Computing Hyperplanes of Near Polygons

Anurag Bishnoi
(joint work with Bart De Bruyn)

Ghent University

anurag.2357@gmail.com

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Near polygons

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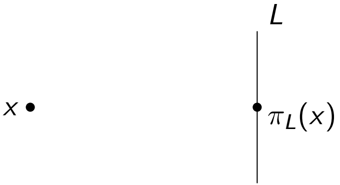
It is a point-line geometry $\mathcal{N}$ that satisfies the following properties:

(NP1) The collinearity graph of $\mathcal{N}$ is connected and has diameter d.

(NP2) For every point x and every line L there exists a unique point $\pi_L(x)$ incident with L that is nearest to x.



Hyperplanes of point-line geometries

A set H of points is called a hyperplane if for every line L, either $L \cap H$ is a singleton or L is contained in H. If no line is contained in H, then it is called an ovoid (or a 1-ovoid). In a near $2d$-gon, the set H_x of points that are distance $< d$ from a point x form a hyperplane, known as a singular hyperplane.

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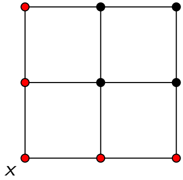
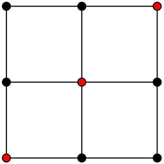


A singular hyperplane

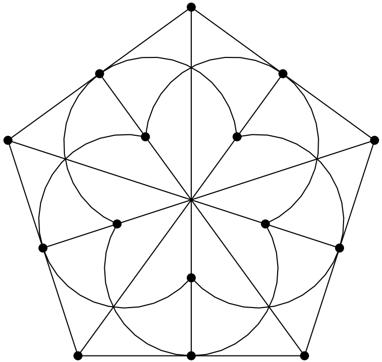


An ovoid

Motivation

Let $\mathcal{N}$ be a near polygon isometrically embedded in another near polygon $\mathcal{N}'$. For every point x of $\mathcal{N}'$ the set $H_x = \{y \in \mathcal{P} : \mathrm{d}(x, y) < m\}$ forms a hyperplane of $\mathcal{N}$, where $m := \max\{\mathrm{d}(x, y) : y \in \mathcal{P}\}$.

Isometric embeddings of near polygons



$W(2)$ with a grid $GQ(2,1)$

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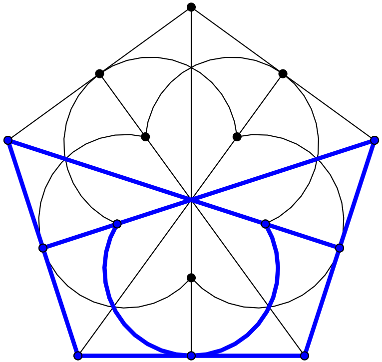

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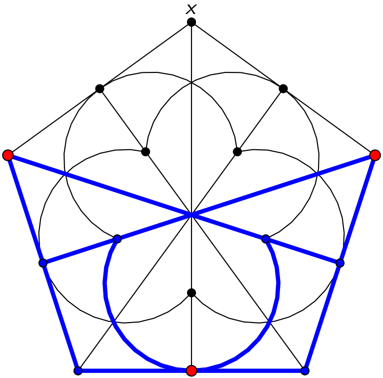

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Main Problem

Given a near polygon $\mathcal{N}$ of order (s, t) with automorphism group G, compute all hyperplanes of $\mathcal{N}$ up to equivalence under the action of G.

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The notion of 1-ovoids is equivalent to exact hitting sets in a hypergraph. Therefore, computing 1-ovoids is equivalent to computing exact covers, which is known to be NP-hard.

Slim geometries

If M is the incidence matrix of $\mathcal{N}$ with rows indexed by lines and columns by points, then a hyperplane corresponds to a 0-1 vector x such that $Mx \in \{1, s+1\}^n$.

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A set H intersects every line in 1 or 3 points $\iff$ H^c intersects every line in 0 or 2 points $\iff$ the characteristic vector v of H^c satisfies $Mv = 0$ over $\mathbb{F}_2$.

Let U be the null space of M over $\mathbb{F}_2$. Then $2^{\dim U} - 1$ is the total number of hyperplanes.

Algorithm 1 pseudocode for computing hyperplanes

Initiate $N := 2^{\dim U} - 1$ and *Hyperplanes* $:= \mathrm{dictionary}()$.
while $N \neq 0$ **do**
  Pick a non-zero vector v in U and let H be the corresponding hyperplane.
  Let $H' := \mathrm{SmallestImageSet}(H)$.
  if H' not in *Hyperplanes* **then**
    Add H' to *Hyperplanes* and put $N := N - \mathrm{Size}(\mathrm{Orbit}_G(H))$.
  end if
end while

A big improvement

Let S be the set of all singular hyperplanes and assume that $\langle S \rangle = U$. Define index i for a hyperplane H to be the minimum number of singular hyperplanes whose "sum" is equal to H. Adding hyperplanes in the increasing order of i gives us a *big* improvement!

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- *To check if a candidate H for S_{i+1} is new, it suffices to compare it with elements of S_{i-1}, S_i and S_{i+1}!*
- *For a fixed $H \in S_i$, we can restrict to elements of S corresponding to the point representatives of the action of $\mathrm{Stab}_G(H)$.*

Test Case: Hall-Janko Near Octagon

The Hall-Janko (or the Cohen-Tits near octagon) is a near octagon of order $(2, 4)$ with its full automorphism group of size 1209600 isomorphic to $J_2 : 2$. It is a regular near octagon giving rise to a distance-regular graph with intersection array $\{10, 8, 8, 2; 1, 1, 4, 5\}$, which uniquely determines the graph.

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Computational Results: It has $2^{28} - 1$ hyperplanes partitioned into 470 equivalence classes under the action of $J_2 : 2$ (≈ 60 mins after all improvements*).
Remark: This gives rise to a binary $[315, 28, 64]$ code with automorphism group $J_2 : 2$, originally discovered by J. D. Key and J. Moori in 2002.

* using RepresentativeAction instead of SmallestImageSet!!!

$G_2(4)$ Near Octagon

There exists a near octagon of order $(2, 10)$ which contains the Hall-Janko near octagon isometrically embedded in it and that has the group $G_2(4) : 2$ as its full automorphism group.

It can be constructed using the conjugacy class of 4095 central involutions of the group $G_2(4) : 2$.

Reference: A. Bishnoi and B. De Bruyn. A new near octagon and the Suzuki tower. `http://arxiv.org/abs/1501.04119`.

Generalized Polygons

Generalized Polygons

A generalized $2d$-gon can be viewed as a near $2d$-gon which satisfies the following additional properties:

(GH1) Every point is incident with at least two lines.

(GH2) Given any two points x, y at distance i from each other, there is a unique neighbour of y that is at distance $i - 1$ from x.

A near 4-gon is a (possibly degenerate) generalized 4-gon, aka, generalized quadrangle.

The incidence graph of a generalized n-gon has a diameter n and girth $2n$. Therefore, it is a (bipartite) Moore graph. The collinearity graph is a distance regular graph. By Feit and Higman, generalized n-gons exist only for $n = 3, 4, 6, 8$ and 12.

Generalized Hexagons

They are near 6-gons in which every pair of points at distance 2 have a unique common neighbour. All known generalized hexagons have order $(q, 1)$, $(1, q)$, (q, q), (q, q^3) or (q^3, q) for prime power q.

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Split Cayley hexagons $H(q)$ of order (q, q) are generalized hexagons with the group $G_2(q)$ of order $q^6(q^6 - 1)(q^2 - 1)$ as an automorphism group.

Known 1-ovoids in generalized hexagons

Every $GH(q,1)$ has 1-ovoids (since the incidence graph of $PG(2,q)$ has a perfect matching). No $GH(s,s^3)$ can have 1-ovoids (De Bruyn - Vanhove, 2013).

- $H(2)$ has 36 1-ovoids, all isomorphic under the action of $G_2(2)$, while its point-line dual $H^D(2)$ has none.
- $H(3) \cong H(3)^D$ has 3888 1-ovoids, all isomorphic under the action of $G_2(3)$.
- $H(4)$ has two non-isomorphic 1-ovoids.

See "Ovoids and Spreads of Finite Classical Generalized Hexagons and Applications" by An De Wispelaere (PhD Thesis).

New Result

Theorem (A. B. and F. Ihringer)

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The dual split Cayley hexagon of order 4 *has no* 1*-ovoids.*

For $H(4)$, Pech and Reichard proved that examples given by An are the only ones in "Enumerating Set Orbits". Exhaustive search with symmetry breaking doesn't seem to work for $H(4)^D$.

Proof

<u>**Main Idea:**</u> *Since $H(4,1)$ is a full subgeometry of $H(4)^D$, every 1-ovoid of $H(4)^D$ gives rise to a 1-ovoid of $H(4,1)$.*
So, fix a subgeometry $\mathcal{H} \cong H(4,1)$ of $H(4)^D$, compute all 1-ovoids of $\mathcal{H}$ up to equivalence under the action of $\mathrm{Stab}(\mathcal{H})$, show that none of them extends to a 1-ovoid of $H(4)^D$.

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To list 1-ovoids, use the Dancing Links algorithm by Knuth for finding exact covers. Every 1-ovoid of $H(4,1)$ corresponds to a perfect matching in the incidence graph of $PG(2,4)$ and hence there are 18534400 of them in total.

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There are 350 different 1-ovoids ($\approx$ 44 min), none of them extends to an ovoid of $H(4)^D$ ($\approx$ 1 min using LP solvers).

Semi-finite generalized polygons

Is there a generalized polygon of order (s, ∞)?

GQ's of order (s, ∞) do not exist for $s = 2, 3$ and 4 (Cameron, Brouwer/Kantor, Cherlin).

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Let $\mathcal{G}$ be a generalized hexagon of order (q, q) contained in a generalized hexagon $\mathcal{G}'$ as a full subgeometry Then points of $\mathcal{G}'$ are at distance 0, 1 or 2 from $\mathcal{G}$ giving rise to three different types of hyperplanes.

Theorem

If a generalized hexagon doesn't have 1-ovoids, then it cannot be contained in any semi-finite generalized hexagon as a full subgeometry.

Theorem (A. B. and B. De Bruyn)

A semi-finite generalized hexagon of order $(2, \infty)$ doesn't contain any subhexagons of order $(2, 2)$.

Lemma (A. B. and B. De Bruyn)

Let L be a line of $\mathcal{G}'$ that doesnt intersect $\mathcal{G}$. Then there exists an integer c_L such that for any distinct points x, y on L we have $|H_x \cap H_y| = q + 1 - c_L$.

Using this and some computations we can also handle $H(3)$ and $H(4)$.

Open Problems

1. Classify 1-ovoids in $H(5)$ and its dual.
2. For $\mathrm{char}(\mathbb{F}_q) \neq 3$, are there any 1-ovoids in $H(q)^D$?
3. Are there any semi-finite hexagons containing a subhexagon of order q? (solved for $q = 2, 3, 4$)
4. Are there any spreads in $GQ(q^2, q^3)$ obtained from the Hermitian variety $H(4, q^2)$? (solved for $q = 2$)
5. Are there any 1-ovoids in Ree-Tits octagons? (solved for order $(2, 4)$)