \times v$ matrix whose (i, j) entry is 1 if point i is in block j and 0 otherwise.

$$\begin{array}{c} \text{point } x_1 \\ \\ \text{point } x_2 \end{array} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The dot product of two distinct rows is λ.

The dot product of a row with itself is $k = \frac{\lambda(v-1)}{k-1}$.

BRC proof

The incidence matrix M of a symmetric (v, k, λ)-design is a $v \times v$ matrix whose (i, j) entry is 1 if point i is in block j and 0 otherwise.

$$\begin{array}{r} \text{point } x_1 \\ \\ \text{point } x_2 \end{array} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The dot product of two distinct rows is λ.

The dot product of a row with itself is $k = \frac{\lambda(v-1)}{k-1}$.

BRC proof

BRC proof

If M is the incidence matrix of a symmetric $(13, k, \lambda)$-design, then

$$MM^T = \begin{pmatrix} k & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\ \lambda & k & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & k & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda & k & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda & k & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda & \lambda & k & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & k & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & k & \lambda & \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & k & \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & k & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & k & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & k & \lambda \\ \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & k \end{pmatrix}.$$

BRC proof

If M is the incidence matrix of a symmetric $(13, k, \lambda)$-design, then

$$MM^T = \begin{pmatrix} k & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\ \lambda & k & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & k & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda & k & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda & k & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda & \lambda & k & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & k & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & k & \lambda & \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & k & \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & k & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & k & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & k & \lambda \\ \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & k \end{pmatrix}.$$

The BRC theorem can be proved by observing that

- $\det(MM^T) = \det(M)^2$ is square; and
- MM^T is rationally congruent to I.

(A is rationally congruent to B if $A = QBQ^T$ for an invertible rational matrix Q.)

Part 2:

Extending BRC to coverings

Pair covering designs

Pair covering designs

Recall a symmetric (v, k, λ)-design has $v = \frac{k(k-1)}{\lambda} + 1$.

Pair covering designs

When $v = \frac{k(k-1)-d}{\lambda} + 1$, there may exist a symmetric (v, k, λ)-covering with an d-regular excess.

Pair covering designs

When $v = \frac{k(k-1)-d}{\lambda} + 1$, there may exist a symmetric (v, k, λ)-covering with an d-regular excess.

A symmetric (v, k, λ)-covering has v points and v blocks, each containing k points. Any two points occur together in *at least* λ blocks.

Pair covering designs

When $v = \frac{k(k-1)-d}{\lambda} + 1$, there may exist a symmetric (v, k, λ)-covering with an d-regular excess.

A symmetric (v, k, λ)-covering has v points and v blocks, each containing k points. Any two points occur together in *at least* λ blocks.

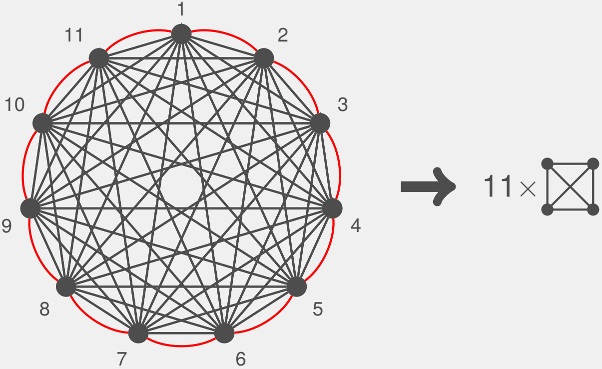
The excess is the multigraph on the point set in which $\mu(xy) = r_{xy} - \lambda$.

Pair covering designs

When $v = \frac{k(k-1)-d}{\lambda} + 1$, there may exist a symmetric (v, k, λ)-covering with an d-regular excess.

A symmetric (v, k, λ)-covering has v points and v blocks, each containing k points. Any two points occur together in *at least* λ blocks.

The excess is the multigraph on the point set in which $\mu(xy) = r_{xy} - \lambda$.



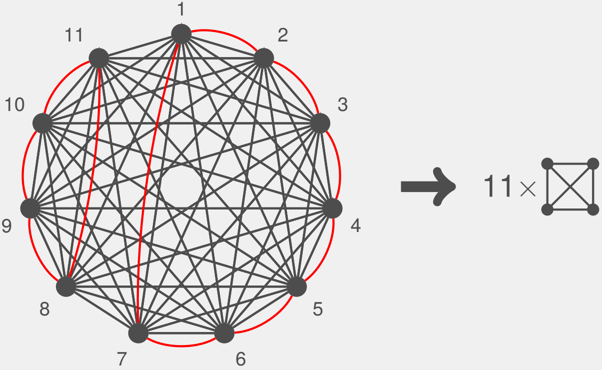
A symmetric $(11, 4, 1)$-covering with a C_{11} excess.

Pair covering designs

When $v = \frac{k(k-1)-d}{\lambda} + 1$, there may exist a symmetric (v, k, λ)-covering with an d-regular excess.

A symmetric (v, k, λ)-covering has v points and v blocks, each containing k points. Any two points occur together in *at least* λ blocks.

The excess is the multigraph on the point set in which $\mu(xy) = r_{xy} - \lambda$.



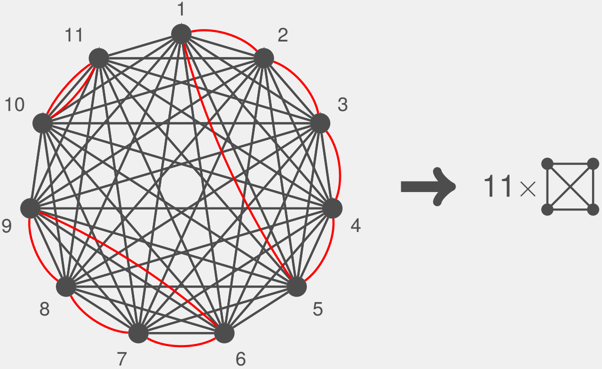
A symmetric $(11, 4, 1)$-covering with a $C_7 \cup C_4$ excess.

Pair covering designs

When $v = \frac{k(k-1)-d}{\lambda} + 1$, there may exist a symmetric (v, k, λ)-covering with an d-regular excess.

A symmetric (v, k, λ)-covering has v points and v blocks, each containing k points. Any two points occur together in *at least* λ blocks.

The excess is the multigraph on the point set in which $\mu(xy) = r_{xy} - \lambda$.



A symmetric $(11, 4, 1)$-covering with a $C_5 \cup C_4 \cup C_2$ excess.

BRC results for coverings

BRC results for coverings

- The Bruck-Ryser-Chowla theorem establishes the non-existence of certain symmetric coverings with empty excesses.

BRC results for coverings

- The Bruck-Ryser-Chowla theorem establishes the non-existence of certain symmetric coverings with empty excesses.
- Bose and Connor (1952) used similar methods to establish the non-existence of certain symmetric coverings with 1-regular excesses.

BRC results for coverings

- The Bruck-Ryser-Chowla theorem establishes the non-existence of certain symmetric coverings with empty excesses.
- Bose and Connor (1952) used similar methods to establish the non-existence of certain symmetric coverings with 1-regular excesses.
- My results concern nonexistence of symmetric coverings with 2-regular excesses.

Degenerate coverings

Degenerate coverings

There is a $(\lambda+4, \lambda+2, \lambda)$-symmetric covering with excess D for every $\lambda \geqslant 1$ and every 2-regular graph D on $\lambda+4$ vertices.

(It has block set $\{V \setminus \{x, y\} : xy \in E(D)\}$.)

What does MM^T look like now?

What does MM^T look like now?

If M is the incidence matrix of a $(11, k, \lambda)$-covering with excess C_{11},

$$MM^T = \begin{pmatrix} k & \lambda+1 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda+1 \\ \lambda+1 & k & \lambda+1 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda+1 & k & \lambda+1 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda+1 & k & \lambda+1 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda+1 & k & \lambda+1 & \lambda & \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda & \lambda+1 & k & \lambda+1 & \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda & \lambda & \lambda+1 & k & \lambda+1 & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda+1 & k & \lambda+1 & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda+1 & k & \lambda+1 & \lambda \\ \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda+1 & k & \lambda+1 \\ \lambda+1 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda+1 & k \end{pmatrix}.$$

What does MM^T look like now?

If M is the incidence matrix of a $(11, k, \lambda)$-covering with excess $C_7 \cup C_4$,

$$MM^T = \begin{pmatrix} k & \lambda+1 & \lambda & \lambda & \lambda & \lambda & \lambda+1 & \lambda & \lambda & \lambda & \lambda \\ \lambda+1 & k & \lambda+1 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda+1 & k & \lambda+1 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda+1 & k & \lambda+1 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda+1 & k & \lambda+1 & \lambda & \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda & \lambda+1 & k & \lambda+1 & \lambda & \lambda & \lambda & \lambda \\ \lambda+1 & \lambda & \lambda & \lambda & \lambda & \lambda+1 & k & \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & k & \lambda+1 & \lambda & \lambda+1 \\ \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda+1 & k & \lambda+1 & \lambda \\ \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda+1 & k & \lambda+1 \\ \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda+1 & \lambda & \lambda+1 & k \end{pmatrix}.$$

What does MM^T look like now?

If M is the incidence matrix of a $(11, k, \lambda)$-covering with excess $C_6 \cup C_3 \cup C_2$,

$$MM^T = \begin{pmatrix} k & \lambda+1 & \lambda & \lambda & \lambda & \lambda+1 & \lambda & \lambda & \lambda & \lambda & \lambda \\ \lambda+1 & k & \lambda+1 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda+1 & k & \lambda+1 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda+1 & k & \lambda+1 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda+1 & k & \lambda+1 & \lambda & \lambda & \lambda & \lambda & \lambda \\ \lambda+1 & \lambda & \lambda & \lambda & \lambda+1 & k & \lambda & \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & k & \lambda+1 & \lambda+1 & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda+1 & k & \lambda+1 & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda+1 & \lambda+1 & k & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & k & \lambda+2 \\ \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda+2 & k \end{pmatrix}.$$

Determinant results (with BBM&S)

Determinant results (with BBM&S)

Based around the observation that $\det(MM^T)$ is square.

Determinant results (with BBM&S)

Based around the observation that $\det(MM^T)$ is square.

Lemma For a (v, k, λ)-covering with a 2-regular excess,

$$\det(MM^T) = (k - \lambda + 2)^{t-1}(k - \lambda - 2)^e \quad \text{(up to a square)},$$

where t is the number of cycles in the excess, and e is the number of even cycles.

Determinant results (with BBM&S)

Based around the observation that $\det(MM^T)$ is square.

Lemma For a (v, k, λ)-covering with a 2-regular excess,

$$\det(MM^T) = (k - \lambda + 2)^{t-1}(k - \lambda - 2)^e \quad \text{(up to a square)},$$

where t is the number of cycles in the excess, and e is the number of even cycles.

Theorem If there exists a nondegenerate symmetric (v, k, λ)-covering with a 2-regular excess, then

- v is even, $k - \lambda - 2$ is square, and the excess has an odd number of cycles; or
- v is even, $k - \lambda + 2$ is square, and the excess has an even number of cycles; or
- v is odd and the excess has an odd number of cycles.

Determinant results (with BBM&S)

Based around the observation that $\det(MM^T)$ is square.

Lemma For a (v, k, λ)-covering with a 2-regular excess,

$$\det(MM^T) = (k - \lambda + 2)^{t-1}(k - \lambda - 2)^e \quad \text{(up to a square)},$$

where t is the number of cycles in the excess, and e is the number of even cycles.

Theorem If there exists a nondegenerate symmetric (v, k, λ)-covering with a 2-regular excess, then

- v is even, $k - \lambda - 2$ is square, and the excess has an odd number of cycles; or
- v is even, $k - \lambda + 2$ is square, and the excess has an even number of cycles; or
- v is odd and the excess has an odd number of cycles.

Corollary There does not exist a nondegenerate symmetric (v, k, λ)-covering with a 2-regular excess if v is even and neither $k - \lambda - 2$ nor $k - \lambda + 2$ is square.

Rational congruence results (with F&H)

Rational congruence results (with F&H)

Based around the observation that $C_p(MM^T) = C_p(I)$ for each prime p.

Rational congruence results (with F&H)

Based around the observation that $C_p(MM^T) = C_p(I)$ for each prime p.

- Computing $C_p(MM^T)$ naively involves calculating the determinant of every leading principal minor of MM^T.
- We give an efficient algorithm for finding $C_p(MM^T)$ (instead involving calculating the first v terms of a recursive sequence).
- We cannot rule out the existence of symmetric coverings for any more entire parameter sets.
- We rule out the existence of many more symmetric coverings with specified excesses.
- We rule out the existence of some more cyclic symmetric coverings.

Example: $(v, k, \lambda) = (11, 4, 1)$

Example: $(v, k, \lambda) = (11, 4, 1)$

Possible excess types:

$[C_{11}]$,
$[C_9 \cup C_2]$, $[C_8 \cup C_3]$, $[C_7 \cup C_4]$, $[C_6 \cup C_5]$,
$[C_7 \cup C_2 \cup C_2]$, $[C_6 \cup C_3 \cup C_2]$, $[C_5 \cup C_4 \cup C_2]$, $[C_5 \cup C_3 \cup C_3]$, $[C_4 \cup C_4 \cup C_3]$,
$[C_5 \cup C_2 \cup C_2 \cup C_2]$, $[C_4 \cup C_3 \cup C_2 \cup C_2]$, $[C_3 \cup C_3 \cup C_2 \cup C_2]$,
$[C_5 \cup C_2 \cup C_2 \cup C_2 \cup C_2]$

Example: $(v, k, \lambda) = (11, 4, 1)$

Possible excess types:

$[C_{11}]$,
$[C_9 \cup C_2]$, $[C_8 \cup C_3]$, $[C_7 \cup C_4]$, $[C_6 \cup C_5]$,
$[C_7 \cup C_2 \cup C_2]$, $[C_6 \cup C_3 \cup C_2]$, $[C_5 \cup C_4 \cup C_2]$, $[C_5 \cup C_3 \cup C_3]$, $[C_4 \cup C_4 \cup C_3]$,
$[C_5 \cup C_2 \cup C_2 \cup C_2]$, $[C_4 \cup C_3 \cup C_2 \cup C_2]$, $[C_3 \cup C_3 \cup C_2 \cup C_2]$,
$[C_5 \cup C_2 \cup C_2 \cup C_2 \cup C_2]$

ruled out by determinant arguments

Example: $(v, k, \lambda) = (11, 4, 1)$

Possible excess types:

$[C_{11}]$,
$[C_9 \cup C_2]$, $[C_8 \cup C_3]$, $[C_7 \cup C_4]$, $[C_6 \cup C_5]$,
$[C_7 \cup C_2 \cup C_2]$, $[C_6 \cup C_3 \cup C_2]$, $[C_5 \cup C_4 \cup C_2]$, $[C_5 \cup C_3 \cup C_3]$, $[C_4 \cup C_4 \cup C_3]$,
$[C_5 \cup C_2 \cup C_2 \cup C_2]$, $[C_4 \cup C_3 \cup C_2 \cup C_2]$, $[C_3 \cup C_3 \cup C_2 \cup C_2]$,
$[C_5 \cup C_2 \cup C_2 \cup C_2 \cup C_2]$

ruled out by determinant arguments

ruled out by rational congruence arguments

Example: $(v, k, \lambda) = (11, 4, 1)$

Possible excess types:

$[C_{11}]$,
$[C_9 \cup C_2]$, $[C_8 \cup C_3]$, $[C_7 \cup C_4]$, $[C_6 \cup C_5]$,
$[C_7 \cup C_2 \cup C_2]$, $[C_6 \cup C_3 \cup C_2]$, $[C_5 \cup C_4 \cup C_2]$, $[C_5 \cup C_3 \cup C_3]$, $[C_4 \cup C_4 \cup C_3]$,
$[C_5 \cup C_2 \cup C_2 \cup C_2]$, $[C_4 \cup C_3 \cup C_2 \cup C_2]$, $[C_3 \cup C_3 \cup C_2 \cup C_2]$,
$[C_5 \cup C_2 \cup C_2 \cup C_2 \cup C_2]$

ruled out by determinant arguments

ruled out by rational congruence arguments

It turns out $[C_{11}]$ and $[C_6 \cup C_3 \cup C_2]$ are realisable and $[C_5 \cup C_3 \cup C_3]$ is not.

Computational rational congruence results

(v, k, λ)	# of excess types	# ruled out by det results	# ruled out by RC results ($p < 10^3$)	# which may exist
$(11, 4, 1)$	14	7	4	3
$(19, 5, 1)$	105	52	43	10
$(29, 6, 1)$	847	423	393	31
$(41, 7, 1)$	7245	3621	3376	248
$(55, 8, 1)$	65121	32555	30746	1820
$(71, 9, 1)$	609237	304604	292475	12158

Theoretical rational congruence results

Theorem There does not exist a symmetric $(\frac{1}{2}p^\alpha(p^\alpha - 1), p^\alpha, 2)$-covering with Hamilton cycle excess when $p \equiv 3 \pmod 4$ is prime, α is odd and $(p, \alpha) \neq (3, 1)$.

That’s all.