Patent Pending

New
Minimum Number of Clues Findings
to
Sudoku-derivative Games up to 5-by-5 Matrices

including

a Definition for Relating Isotopic Patterns

and

a Technique for Cataloging and Counting Distinct Isotopic Patterns for the 5-by-5 Matrix

Brian Diamond

Games and fun

keep us

learning.

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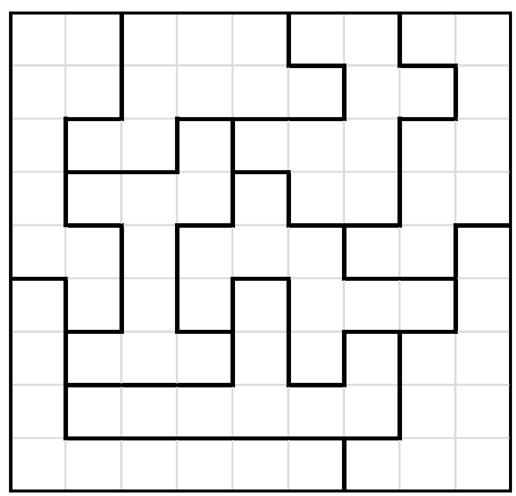
		3	8			7		
				2				
4					1		3	5
					7		2	
1				6				4
	2		3					
5	6		2					9
				3				
		7			4	8		

Minimum number of clues (MCN) for standard Sudoku determined to be 17.

- G. McGuire, B. Tugemann, G. Civario, January, 2012

But how many clues would be sufficient (MNC) if the 9-by-9 matrix consisted of diversely-shaped, interconnected "pieces" consisting of nine contiguous squares?

For example:



Let's state the object of the old game:

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With traditional Sudoku, given a 9-by-9 matrix and a set of filled-in starting squares, the object of the game is to deductively complete the puzzle such that each row, column, and 3-by-3 sub-region contains precisely one of each "color" (numeral, etc.).

Let's define the object of the new game:

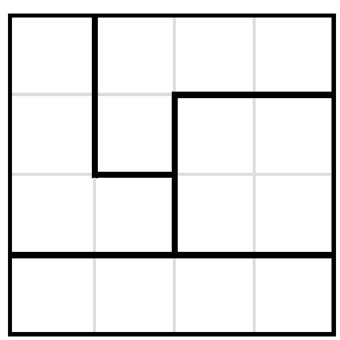
A **puzzle** comprising a particular **pattern** (i.e.: an n-by-n matrix containing n regions each comprising n contiguous squares) is solved when a ***minimum starting set*** of squares – that is, the fewest number of squares with their provided initial colors (values) - is identified which satisfies the following condition: that those squares are sufficient to ***unambiguously*** complete the remaining squares of the puzzle by means of a sequence of deductive steps whereby, upon completion of the puzzle, each row, column, and region contains each of the n distinct colors – and not by merely guessing the remaining values in order to obtain the final condition.

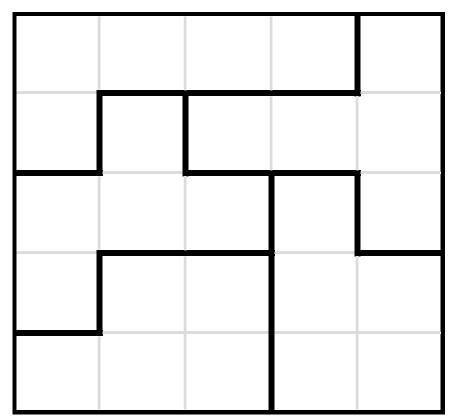
That is, the puzzle must be completed from the starting set in exactly one way.

The game is played by selecting various candidate minimum starting sets and attempting to color squares deductively until a solution is found which uniquely completes the n-by-n matrix according to the above conditions.

Before we enter into further mathematical discussion, it is best to visualize how my wife presents the game to fellow commuters on the Long Island Railroad:

She shows them a blank pattern similar to the ones below and simply tells them that the object of the game is for them to try to supply a *minimum set* of starting squares, along with their initial values, from which they must attempt to compete the puzzle in the same style of play as Sudoku. If they get stuck, then they start over until they find a solution – the solution being to find the starting squares and their values – and not simply haphazardly complete the underlying Latin square. It's just like they're playing Sudoku , except now *they are the ones providing the starting configuration*!

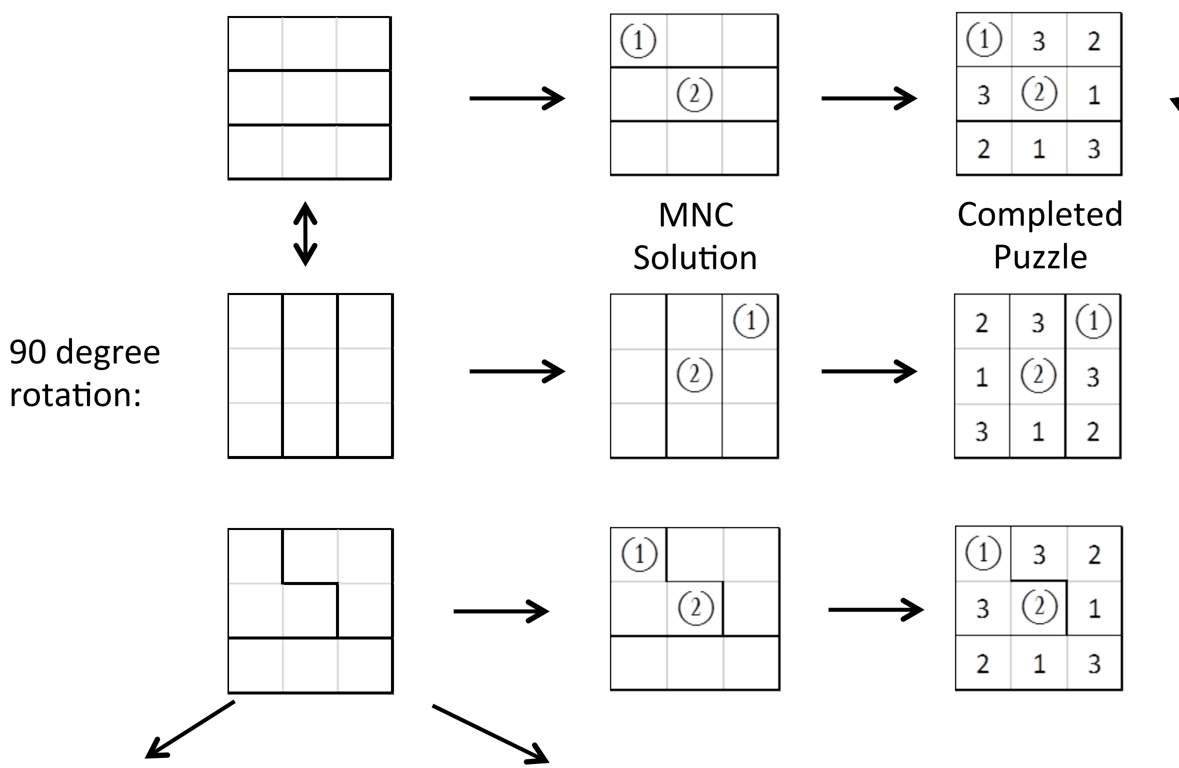




Consider 3-by-3 matrices:

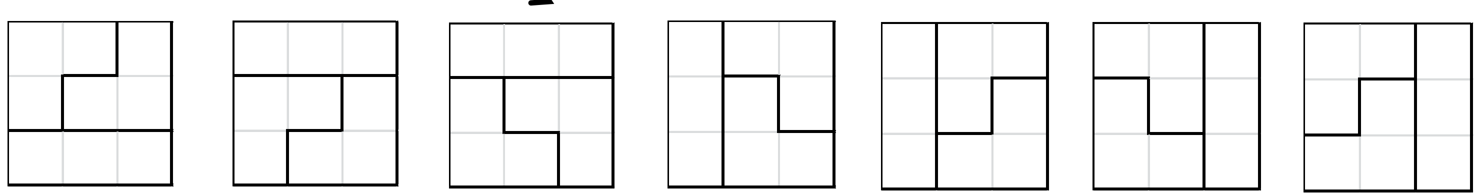
How many isotopically <u>distinct</u> patterns are there?

That is, once an MNC starting set of squares solves one pattern, then the starting set can be trivially rearranged to solve a symmetric pattern.



These two patterns are clearly isotopic to each other – and so are their solutions!

All 8 of these patterns are isotopic to each other by reflection and rotation – and so their solutions can be trivially generated.



Thus we see there are two isotopically distinct patterns for 3-by 3matrices.

More importantly, we see that it is sufficient to provide **two** starting squares – given the proper square locations and “colors” within each starting square – to be able to complete all **3-by-3** patterns “Sudoku”-style. That is, the minimum number of clues for both isotopically distinct sets is **2**.

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The fifteen isotopically distinct patterns along with (*non-unique)* minimum starting set solutions for all 4-by-4 patterns.

4	1	3	2
①	4	②	3
2	③	4	1
3	2	①	4

(1)

1	④	3	2
2	3	4	①
③	2	1	4
4	1	②	3

(2)

2	③	1	4
4	1	②	3
3	2	4	①
1	4	3	2

(3)

2	4	3	①
1	③	4	2
4	1	②	3
3	2	1	4

(4)

2			①
1	2		
	1	②	
		1	2

(5)

4	2	3	①
1	4	②	3
2	③	1	4
3	1	4	2

(6)

2	4	3	①
4	1	②	3
1	③	4	2
3	2	1	4

(7)

		②	1
	2	1	
2	①		
1			2

(8)

1	④	3	2
2	3	4	①
③	2	1	4
4	1	②	3

(9)

2	①	3	4
4	3	②	1
1	2	4	③
3	4	1	2

(10)

2	①	3	4
4	3	②	1
1	2	4	③
3	4	1	2

(11)

2		1	
	1		②
①		2	
	2		1

(12)

		1	X
	1	X	
①	2		
②			1

(13)

2	4	3	1
3	①	②	4
4	③	1	2
1	2	4	3

(14)

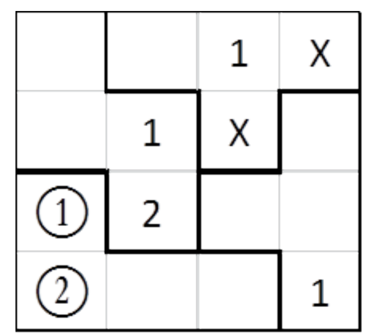
4	3	②	1
2	4	1	③
3	①	4	2
1	2	3	4

(15)

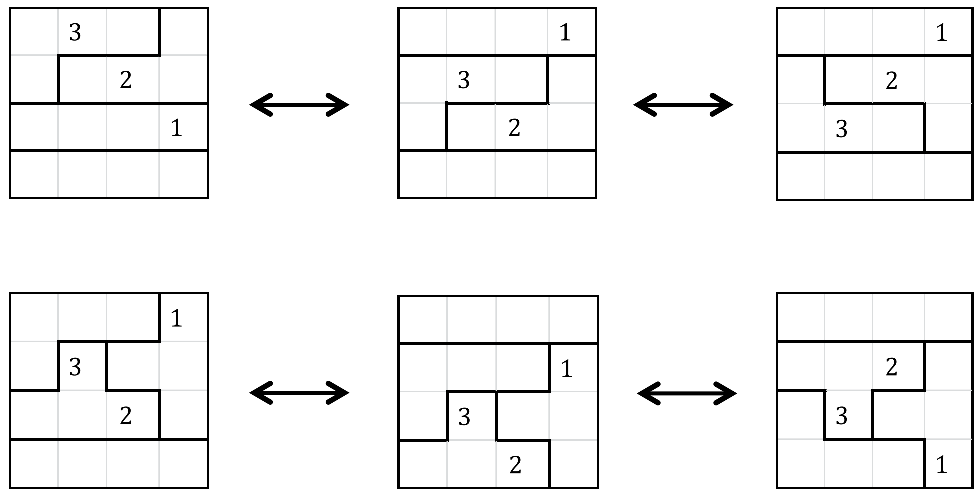
Now notice that some **4-by-4** patterns require minimal starting sets of **three** squares and other **4-by-4** patterns require minimal starting sets of **four** squares.

What correlation can we make?

Also observe that puzzle 13 is *unsolvable*. That is, not only can a starting set *not* be found, but there is no way to color this puzzle with four colors in each row, column, and piece.



Observe how the following sets of patterns are clearly similar but vary by swapping row positions. Also notice how the MNC solutions trivially change with the swapped rows.



Therefore we identify three types of symmetric variation which define whether two patterns are considered *isotopic* within this new game:

- Reflection (vertical or horizontal)
- Rotation (by 90, 180, or 270 degrees or not at all)
- The swapping or rows or columns whereby pieces still contain the same number of squares. Although the swapping may change the shape or position of the pieces affected, the relative position of the swapped squares within the rows, columns, and pieces remains the same.

How do I proceed with Identifying all distinct patterns for the 5-by-5 puzzle (which I have named "Quintoku") ?

Dr. Anton Betten, CSU, has determined by computer that there are **4006** ways to partition a 5-by-5 matrix into patterns consisting of 5 regions ("pieces") containing 5 contiguous squares each – ***not considering duplication by isotopic symmetry***.

- How can I possibly determine the MNC for each pattern?
- I don't want to solve all 4006 patterns anyway. There must be a consistent way to combine several patterns into one distinct isotopic group.
- I need a methodology for cataloging patterns in order to be certain I have accounted for every isotopic group.
- Goodness – and I have to solve the game for every distinct isotopic pattern, too??

Unfortunately, Brian can't find anything better to do.

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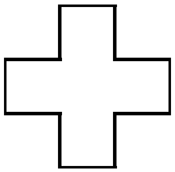
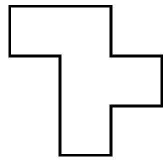
Introducing the diverse regions, or “pieces” for the 5-by-5 Quintoku game:

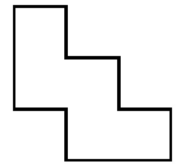
“Plus”,



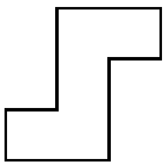
“Jig” (short for “jigsaw”),



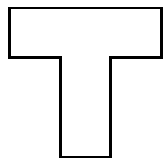
“Cass”, (short for “Cassiopeia”),



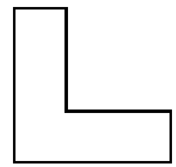
“Zig” (short for zig-zag),



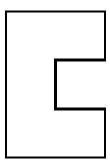
“T”,



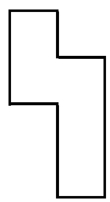
“L”,



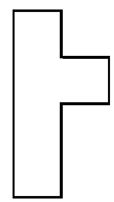
“C”,



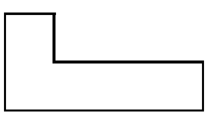
“Bolt”,



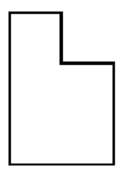
“Turn” (short for turnstyle),



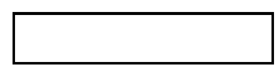
“J”,



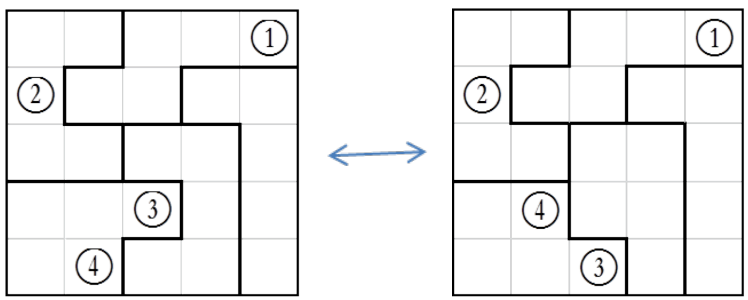
“Utah”,



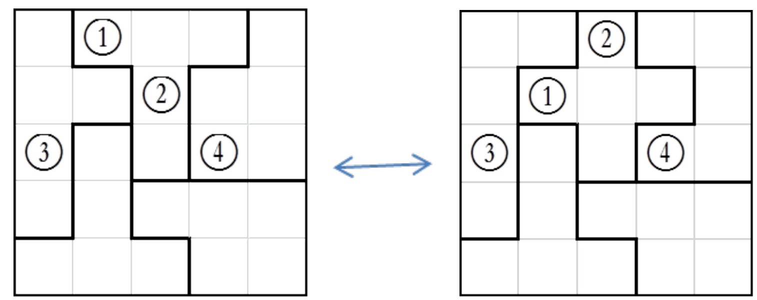
and finally “Bar”



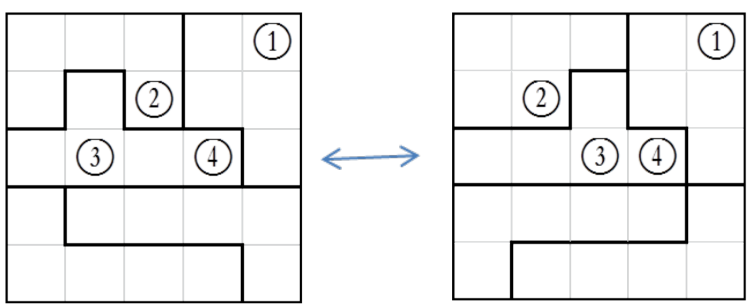
Notice that besides the readily identifiable rotations and reflections which would make two 5-by-5 patterns isotopic, there *can exist* row and column swaps which change the shape of the pieces but which maintain the isotopic relationship.



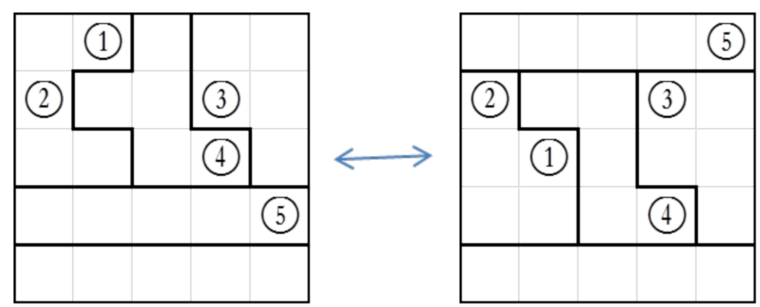
(a)



(b)



(c)



(d)

A Technique for Cataloging and Counting Distinct Isotopic Patterns

After solving more than one hundred puzzles, I created a lexicographical order of the twelve pentomino piece shapes (see slide 15) based on the relative *infrequency* of the appearance of the pieces.

My cataloging of patterns utilized the following hierarchical methodology:

- I initially distinguished the patterns by the number of straight partitions each contains. Thus I first identified patterns containing no straight partitions first, and proceeded down to the final "blank" pattern containing only all straight partitions.
- Using the lexicographical sequence of the infrequency of pieces, I identified the patterns containing the successively more frequently occurring pieces while eliminating all recurring patterns containing previously catalogued pieces.
- For the piece under consideration, I successively placed it in non-symmetric locations within the 5-by-5 matrix.
- I performed a recursive tree search placing the remaining 2nd, 3rd, 4th, and 5th pieces onto the matrix in unique locations.
- Lastly, patterns were scrutinized to see if they were isotopic to any previously identified patterns.

Question: Can the MNC starting set for a contiguous partition of an n-by-n matrix ever be less than $n-1$?

NO! By filling in n occurrences of the first $n-2$ colors in the manner of a Latin square, two unfilled squares still occupy each row and column, and these unfilled squares connect a bipartite path (or set of paths) for the lost two colors. Thus, an $n-1\uparrow st$ color is always necessary.

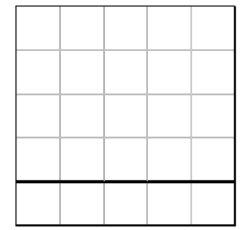
2		3	1	
	1	2	3	
1	3			2
3			2	1
	2	1		3

So – after several years of doodling – what MNC findings were I able to determine for all 5-by-5 Quintoku patterns ?

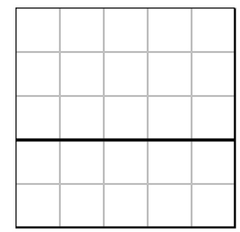
Findings for 5-by-5 Quintoku puzzles

Finding: Of the 148 distinct Quintoku patterns containing no straight partition, minimum starting sets of size **4** exist for all save *eigh*t unsolvable patterns. (Reached March 2, 2014)

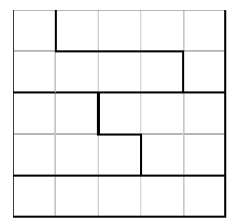
Finding: All distinct 87 Quintoku patterns containing a single straight partition which subdivides that game into 4-by-5 and 1-by-5 sub-regions can be solved with minimum starting sets of size **4**. (Reached June 7, 2014)



Finding: All distinct 22 Quintoku patterns containing a straight partition which subdivide that game into 3-by-5 and 2-by-5 sub-regions can be solved with minimum starting sets of size **4**.



Finding: The fifteen *degenerate* 5-by-5 patterns (containing two or more straight partitions) can all be solved with minimum starting sets of size **5** or **6**. In particular, the pattern containing only straight partitions (the "*blank*" pattern) can be solved with a minimum starting set of size **6**.



Finding: There exist 272 isotopically distinct Quintoku patterns. That is, there are 272 ways to partition a 5-by-5 matrix into contiguous pieces each of five squares in size according to the definition of Quintoku-isotopic.

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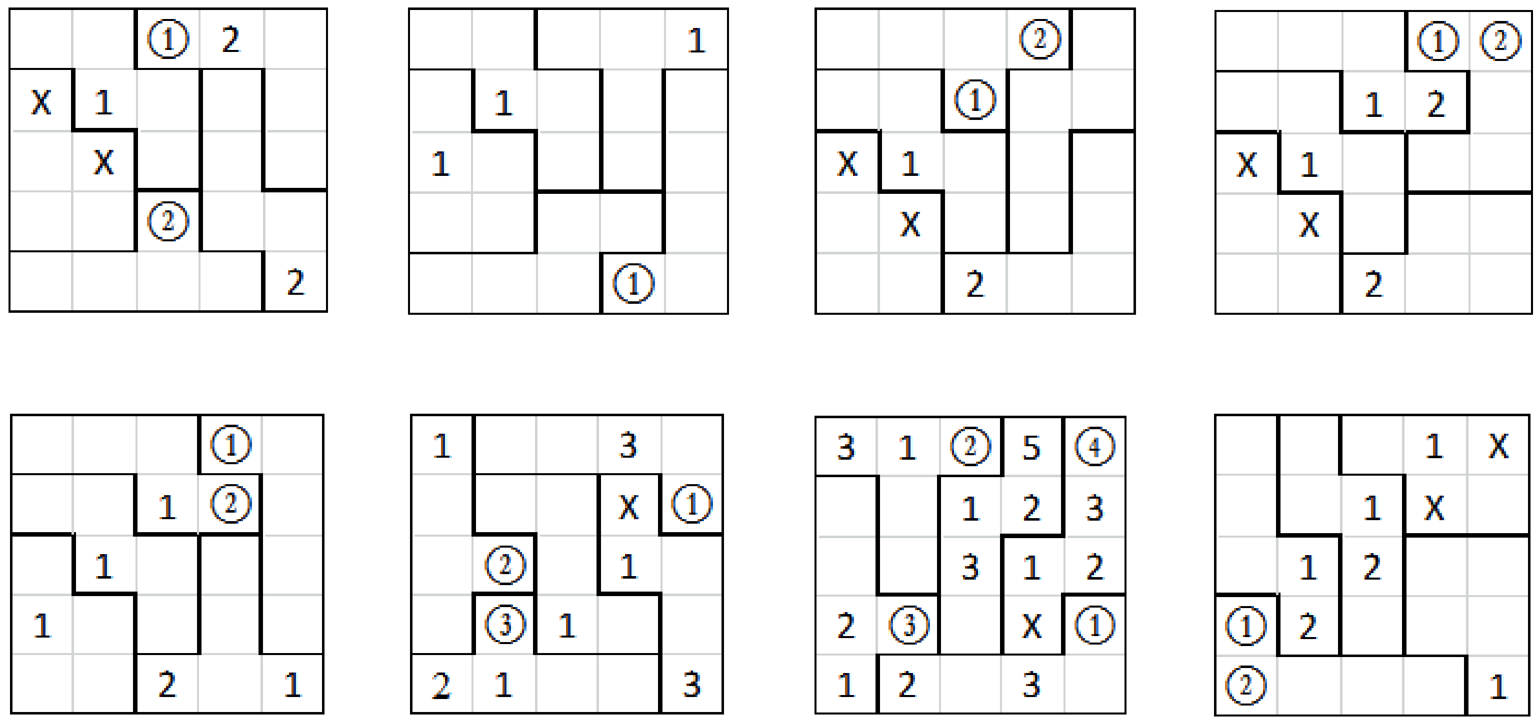
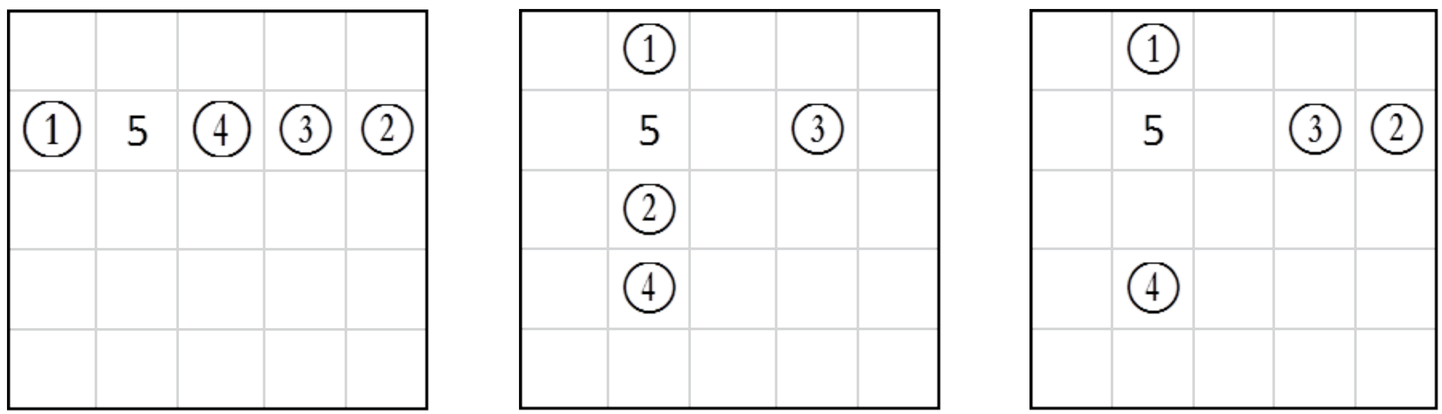
The eight unsolvable 5-by-5 patterns



Observe that the "blank" pattern clearly requires an MNC of more than 4 starting squares.



<u>Theorem</u>: All n-by-n matrices can be partitioned into a “step-wise” partition which can be solved with a starting set size of $n-1$.

3x3

2	3	1
3	①	②
1	2	3

4x4

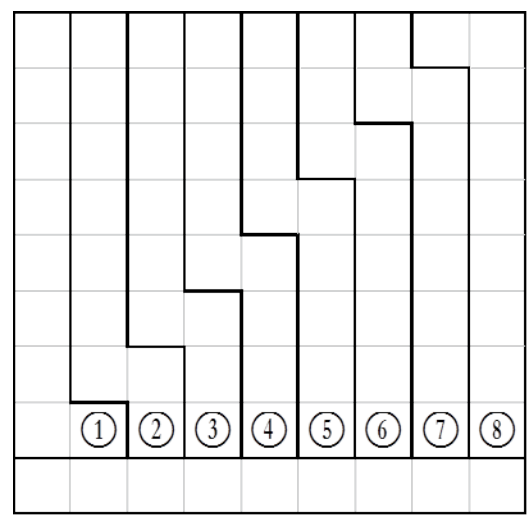
2	3	4	1
3	4	1	2
4	①	②	③
1	2	3	4

5x5

2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	①	②	③	④
1	2	3	4	5

In sequence, each starting square respectively “automatically” fills in a total of n squares. The last unfilled n squares have to be filled in by the $n{\uparrow}th$ color!

9x9



Expanding the Game

- Permit wrapping: where pieces maybe considered contiguous by connecting the top and bottom rows and connecting left and right columns ("toroidal")
- Considering higher order matrices
- Non-contiguous pieces
- Cubic manifestation?

Thanks!