COMPUTING INTERSECTION NUMBERS OF CHERN CLASSES

DAN BATES, DAVID EKLUND, AND CHRIS PETERSON

Abstract. Let $Z \subset \mathbb{P}^r$ be a smooth variety of dimension n and let $c_0, \dots, c_n$ be the Chern classes of Z. We present an algorithm to compute the degree of any monomial in $\{c_0, \dots, c_n\}$. The method is based on intersection theory and may be implemented as a numeric or as a symbolic algorithm.

1. Introduction

This paper presents an algorithm to compute the degree of every monomial in the Chern classes of a smooth irreducible scheme $Z \subset \mathbb{P}^r$ from an ideal defining Z. Intersection theory, residuation, and numerical homotopy provide the theoretical support and enable implementation using symbolic tools, numeric tools or a combination. In addition, we present an algorithm to compute the degrees of the Chern classes of Z which refines the method presented in [6]. Many well known invariants of a smooth variety can be expressed in terms of the intersection numbers of Chern classes of the variety. The fundamental nature of these Chern numbers and the speed with which certain types of output could be produced by numerical homotopy methods provided the motivation to develop the algorithms found in this paper.

From the generators of the ideal of a smooth variety Z, there are known methods for computing the Chern classes of Z through a purely symbolic computation. One of the earliest effective algorithms was developed by Aluffi [2]. More generally, Aluffi's algorithm computes the push-forward to $\mathbb{P}^n$ of the Chern-Schwartz-MacPherson class of the support of a projective scheme Z' from the ideal of Z' in $\mathbb{P}^n$. If Z' is smooth, this is the same thing as computing the degrees of the Chern classes of Z'. His approach, which works for arbitrary closed subschemes of $\mathbb{P}^n$, is to reduce the computation to the hypersurface case and then to compute certain Segre classes. He has implemented the algorithm in *Macaulay2* [11] and the code is available at the author's home page. The algorithm proposed in the present paper, while also rooted in intersection theory, differs in several ways from the above description both in application and use of tools. For instance, we approach the problem through polar classes and Vogel cycles which enables the use of tools that can be implemented in both a symbolic setting and a numeric setting. One advantage of an implementation in a numeric setting is that the algorithms will work well whether the generators are sparse or dense and whether they have rational, algebraic or transcendental coefficients. This is not generally the case with purely

2000 *Mathematics Subject Classification.* 13Dxx, 13Pxx, 14Qxx, 14M06, 65H10, 65E05.
Key words and phrases. homotopy continuation, numerical algebraic geometry, polynomial system, linkage.
The first author was partially supported by NSF grant DMS-0914674. The second author was partially supported by CIAM, KTH. The third author was partially supported by the AFOSR and NSF grant DMS-0901770.

symbolic methods. Furthermore, in the numerical setting utilized in this paper, the degrees of monomials in the Chern classes can still be extracted from the generators of the ideal of the variety (with high probability) even when the generators have small inaccuracies in their coefficients. Perhaps the most important feature is that the numerical homotopy algorithms, that play a dominant role in the numerical implementation, are trivially parallelizable. This allows the algorithm to be implemented on a cluster or to take advantage of a cloud computing environment. In addition, the algorithms will benefit from an increase in the number of cores as well as from an increase in clock speed.

The paper is organized as follows. In Section 2, we provide theoretical background and results that support the algorithm to compute intersection numbers of Chern classes. The algorithm itself is presented in Section 3. Section 4 contains a collection of concrete examples illustrating both a numeric and a symbolic implementation of the algorithm. In Section 5, we formulate a separate algorithm to compute the degrees of the Chern classes. Finally, Section 6 is a collection of standard formulae involving degrees of monomials of Chern classes which serve to motivate the development of the algorithms.

2. Definitions and background

In this section we go through some results from intersection theory that we need for the computational procedure that is the subject of this paper. The following definition sets the notation that will be used in the sequel.

Definition 2.1. Let $X \subseteq \mathbb{P}^r$ be a complex projective variety of dimension n.

(1) Let $A_*(X)$ denote the Chow ring of X ($A_*(X) = \bigoplus_k A_k(X)$ where $A_k(X)$ denotes the Chow group of cycles of dimension k on X modulo rational equivalence). For $\alpha \in A_*(X)$, $\{\alpha\}_k$ denotes the component of α in $A_k(X)$.

(2) Let $\beta \in A_k(X)$ where β is represented by $\sum_{i=1}^m a_i V_i$ with $a_1, \dots, a_m \in \mathbb{Z}$ and with $V_1, \dots, V_m \subseteq X$ irreducible subvarieties of dimension k. The degree of β is defined by $\deg(\beta) = \sum_i a_i \deg(V_i)$, where $\deg(V_i)$ is the degree of the projective variety V_i. Let $H \in A_*(X)$ be the hyperplane class. Note that $\deg(\beta) = \deg(H^k\beta)$ and that if $H^k\beta$ is represented by $\sum_{i=1}^t b_i p_i$, where $p_1, \dots, p_t \in X$ and $b_1, \dots, b_t \in \mathbb{Z}$, then $\deg(\beta) = \sum_{i=1}^t b_i$.

(3) If X is smooth then we let T_X denote the tangent bundle of X. If X is regularly embedded in a variety Y then we let $N_X Y$ denote the normal bundle of X in Y.

(4) Given a vector bundle E of rank ρ on a smooth variety, $c_i(E)$ denotes the i^{th} Chern class of E, $c_i(E) = 0$ for $i > \rho$, and $c(E) = 1 + c_1(E) + \cdots + c_\rho(E)$ denotes the total Chern class of E. Suppose that X is smooth. The i^{th} Chern class of X is defined as $c_i(T_X)$ and is denoted c_i.

(5) Suppose that X is smooth. In this paper we will refer to the degree of any monomial m in the Chern classes of X as a Chern number of X. This is not the standard definition, which in addition requires that $m \in A_0(X)$. Note that if $m \in A_k(X)$ and $0 < k$, then $\deg(m)$ possibly depends on the embedding of X in a projective space.

(6) For a subscheme $Y \subseteq X$, we let $[Y] \in A_*(X)$ denote the cycle class corresponding to Y. If $X = \emptyset$, then $[X] = 0$.

2.1. **Polar classes.** Let $Z \subset \mathbb{P}^r$ be a smooth connected scheme of dimension $n < r$ defined by an ideal $J = (f_1, \dots, f_t)$. For $0 \le j \le n$ and $L \subseteq \mathbb{P}^r$ a linear subspace of dimension $(r - n - 2) + j$, let

$$Z(L) = \{z \in Z : \dim(T_z Z \cap L) \ge j - 1\}.$$

Here we set $\dim(\emptyset) = -1$. If Z has codimension 1 and $j = 0$, then L is the empty set (rather than a linear subspace). Let $l_1, \dots, l_{n-j+2}$ be linear equations defining L. There is a scheme structure on $Z(L)$ imposed by the sum of the ideal J and the ideal generated by the $(r - j + 2) \times (r - j + 2)$-minors of the Jacobian matrix of $\{f_1, \dots, f_t, l_1, \dots, l_{n-j+2}\}$. Note that the dimension of Z forces all of the $(r-n+1) \times (r-n+1)$-minors of the Jacobian matrix of $f_1, \dots, f_t$ to vanish modulo the ideal J. As a consequence, to determine the scheme structure on $Z(L)$, it is enough to take the sum of the ideal J and the ideal generated by the subcollection of $(r-j+2) \times (r-j+2)$-minors of the Jacobian matrix of $\{f_1, \dots, f_t, l_1, \dots, l_{n-j+2}\}$ that involve the final $n - j + 2$ rows. For general L, $Z(L)$ is either empty or of pure codimension j in Z and the class $[Z(L)] \in A_*(Z)$ does not depend on L. In fact, for general L,

$$[Z(L)] = \sum_{i=0}^{j} (-1)^i \binom{n - i + 1}{j - i} H^{j-i} c_i,$$

where $H \in A_{n-1}(Z)$ is the hyperplane class (see [9, Example 14.4.15]). In this setting, $Z(L)$ is called a j^{th} *polar locus* of Z and $[Z(L)]$ is called the j^{th} *polar class*.

2.2. **A recursive formula for Chern numbers.** Let $Z \subset \mathbb{P}^r$ be a smooth connected n-dimensional subvariety with Chern classes $c_0, \dots, c_n$. Let $I = (i_1 \dots i_n) \in \mathbb{N}^n$ be a vector of non-negative integers, let $||I|| = \sum_{j=1}^n j \cdot i_j$, and let $c^I = c_1^{i_1} c_2^{i_2} \dots c_n^{i_n}$. When $||I|| \le n$ then $c^I \in A_k(Z)$ where $k = n - ||I||$. Let $\mathbb{G}\mathrm{r}(l, r)$ denote the Grassmannian of projective l-planes in $\mathbb{P}^r$. Given an $I = (i_1 \dots i_n) \in \mathbb{N}^n$, let

$$\mathbb{G}_I = \prod_{\substack{1 \le j \le n \\ 1 \le l \le i_j}} \mathbb{G}\mathrm{r}(r - n + j - 2, r).$$

For each $j = 1, \dots, n$, let $L_{j1}, \dots, L_{ji_j}$ be linear spaces of dimension $r - n + j - 2$ and let $\mathbb{L} \in \mathbb{G}_I$ denote the corresponding element. Put

$$Z_I = Z_I(\mathbb{L}) = \bigcap_{\substack{1 \le j \le n \\ 1 \le l \le i_j}} Z(L_{jl}).$$

If I is the zero-vector, then $Z_I = Z$. We say that a property holds for a generic Z_I if it holds for $Z_I(\mathbb{L})$ for generic $\mathbb{L} \in \mathbb{G}_I$.

Lemma 2.2. *A generic Z_I is either empty or has the expected dimension $n - ||I||$.*

Proof. Note first that since Z is smooth, no component of a generic Z_I has dimension smaller than $n - ||I||$ (see [9, Section 8.2]). We show that for any fixed irreducible closed set $W \subseteq Z$ and for any j with $1 \le j \le n$, the polar locus $Z(L)$ intersects W properly for a general linear space L of dimension $r - n + j - 2$. It follows that the linear spaces L_{jl} may be chosen so that the successive intersections of the polar loci $Z(L_{jl})$ are all proper. We have the incidence set

$Z \times \mathbb{G}_I \supseteq \mathcal{I} = \{(z, \mathbb{L}) : z \in Z_I(\mathbb{L})\}$ with the projection $\pi : \mathcal{I} \to \mathbb{G}_I$. By upper semi-continuity of the dimension of the fibers of π, it follows that a generic Z_I has the expected dimension, or is empty.

Let $\dim(W) = k$ and $l = r - n + j - 2$. Since $\dim(Z(L)) = n - j$ we have to show that every component of $W \cap Z(L)$ has dimension $k - j$. Note that no component of $W \cap Z(L)$ has dimension less than $k - j$ since Z is smooth. We may assume that $k < j$ since otherwise we can replace W by $W \cap V$ for a general linear space V in $\mathbb{P}^r$ of codimension $k - j + 1$. In the case $k < j$, we have to show that $W \cap Z(L) = \emptyset$ for general L of dimension l. Consider the Gauss map $\gamma : Z \to \mathbb{G}\mathrm{r}(n, r)$ given by $z \mapsto T_z Z$. Then $\gamma(W)$ is an irreducible closed set of dimension at most k. For $L \in \mathbb{G}\mathrm{r}(l, r)$, the locus $\Sigma(L, j) = \{\Lambda \in \mathbb{G}\mathrm{r}(n, r) : \dim(\Lambda \cap L) \geq j - 1\}$ is an irreducible closed set of dimension $(r - n)(n + 1) - j$ (see [13, Example 11.42]). Now fix $L \in \mathbb{G}\mathrm{r}(l, r)$. The projective general linear group $\mathrm{PGL}(\mathbb{C}^{r+1})$ acts transitively on $\mathbb{G}\mathrm{r}(n, r)$ and for $g \in \mathrm{PGL}(\mathbb{C}^{r+1})$, $g\Sigma(L, j) = \Sigma(gL, j)$. By Kleiman's transversality theorem [14], a generic translate $g\Sigma(L, j)$ does not intersect $\gamma(W)$ since $\dim(\gamma(W)) + \dim(\Sigma(L, j)) \leq k + (r - n)(n + 1) - j < (r - n)(n + 1) = \dim(\mathbb{G}\mathrm{r}(n, r))$. Hence $W \cap Z(gL) = \emptyset$ for generic $g \in \mathrm{PGL}(\mathbb{C}^{r+1})$. ☐

For $I \in \mathbb{N}^n$, let $|I| = \sum_{i \in I} i$. We will use the following ordering of $\mathbb{N}^n$: $K < I$ if $|K| < |I|$, or if $|K| = |I|$, $K \neq I$, and the rightmost nonzero entry in the vector $I - K$ is positive. This is the order in which the degrees of the monomials c^I are computed in the method.

We can now reduce the computation of $\deg(c^I)$ to the computation of $\deg(Z_I)$, for a generic Z_I, and the computation of $\deg(c^K)$ for $K < I$. In other words, ordering the computation in this way leads to a triangular system which can be solved step by step in a very straightforward manner. The following proposition makes this explicit.

Proposition 2.3. *For a generic Z_I,*

$$\deg(c^I) = (-1)^{|I|}(\deg(Z_I) - \sum_{0 \leq K < I} a_K \deg(c^K)),$$

for some $a_K \in \mathbb{Z}$.

Proof. Put $a_{ij} = (-1)^i \binom{n-i+1}{j-i}$, for $0 \leq i \leq j \leq n$. Then $[Z(L_{jl})] = \sum_{i=0}^{j} a_{ij} H^{j-i} c_i$ for $j = 1, \ldots, n$ and $l = 1, \ldots, i_j$. Hence, by Lemma 2.2, the degree of Z_I is equal to the degree of the class $\Pi_{j=1}^{n} (\sum_{i=0}^{j} a_{ij} H^{j-i} c_i)^{i_j}$. Therefore, $\deg(Z_I)$ may be written as a linear combination $\sum a_K \deg(c^K)$ over some set of $K \in \mathbb{N}^n$. Since $a_{jj} = (-1)^j$ for all j, $K = I$ occurs in this sum with coefficient $(-1)^{|I|}$. It remains to see that $K < I$ for all other terms. The integers a_K should satisfy

$$\deg \left(\prod_{j=1}^{n} \left(\sum_{i=0}^{j} a_{ij} H^{j-i} c_i \right)^{i_j} \right) = \sum_{0 \leq K \leq I} a_K \deg(c^K). \tag{1}$$

Observe that

$$\left(\sum_{i=0}^{j} a_{ij} H^{j-i} c_i \right)^{i_j} = \sum_{l_{0j} + \cdots + l_{jj} = i_j} \frac{i_j!}{l_{0j}! \ldots l_{jj}!} a_{0j}^{l_{0j}} \ldots a_{jj}^{l_{jj}} H^N c_1^{l_{1j}} \ldots c_j^{l_{jj}},$$

where $0 \leq l_{ab}$ for all a, b and $N = \sum_{i=0}^{j}(j-i)l_{ij}$. Given $K \in \mathbb{N}^n$, a_K may be expressed as a sum of products, where the sum is over all upper triangular $(n \times n)$-matrices $L = (l_{ab})_{1 \leq a,b \leq n}$ of non negative integers such that the sum of its columns is K and the sum of its rows is $I - L_0$ for some $L_0 \in \mathbb{N}^n$. To such a matrix we adjoin a first row of non-negative integers $L_0 = (l_{01}, \ldots, l_{0n})$ so that the sum of the rows of the resulting $((n+1) \times n)$-matrix L' is I. Letting Δ be the set of such extended matrices we have

$$a_K = \sum_{L' \in \Delta} \prod_{j=1}^{n} \frac{i_j!}{l_{0j}!\, l_{1j}! \ldots l_{jj}!} a_{0j}^{l_{0j}} \ldots a_{jj}^{l_{jj}}. \tag{2}$$

Suppose $K = (k_1, \ldots, k_n)$ is such that $a_K \deg(c^K)$ occurs in the sum (1), so that there exists an $(n \times n)$-matrix L as above. Then $|K| \leq |I|$ since $|K|$ is the sum of all the entries of L and $|I| = |K| + |L_0|$. If $|I| = |K|$, then L_0 is the zero-vector and $k_n = l_{nn} = i_n - \sum_{j=1}^{n-1} l_{jn}$. Hence $k_n \leq i_n$, with equality exactly when $l_{1n} = \cdots = l_{n-1,n} = 0$. Continuing in the same manner, one sees that $K \leq I$, with equality exactly when L is the diagonal matrix with diagonal equal to I. □

3. Computing Chern numbers

By Proposition 2.3, we may compute the Chern numbers of a smooth projective variety Z recursively (given that we can compute $\deg(Z_I)$). As input, we assume equations defining Z. The computational tool that we need is a procedure to compute the degree of a projective scheme given equations defining it. The method presented below may be implemented using numerical homotopy methods [1] or symbolic algorithms. The former methods yield probabilistic algorithms for extracting data from systems of polynomials. For example, the module `LocalDimFinder` in *Bertini* [3] can be used to compute the degree of a scheme with high probability. If symbolic methods are used, the degree of a scheme may be computed via the Hilbert polynomial using freely available software such as *Macaulay2* or *Singular* [11, 5]. The result is a probabilistic algorithm for computing Chern numbers.

Algorithm 1. *Chern_Numbers*$(\{F_1, F_2, \ldots, F_t\})$

Input: A set of t homogeneous polynomials $F_1, F_2, \ldots, F_t$ in $\mathbb{C}[z_0, z_1, \ldots, z_r]$ that generate an ideal which defines a smooth connected scheme Z.

Output: The degrees of all the monomials in $\{c_0, \ldots, c_n\}$, where n denotes the dimension of Z and c_i denotes the i^{th} Chern class of Z.

Algorithm:

Determine the dimension of Z and store in n.
For $I = (i_1, \ldots, i_n) \in \mathbb{N}^n$, $||I|| \leq n$, in increasing order:
  For $j = 1, \ldots, n$:
    For $k = 1, \ldots, i_j$:
      Let $s = n - j + 2$.
      Let $l_1, \ldots, l_s$ be s random linear forms in $\mathbb{C}[x_0, \ldots, x_r]$.
      Compute the Jacobian matrix, J, of $(F_1, \ldots, F_t, l_1, \ldots, l_s)$.
      Compute the set M_{jk} of $(r - j + 2) \times (r - j + 2)$-minors of J.
    Next k.

For $i = 0, \ldots, j$
Let $a_{ij} = (-1)^i \binom{n-i+1}{j-i}$
Next i.
Next j.
For $K = 0, \ldots, l$
Compute a_K as in (2).
Next K.
Let N be the ideal generated by $\{F_1, \ldots, F_t\}$ and M_{jk} for all j, k.
Compute the degree D of the scheme defined by N.
Compute $\deg(c^I) = (-1)^{|I|}(D - \sum_{0 \leq K < I} a_K \deg(c^K))$.
Next I.

Remark 3.1. Let $Z \subset \mathbb{P}^r$ be a smooth connected n-dimensional subscheme defined by an ideal I. For a vector bundle E on Z, we will use $s_p(E)$ to denote the p^{th} Segre class of E. The method to compute Chern numbers presented in this section is based on the observation that the polar classes may be expressed in terms of the Chern classes. Another suite of subvarieties that can be used for this purpose are certain loci associated with the conormal bundle of Z in $\mathbb{P}^r$, which we denote by $(N_Z\mathbb{P}^r)^*$. Let m be an integer, let L be the line bundle on Z that corresponds to the sheaf $\mathcal{O}_Z(m)$ and put $E = (N_Z\mathbb{P}^r)^* \otimes L$. Suppose m is big enough that E is globally generated and Z is cut out as a scheme by degree m polynomials in I. Let $s_1, \ldots, s_{r-1}$ be global sections of E. For $j \leq r-1$ and $z \in Z$, let $\langle s_1(z), \ldots, s_j(z)\rangle$ denote the span of $\{s_1(z), \ldots, s_j(z)\}$. For $1 \leq p \leq n$, let

$$\Omega_p = \{z \in Z : \dim(\langle s_1(z), \ldots, s_{r-n+p-1}(z)\rangle) < r-n\}.$$

There is a scheme structure on Ω_p defined by the vanishing of appropriate determinants. Then, for generic $s_1, \ldots, s_{r-1}$, Ω_p has pure codimension p and $[\Omega_p] = (-1)^p s_p(E)$ (see [9, Example 14.3.2]). By applying the conormal sequence and [9, Example 3.1.1], one can show that

$$[\Omega_p] = \sum_{i=0}^{p} a_{ip} H^{p-i} c_i,$$

where $a_{ip} = \sum_{l=0}^{p-i} (-1)^l \binom{r+l}{l}\binom{p-n+r-1}{p-i-l} m^{p-i-l}$. Equations for Ω_p are given by the ideal I together with the $(r-n) \times (r-n)$-minors of the Jacobian matrix of $(f_1, \ldots, f_{r-n+p-1})$, where $f_1, \ldots, f_{r-1}$ are generic elements of I of degree m.

The classes $[\Omega_p]$ may also be viewed as pieces of the *Vogel cycle* associated to $\mathbb{P}^r$ and sections of $\mathcal{O}_{\mathbb{P}^r}(m)$ that define Z (see [8, Section 2.1]).

4. Examples

In this section, we provide examples utilizing Algorithm 1 to compute the full set of Chern numbers of smooth varieties embedded in a projective space. The examples illustrate both a numeric implementation and a symbolic implementation. In the examples, if Z is an n-fold then I is an ordered n-tuple, Z_I is an intersection of polar loci with the number of each type of polar locus determined by the entries of I, and c^I is a monomial in the Chern classes written in multinomial notation. For instance, if $Z \subseteq \mathbb{P}^r$ is a threefold and if $I = (1, 2, 3)$ then

$$Z_I = Z(L_1) \cap Z(L_2) \cap Z(L_2') \cap Z(L_3) \cap Z(L_3') \cap Z(L_3'')$$

and $c^I = c_1 \cdot c_2^2 \cdot c_3^3$, where $\dim(L_1) = r-4$, $\dim(L_2) = \dim(L_2') = r-3$ and $\dim(L_3) = \dim(L_3') = \dim(L_3'') = r-2$. To avoid confusion, we note in passing that $c_1 \cdot c_2^2 \cdot c_3^3 = 0$ on a threefold. If I is the n-tuple all of whose entries are zero then we let $Z_I = Z$ and $c^I = c_0$.

4.1. **Surfaces.** We illustrate Algorithm 1 in the case of a smooth connected surface $Z \subset \mathbb{P}^r$. Due to the simplicity of the case when $r = 3$ (the hypersurface case), we will assume $r \geq 4$. Let L_0, L_1, L_2 be generic linear spaces with $\dim(L_j) = (r-4)+j$. Let $H \in A_*(Z)$ denote the hyperplane class. Then

$$\begin{aligned}
[Z(L_0)] &= c_0, \\
[Z(L_1)] &= 3Hc_0 - c_1 = 3H - c_1, \\
[Z(L_2)] &= 3H^2c_0 - 2Hc_1 + c_2 = 3H^2 - 2Hc_1 + c_2.
\end{aligned}$$

Note that $\deg(H^2) = \deg(c_0) = \deg(Z)$. Clearly we can solve for the degrees of c_0, c_1 and c_2 as soon as we have computed the degrees of $Z(L_0)$, $Z(L_1)$ and $Z(L_2)$. Now we turn to c_1^2. Suppose L_1' is another linear space of dimension $r-3$ sufficiently generic that $Z(L_1')$ has codimension 1 in Z and intersects $Z(L_1)$ properly. Then

$$[Z(L_1) \cap Z(L_1')] = (3H - c_1)(3H - c_1) = 9H^2 - 6Hc_1 + c_1^2,$$

and we may solve for $\deg(c_1^2)$ from the degrees of Z, c_1 and $Z(L_1) \cap Z(L_1')$.

Example 4.1. The following example of a surface in $\mathbb{P}^4$ is taken from [4, Appendix B 6.2]. It can be constructed from the Horrocks-Mumford rank 2 bundle E on $\mathbb{P}^4$ as follows. Let S be a degree 10 Abelian surface arising as the vanishing locus of a regular section of E. Take the residual of S in a complete intersection of two quintics from the ideal of S to obtain a degree 15 surface Z. Then Z is a quasi-complete intersection (i.e. it is cut out as a scheme by three equations) and the ideal J of Z is generated in degree 5. A numerical implementation of Algorithm 1 using *Bertini* [3] determined that

$$\deg(Z(L_0)) = 15, \deg(Z(L_1)) = 70, \deg(Z(L_2)) = 120, \deg(Z(L_1) \cap Z(L_1')) = 260.$$

This leads to the following triangular system of linear equations:

$$\begin{array}{rllll}
\deg(H^2) & & & & = 15 \\
3\deg(H^2) & -\deg(c_1 \cdot H) & & & = 70 \\
3\deg(H^2) & -2\deg(c_1 \cdot H) & +\deg(c_2) & & = 120 \\
9\deg(H^2) & -6\deg(c_1 \cdot H) & & +\deg(c_1^2) & = 260.
\end{array}$$

This system has the unique solution

$$\deg(Z) = \deg(H^2) = 15,\ \deg(c_1) = -25,\ \deg(c_2) = 25,\ \deg(c_1^2) = -25.$$

We summarize this data in the following table.

I	Z_I	$\deg(Z_I)$	c^I	$\deg(c^I)$
$(0,0)$	$Z(L_0) = Z$	15	c_0	15
$(1,0)$	$Z(L_1)$	70	c_1	-25
$(0,1)$	$Z(L_2)$	120	c_2	25
$(2,0)$	$Z(L_1) \cap Z(L_1')$	260	c_1^2	-25

Example 4.2. The following example of a surface in $\mathbb{P}^4$ is taken from [4, Appendix B 4.15] and was originally constructed in [16]. The surface $Z \subset \mathbb{P}^4$ is a non-minimal K3-surface, i.e. a birational model of a K3-surface, which has maximal degree

among the known non-minimal K3-surfaces in $\mathbb{P}^4$. The ideal J of Z is generated in degrees 5 and 6. The table shows the result of executing Algorithm 1, implemented in a symbolic setting, using *Macaulay2* [11] (over a finite field).

I	$\deg(Z_I)$	c^I	$\deg(c^I)$
$(0,0)$	14	c_0	14
$(1,0)$	64	c_1	-22
$(0,1)$	125	c_2	39
$(2,0)$	243	c_1^2	-15

The final column of the table is found by solving a system of equations (like in the previous example).

4.2. **Threefolds.** We illustrate Algorithm 1 in the case of a smooth connected threefold $Z \subset \mathbb{P}^r$. Due to the simplicity of the case when $r = 4$ (the hypersurface case), we will assume $r \geq 5$. Let L_0, L_1, L_2, L_3 be generic linear spaces with $\dim(L_j) = (r-5)+j$. Let $H \in A_*(Z)$ denote the hyperplane class. Then

$$\begin{aligned}
&[Z(L_0)] = c_0,\\
&[Z(L_1)] = 4Hc_0 - c_1 = 4H - c_1,\\
&[Z(L_2)] = 6H^2c_0 - 3Hc_1 + c_2 = 6H^2 - 3Hc_1 + c_2,\\
&[Z(L_3)] = 4H^3c_0 - 3H^2c_1 + 2Hc_2 - c_3 = 4H^3 - 3H^2c_1 + 2Hc_2 - c_3.
\end{aligned}$$

Note that $\deg(H^3) = \deg(c_0) = \deg(Z)$. Clearly we can solve for the degrees of c_0, c_1, c_2 and c_3 as soon as we have computed the degrees of $Z(L_0)$, $Z(L_1)$, $Z(L_2)$ and $Z(L_3)$. The other monomials in the Chern classes are c_1^2, c_1c_2 and c_1^3. Their degrees can be determined from the degrees of $Z(L_1) \cap Z(L_1')$, $Z(L_1) \cap Z(L_2)$ and $Z(L_1) \cap Z(L_1') \cap Z(L_1'')$ where L_1' and L_1'' are generic linear spaces of dimension $r-4$. Altogether, we have the following triangular system of equations

$$\begin{aligned}
&\deg([Z(L_0)]) && = H^3\\
&\deg([Z(L_1)]) && = 4H^3 - H^2c_1\\
&\deg([Z(L_2)]) && = 6H^3 - 3H^2c_1 + Hc_2\\
&\deg([Z(L_3)]) && = 4H^3 - 3H^2c_1 + 2Hc_2 - c_3\\
&\deg([Z(L_1) \cap Z(L_1')]) && = 16H^3 - 8H^2c_1 + Hc_1^2\\
&\deg([Z(L_1) \cap Z(L_2)]) && = 24H^3 - 18H^2c_1 + 4Hc_2 + 3Hc_1^2 - c_1c_2\\
&\deg([Z(L_1) \cap Z(L_1') \cap Z(L_1'')]) && = 64H^3 - 48H^2c_1 + 12Hc_1^2 - c_1^3.
\end{aligned}$$

Example 4.3. Let J be an ideal generated by all (4×4)-minors of a (4×5)-matrix of generic linear forms in $\mathbb{C}[x_0, \ldots, x_5]$ and let $Z \subset \mathbb{P}^5$ be the 3-fold defined by J. The table shows the result of executing Algorithm 1 implemented in [3].

I	$\deg(Z_I)$	c^I	$\deg(c^I)$
$(0,0,0)$	10	c_0	10
$(1,0,0)$	40	c_1	0
$(0,1,0)$	105	c_2	45
$(0,0,1)$	176	c_3	-46
$(2,0,0)$	155	c_1^2	-5
$(1,1,0)$	381	c_1c_2	24
$(3,0,0)$	586	c_1^3	-6

The final column of the table is obtained by solving the triangular system of equations determined by the second column.

5. The degrees of the Chern classes

In this section, we shall formulate a method to compute the degrees of the Chern classes which is a modification, and an improvement, of the procedure described in [6]. We first need to go through some results on generic residual intersections and derive a recursive formula for the degrees of the Chern classes.

Let I be a homogeneous ideal defining a scheme $X \subset \mathbb{P}^r$, let I_d denote the degree d component of I, and let $\bar{I}$ denote the saturation of I with respect to the maximal ideal. Thus $\bar{I}$ denotes the homogeneous ideal, I_X, of X. For $d \in \mathbb{N}$, we say that X is cut out as a scheme by degree d polynomials in I if the saturation (with respect to the maximal ideal) of the ideal generated by I_d is equal to I_X.

In the following lemma and its proof, we let $\dim(\emptyset) = -\infty$.

Lemma 5.1. *Let $I \subset \mathbb{C}[x_0, \ldots, x_r]$ be an ideal that defines a smooth equidimensional scheme $Z \subset \mathbb{P}^r$ of dimension n. Fix integers $(n_1, \ldots, n_r)$ such that for each i, Z is cut out as a scheme by degree n_i polynomials in I. For $G_i \in I_{n_i}$, let $X_i = \{G_i = 0\}$ and for $p \in Z$, let T_pX_i denote the tangent space of X_i at p. For $0 < i \leq r$ and $0 \leq k < r - n$, put*

$$T(i, k) = \{p \in Z : \dim(T_pX_1 \cap \cdots \cap T_pX_i) \geq r - k\}.$$

Then, for general forms $(G_1, \ldots, G_r)$ in I with $\deg(G_i) = n_i$,

$$\dim(T(i, k)) \leq n + k - i.$$

Proof. The bound, $\dim(T(i, k)) \leq n + k - i$, is obvious for $i \leq k$. We proceed by induction on i. In the case $i = 1$ and $k = 0$, the claim is that for general $G_1 \in I_{n_1}$, X_1 is nonsingular at some point of Z. This is clear since the ideal generated by I_{n_1} defines the smooth scheme Z. Now fix i, $0 < i < r$, and suppose that $\dim(T(i, k)) \leq n + k - i$ for $0 \leq k < r - n$. Note that $T(i + 1, k) \subseteq T(i, k)$ and $T(i, k - 1) \subseteq T(i, k)$. If $0 < k < r - n$, then any irreducible component of $T(i, k)$ which is contained in $T(i, k - 1)$ has dimension less than or equal to $\dim(T(i, k - 1)) \leq n + k - 1 - i = n + k - (i + 1)$, and thus such a component already satisfies the bound in the case $(i + 1, k)$. Let $0 \leq k < r - n$ and let $C \subseteq T(i, k)$ be an irreducible component (if $T(i, k)$ is empty we are done). Suppose that $k = 0$ or that C is not contained in $T(i, k - 1)$. It follows that there is a $p \in C$ with $\dim(T_pX_1 \cap \cdots \cap T_pX_i) = r - k$, which is bigger than n since we have assumed that $k < r - n$. Since Z is smooth at p, we have that for a general $G_{i+1} \in I_{n_{i+1}}$, the tangent space T_pX_{i+1} does not contain $T_pX_1 \cap \cdots \cap T_pX_i$. In other words, we have $\dim(C \cap T(i + 1, k)) < \dim(C)$. Thus, for a general G_{i+1}, we have that $\dim(T(i + 1, k)) \leq n + k - (i + 1)$. □

Theorem 5.2. *Let $I \subset \mathbb{C}[x_0, \ldots, x_r]$ be a non-zero ideal, that defines a smooth equidimensional scheme $Z \subset \mathbb{P}^r$, and let $\{F_1, \ldots, F_t\}$ be a set of non-zero generators of I. Put $d = \max\{\deg(F_1), \ldots, \deg(F_t)\}$ and fix integers $n_1, \ldots, n_r$ with $n_i \geq d$ for all i. If $G_1, \ldots, G_r$ are general forms in I with $\deg(G_i) = n_i$, then the ideal $J = (G_1, G_2, \ldots, G_r)$ defines a disjoint union $Z \cup S$, where S is either empty or a reduced zero-scheme.*

Proof. Observe that since $d \leq n_i$, the ideal generated by I_{n_i} is equal to $\bigoplus_{e \geq n_i} I_e$. Since the ideal $\bigoplus_{e \geq n_i} I_e$ and I have the same saturation, namely the homogeneous ideal of Z, the scheme Z is cut out by degree n_i polynomials in I.

Consider first the set theoretic content of the theorem, that is that for general forms $G_1,\dots,G_r$ in I with $\deg(G_i)=n_i$ the zeroset of the ideal $J=(G_1,\dots,G_r)$ is $Z\cup S$ where S is finite. If $G_i\in I_{n_i}$ we let $X_i=\{G_i=0\}$. A general $G_1\in I_{n_1}$ is non-zero, and thus $X_1\setminus Z$ is empty or has codimension 1 in $\mathbb{P}^r$. Since the common zeros of I_{n_2} is Z, a general $G_2\in I_{n_2}$ is such that $(X_1\cap X_2)\setminus Z$ is empty or has codimension 2 in $\mathbb{P}^r$. By induction, the residual S to Z at the r^{th} step is finite.

We proceed to the scheme theoretic content of the theorem. First we shall argue that S is reduced. We will use a scheme theoretic version of Bertini's theorem, see [8, Corollary 3.4.9]. For a general $G_1\in I_{n_1}$, the hypersurface X_1 is smooth outside Z (by Bertini's theorem applied to the linear system on $\mathbb{P}^r\setminus Z$ given by I_{n_1}). Similarly, I_{n_2} defines a base point free linear system on $X_1\setminus Z$, and thus for a general $G_2\in I_{n_2}$, $(X_1\cap X_2)\setminus Z$ is smooth. By induction, the residual S at the r^{th} step is reduced.

Let $n=\dim(Z)$ and let X denote the scheme defined by J, that is $X=X_1\cap\cdots\cap X_r$. It remains to show that for general $(G_1,\dots,G_r)$, the X_i intersect smoothly in Z, or in other words that for $p\in Z$ the local ring of X at p is regular. By the Jacobian Criterion, see [7, Corollary 16.20], this is the same as showing that the Jacobian matrix of $(G_1,\dots,G_r)$ has rank $r-n$ at p. With notation as in Lemma 5.1, this is equivalent to $T(r,r-n-1)=\emptyset$, which is implied by the lemma. □

Suppose that $Z\subset\mathbb{P}^r$ is a smooth, connected, n-dimensional variety defined by an ideal $I=(F_1,\dots,F_t)\subseteq\mathbb{C}[z_0,\dots,z_r]$, where $F_i\neq 0$ for all i. Let $c_0,\dots,c_n$ be the Chern classes of Z and let $d=\max\{\deg(F_1),\dots,\deg(F_t)\}$. Let $L\subseteq\mathbb{P}^r$ be a linear subspace of dimension s, where $r-n\le s\le r$, defined by linear forms $H_{s+1},\dots,H_r$. Let $k=s+n-r$. Note that for general L, $Z\cap L$ is equidimensional and smooth by Bertini's theorem. Moreover, if $0<k$, $Z\cap L$ is connected for general L by [12, Corollary III.7.9]. For generic forms $G_1,\dots,G_s\in I_d$ and generic linear forms $H_{s+1},\dots,H_r$, the ideal generated by $\{G_1,\dots,G_s,H_{s+1},\dots,H_r\}$ defines a disjoint union $(Z\cap L)\cup S$, where S is empty or a reduced zero scheme. This follows by applying Theorem 5.2 with Z replaced by $Z\cap L$, I replaced by the image of I in $\mathbb{C}[x_0,\dots,x_r]/(H_{s+1},\dots,H_r)$ and $\mathbb{P}^r$ replaced by $L\cong\mathbb{P}^s$. In a similar vein to Proposition 2.3, we reduce the computation of $\deg(c_k)$ to the computation of $\deg(S)$ and $\deg(c_i)$ for $i<k$.

Proposition 5.3. *With notation as above,*

$$\deg(c_k)=d^s-\deg(S)-\sum_{i=0}^{k-1}b_{ik}\deg(c_i),$$

where $b_{ik}=\sum_{l=0}^{k-i}(-1)^l\binom{r+l}{l}\binom{s}{k-i-l}d^{k-i-l}$ *and* $k=s+n-r$.

Proof. If $k=0$ the statement is that $\deg(Z)+\deg(S)=d^{r-n}$, which is a consequence of Bézout's theorem.

Suppose $0<k$. Let $X_i=\{G_i=0\}\cap L$ and let $(X_1\cdot\ldots\cdot X_s)^{Z\cap L}$ denote the equivalence of $Z\cap L$ in $(X_1\cdot\ldots\cdot X_s)$, see [9, Chapter 9]. Let N_i denote the restriction to $Z\cap L$ of the normal bundle of X_i in $L\cong\mathbb{P}^s$. By [9, Proposition 9.1.1] (with Z replaced by $Z\cap L$ and $Y=V=L\cong\mathbb{P}^s$),

$$(X_1\cdot\ldots\cdot X_s)^{Z\cap L}=\{(\Pi_{i=1}^s c(N_i))c(T_{\mathbb{P}^s}|_{Z\cap L})^{-1}c(T_{Z\cap L})\}_0.$$

By [9, Proposition 9.1.2],

$$\deg((X_1\cdot\ldots\cdot X_s)^{Z\cap L})+\deg(S)=d^s.$$

We shall rewrite the latter formula in terms of the Chern classes of Z. Note that the divisor class on $Z\cap L$ which corresponds to N_i is $dH_{Z\cap L}$, where $H_{Z\cap L}\in A_*(Z\cap L)$ is the hyperplane class. Also, $c(T_{\mathbb{P}^s}|_{Z\cap L})=(1+H_{Z\cap L})^{s+1}$. Now let $i: Z\cap L\to Z$ be the inclusion. Since we have an exact sequence

$$0\to T_{Z\cap L}\to i^*T_Z\to N_{Z\cap L}Z\to 0,$$

we have that $c(T_{Z\cap L})=c(i^*T_Z)c(N_{Z\cap L}Z)^{-1}$. Finally, observe that $H_{Z\cap L}=i^*H_Z$, with $H_Z\in A_*(Z)$ the hyperplane section, and $c(N_{Z\cap L}Z)=(1+H_{Z\cap L})^{r-s}=(1+i^*H_Z)^{r-s}$. Collecting these observations,

$$(X_1\cdot\ldots\cdot X_s)^{Z\cap L}=\{(1+di^*H_Z)^s(1+i^*H_Z)^{-(s+1)}c(i^*T_Z)(1+i^*H_Z)^{s-r}\}_0.$$

Let $\omega=(1+dH_Z)^s(1+H_Z)^{-(r+1)}c(T_Z)$. Using that $i^*: A_*(Z)\to A_*(Z\cap L)$ is a ring homomorphism and $c(i^*T_Z)=i^*c(T_Z)$ we get $(X_1\cdot\ldots\cdot X_s)^{Z\cap L}=\{i^*\omega\}_0$. Since degrees are preserved under pushforward by i,

$$\begin{aligned}\deg((X_1\cdot\ldots\cdot X_s)^{Z\cap L})&=\deg(i_*\{i^*\omega\}_0)=\deg(i_*i^*\{\omega\}_{r-s})=\\ \deg([Z\cap L]\{\omega\}_{r-s})&=\deg(H_Z^{r-s}\{\omega\}_{r-s})=\deg(\{\omega\}_{r-s}).\end{aligned}$$

In conclusion, $d^s-\deg(S)=\deg(\{\omega\}_{r-s})$ and since $b_{kk}=1$ it remains to check that $\deg(\{\omega\}_{r-s})=\sum_{i=0}^k b_{ik}\deg(c_i)$. For this we write out ω,

$$\omega=(\sum_{j=0}^{s}\binom{s}{j}d^jH_Z^j)(\sum_{l=0}^{n}(-1)^l\binom{r+l}{l}H_Z^l)(\sum_{i=0}^{n}c_i).$$

The $(r-s)$-dimensional component of ω is a cycle class of codimension $n-(r-s)=k$. If $0\le i\le k$, the terms in $\{\omega\}_{r-s}$ of codimension k that involve c_i correspond to $j=k-i-l$. Summing over l, we get $\sum_{l=0}^{k-i}(-1)^l\binom{r+l}{l}\binom{s}{k-i-l}d^{k-i-l}=b_{ik}$. □

The above recursive formula for the degrees of the Chern classes of Z leads to the following probabilistic algorithm for computing these numbers. The procedure is to repeatedly take collections of random hypersurfaces of high enough degree that contain Z, and then compute the degree of the residual to Z in their intersection. We start with the codimension of Z number of such hypersurfaces. We then step by step increase the number of hypersurfaces by one until their number is equal to r, the dimension of the ambient space. At each step, the degree of a Chern class may be computed using the recursive formula.

Algorithm 2. *Degrees_of_Chern_Classes*$(\{F_1,F_2,\ldots,F_t\})$

Input: A set of homogeneous polynomials $\{F_1,F_2,\ldots,F_t\}\subset\mathbb{C}[z_0,z_1,\ldots,z_r]$, all non-zero, that generate an ideal J which defines a smooth connected scheme Z.

Output: $\{\deg(c_0),\ldots,\deg(c_n)\}$ where n denotes the dimension of Z and c_i denotes the i^{th} Chern class of Z.

Algorithm:
Determine the dimension of Z and store in n.
Let $d=\max\{\deg(F_1),\ldots,\deg(F_t)\}$.
For $k=0$ to n.
  Let $s=r-n+k$.
  Let $G_1,\ldots,G_s$ be s random elements in J_d.

Let $H_{s+1}, \dots, H_r$ be $r-s$ random linear forms.
Let Y be the scheme defined by $(G_1, \dots, G_s, H_{s+1}, \dots, H_r)$.
Compute the number of isolated points D of $Y \setminus Z$.
For $i = 0, \dots, k$
Compute $b_{ik} = \sum_{l=0}^{k-i} (-1)^l \binom{r+l}{l} \binom{s}{k-i-l} d^{k-i-l}$.
Next i
Compute $\deg(c_k) = d^s - D - \sum_{i=0}^{k-1} b_{ik} \deg(c_i)$.
Next k.

In connection with Algorithm 2 we remark that:

(1) If $k > 0$ then no point of the intersection with Z, of a linear space defined by generic linear forms $H_{s+1}, \dots, H_r$, is isolated. The number D is thus simply the number of isolated points of Y (except in the case $k = 0$ where we have to compute the number of isolated points of Y which are not on Z). Note that the number D is the degree of the residual to Z in the scheme defined by $G_1 = \cdots = G_s = 0$.
(2) Algorithm 2 improves the method to compute the degrees of Chern classes presented in [6]. If homotopy methods are used then fewer paths are tracked in Algorithm 2 than in the method found in [6]. If symbolic methods are used then we note that, in Algorithm 2, we don't take any elements from the ideal of degree higher than d whereas the method presented in [6] involves elements of degree $d+1$. In symbolic computations, this difference in degree can be the determining factor in whether a computation terminates.

6. Motivation

Many well known invariants of a smooth variety can be expressed in terms of the Chern numbers of the variety. The fundamental nature of Chern numbers and the speed with which certain types of output could be produced by numerical homotopy methods provided the motivation to develop Algorithm 1. For reference, we collect some standard formulae that involve the degree of monomials in the Chern classes of a variety.

6.1. **Double point formulae.** Let $Z \subset \mathbb{P}^r$ be a smooth variety of dimension n. Suppose, for a moment, that $2n+1 < r$. The secant variety of Z has dimension at most $2n+1$ and hence, by projecting from a generic point, we can embed Z in a projective space of smaller dimension. Every smooth projective variety of dimension n may thus be embedded in $\mathbb{P}^{2n+1}$. If $r < 2n+1$ the invariants of Z satisfy certain constraints. As a consequence, in these cases a procedure to compute the Chern numbers can be simplified since only some of these numbers need to be computed and the others may be derived from the constraint formulae. The purpose of this section is to go through the derivation of these formulae and give them explicitly for surfaces in $\mathbb{P}^4$ and threefolds in $\mathbb{P}^5$ and $\mathbb{P}^6$.

Recall first that $c(\mathbb{P}^r) = (1+h)^{r+1}$, where h is the hyperplane class. Let $N = N_Z\mathbb{P}^r$ and let $i : Z \hookrightarrow \mathbb{P}^r$ be the inclusion. We have an exact sequence

$$0 \to T_Z \to i^* T_{\mathbb{P}^r} \to N \to 0,$$

and hence $c(i^*T_{\mathbb{P}^r}) = c(T_Z)c(N)$. Note that $c(i^*T_{\mathbb{P}^r}) = c(T_{\mathbb{P}^r}|_Z) = (1+H)^{r+1}$, where $H \in A_*(Z)$ is the hyperplane class. Hence, for $0 \le k \le r+1$,

$$\binom{r+1}{k} H^k = \sum_{i=0}^{k} c_i(N) c_{k-i}. \tag{3}$$

On the other hand, by [12, A.C7],

$$c_{r-n}(N) = i^*(i_*([Z])).$$

Since $i_*([Z]) = \deg(Z) h^{r-n}$, it follows that

$$c_{r-n}(N) = \deg(Z) H^{r-n}. \tag{4}$$

Equation (3) gives an expression for $c_k(N)$ in terms of H, the Chern classes of Z and $c_i(N)$, $i < k$. For $k = 1$, we get $c_1(N) = (r+1)H - c_1$, and hence by induction we can write $c_{r-n}(N)$ in terms of H and the Chern classes of Z. Combining this with (4), we get a relation among H and the Chern classes of Z. If $r < 2n+1$ this relation is nontrivial since c_{r-n} only appears once in the derivation, namely in (3) for $k = r - n$.

Remark 6.1. Here are some low dimensional cases of the numerical conditions that follow from (3) and (4):

n	r	Formula
2	4	$\deg(Z)^2 - 10\deg(Z) + \deg(c_2) + 5\deg(c_1) - \deg(c_1^2) = 0$
3	5	$15\deg(Z) - \deg(c_2) - 6\deg(c_1) + \deg(c_1^2) - \deg(Z)^2 = 0$
3	5	$15\deg(c_1) - \deg(c_1c_2) - 6\deg(c_1^2) + \deg(c_1^3) - \deg(Z)\deg(c_1) = 0$
3	6	$35\deg(Z) - \deg(c_3) - 7\deg(c_2) + 2\deg(c_1c_2) - 21\deg(c_1) + 7\deg(c_1^2) - \deg(c_1^3) - \deg(Z)^2 = 0$

Proof. For Z a surface in $\mathbb{P}^4$, (3) and (4) give

$$\begin{aligned} 5H &= c_1 + c_1(N), \\ 10H^2 &= c_2 + c_1c_1(N) + c_2(N), \\ c_2(N) &= \deg(Z)H^2. \end{aligned}$$

Eliminating $c_1(N)$ and $c_2(N)$ and taking degrees the first row of the table follows.

For a threefold $Z \subset \mathbb{P}^5$, we get

$$\begin{aligned} 6H &= c_1 + c_1(N), \\ 15H^2 &= c_2 + c_1c_1(N) + c_2(N), \\ c_2(N) &= \deg(Z)H^2. \end{aligned}$$

This implies that

$$15H^2 - c_2 - 6Hc_1 + c_1^2 - \deg(Z)H^2 = 0.$$

Multiplying both sides by H and taking degrees gives the second row in the table. The third row follows from multiplying both sides by c_1 and taking degrees.

The formula for a threefold in $\mathbb{P}^6$ is shown in the same manner. □

Remark 6.2. By Remark 6.1, a surface Z in $\mathbb{P}^4$ satisfies

$$\deg(Z)^2 - 10\deg(Z) + \deg(c_2) + 5\deg(c_1) - \deg(c_1^2) = 0.$$

Severi's double point formula [17] says that the quantity on the left hand side, for a smooth surface in $Z \subset \mathbb{P}^5$, is the number of secant lines that go through a general point of $\mathbb{P}^5$.

6.2. **Chern numbers and the Hilbert polynomial.** Let $P_Z \in \mathbb{Q}[x]$ be the Hilbert polynomial of a smooth n-dimensional variety $Z \subset \mathbb{P}^r$ and let $H \in A_{n-1}(Z)$ be the hyperplane section. Recall our definition of a Chern number as the degree of any monomial in the Chern classes of Z. Using the Hirzebruch-Riemann-Roch formula (see [12, Theorem A.4.1]), one can express the coefficients of P_Z in terms of the Chern numbers of Z.

We shall first introduce some notation which may be found in [12, Appendix A]. Define the Chern polynomial c_t by $c_t = c_0 + tc_1 + t^2c_2 + \cdots + t^n c_n$. Introduce formal symbols a_i by $c_t = \Pi_{i=1}^n (1 + a_i t)$. The Todd class is defined as

$$\mathrm{td}(T_Z) = \prod_{i=1}^{n} \frac{a_i}{1 - e^{-a_i}}.$$

Here $\frac{x}{1-e^{-x}} = 1 + \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \ldots$, and hence the Todd class begins

$$\mathrm{td}(T_Z) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 - \frac{1}{720}(c_1^4 - 4c_1^2c_2 - 3c_2^2 - c_1c_3 + c_4) + \ldots$$

Let $x \in \mathbb{Z}$. Using the Hirzebruch-Riemann-Roch formula, we may now write the value of the Hilbert polynomial at x in terms of Chern numbers,

$$P_Z(x) = \chi(xH) = \deg(\{\mathrm{ch}(xH) \cdot \mathrm{td}(T_Z)\}_n),$$

where

$$\mathrm{ch}(xH) = \sum_{i=0}^{n} \frac{(xH)^i}{i!},$$

and $\{q\}_n$ denotes the component of degree n of an element q in the graded ring $A_*(Z) \otimes \mathbb{Q}$.

Remark 6.3. We give the formula explicitly in the cases where Z is a curve, surface and threefold, respectively:

$$P_Z(x) = \deg(Z)x + \frac{\deg(c_1)}{2},$$
$$P_Z(x) = \frac{\deg(Z)}{2}x^2 + \frac{\deg(c_1)}{2}x + \frac{1}{12}(\deg(c_1^2) + \deg(c_2)),$$
$$P_Z(x) = \frac{\deg(Z)}{6}x^3 + \frac{\deg(c_1)}{4}x^2 + \frac{1}{12}(\deg(c_1^2) + \deg(c_2))x + \frac{1}{24}\deg(c_1c_2).$$

The following example shows that the Chern numbers of a smooth variety in $\mathbb{P}^r$ cannot in general be computed from its Hilbert polynomial.

Example 6.4. This is an example, which appeared in [10], of two embedded projective varieties with the same Hilbert polynomial but different Chern numbers. Let $Z_1 \subset \mathbb{P}^5$ be $\mathbb{P}^1 \times \mathbb{P}^1$ embedded with $\mathcal{O}(1,2)$ and let $Z_2 \subset \mathbb{P}^5$ be $\mathbb{P}^2$ embedded by $\mathcal{O}(2)$ (the 2-Veronese surface). In other words, $Z_1 = \mathrm{im}(\phi_1)$ and $Z_2 = \mathrm{im}(\phi_2)$, where

$$\phi_1 : \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^5 : (x, y; z, w) \mapsto (xz^2, yz^2, xzw, yzw, xw^2, yw^2),$$
$$\phi_2 : \mathbb{P}^2 \hookrightarrow \mathbb{P}^5 : (x, y, z) \mapsto (x^2, xy, xz, y^2, yz, z^2).$$

We shall verify that

$$P_{Z_1}(x) = P_{Z_2}(x) = 2x^2 + 3x + 1,$$

but that $\deg(c_2(T_{Z_1})) = 4$ while $\deg(c_2(T_{Z_2})) = 3$.

The arithmetic genus p_a of a smooth surface is a birational invariant and since Z_1 and Z_2 are rational, $p_a(Z_1) = p_a(Z_2) = 0$. Let L_1 and L_2 be the divisor classes on Z_1 defined by lines of the two rulings of $\mathbb{P}^1 \times \mathbb{P}^1$ and let L be the divisor class on Z_2 corresponding to $\mathcal{O}_{\mathbb{P}^2}(1)$. Then $L_1^2 = L_2^2 = 0$ and $L_1 L_2 = L^2 = 1$. The canonical classes are given by $K_{Z_1} = -2(L_1 + L_2)$ and $K_{Z_2} = -3L$. Observe that $c_1 = -K$, that Z_1 is embedded by $L_1 + 2L_2$ and that Z_2 is embedded by $2L$. Hence

$$\deg(c_1(T_{Z_1})) = \deg((L_1+2L_2)(2L_1+2L_2)) = 6, \quad \deg(c_1(T_{Z_2})) = \deg(2L\cdot 3L) = 6,$$

$$\deg(Z_1) = \deg((L_1 + 2L_2)^2) = 4, \quad \deg(Z_2) = \deg((2L)^2) = 4.$$

By Remark 6.3, and the fact that $P_{Z_i}(0) = p_a(Z_i) + 1$, it follows that

$$P_{Z_1}(x) = P_{Z_2}(x) = 2x^2 + 3x + 1.$$

However,

$$\deg(c_1(T_{Z_1})^2) = \deg((2L_1 + 2L_2)^2) = 8, \quad \deg(c_1(T_{Z_2})^2) = \deg((3L)^2) = 9.$$

The constant term of the Hilbert polynomial of a surface is $\frac{1}{12}(\deg(c_1^2) + \deg(c_2))$, thus we have $12 = \deg(c_1^2) + \deg(c_2)$ for both Z_1 and Z_2. It follows that

$$\deg(c_2(T_{Z_1})) = 4, \quad \deg(c_2(T_{Z_2})) = 3.$$

Remark 6.5. The Hirzebruch-Riemann-Roch formula shows that the Hilbert polynomial of a variety can be obtained from the Chern numbers of the variety. Example 6.4 illustrates that there are situations where the Chern numbers are not computable from the Hilbert polynomial. Generally speaking, this leads us to conclude that the Chern numbers are a finer set of invariants than the Hilbert polynomial. However, in the special setting of surfaces in $\mathbb{P}^4$ and threefolds in $\mathbb{P}^5$, the extra constraints found in Remark 6.1 allow one to recover the Chern numbers from the Hilbert polynomial.

6.3. **Some common invariants.** Let Z be a smooth projective variety. Examples of common invariants that may be expressed in terms of Chern numbers include the arithmetic genus $p_a = (-1)^n(P_Z(0) - 1)$, the Euler characteristic of the structure sheaf $\chi(\mathcal{O}_Z) = P_Z(0)$, the topological Euler characteristic $e(Z)$, the genus g in the case of curves and the sectional genus π in the case of surfaces. The following table gives some examples.

dim(Z)	Invariant
1	$p_a = g = 1 - \frac{1}{2}\deg(c_1)$
1	$\chi(\mathcal{O}_Z) = \frac{1}{2}\deg(c_1)$
2	$p_a = \frac{1}{12}(\deg(c_1^2) + \deg(c_2)) - 1$
2	$\chi(\mathcal{O}_Z) = \frac{1}{12}(\deg(c_1^2) + \deg(c_2))$
2	$\pi = \frac{1}{2}(\deg(Z) - \deg(c_1)) + 1$
3	$p_a = 1 - \frac{1}{24}\deg(c_1 c_2)$
3	$\chi(\mathcal{O}_Z) = \frac{1}{24}\deg(c_1 c_2)$
n	$e(Z) = \deg(c_n)$

The formulae for p_a and $\chi(\mathcal{O}_Z)$ follow from the discussion in Section 6.2. The formula $e(Z) = \deg(c_n)$ is shown in [15]. When Z is a surface, the expression for π is derived as follows. The sectional genus π of Z is the genus of a general hyperplane section. Let H be the hyperplane class of Z. By the Adjunction formula (see [12, Proposition V.1.5]), $2\pi - 2 = \deg(H(H - c_1))$ and thus

$$\pi = \frac{1}{2}\deg(H^2 - Hc_1) + 1 = \frac{1}{2}(\deg(Z) - \deg(c_1)) + 1.$$

References

[1] E. Allgower, K. Georg, *Introduction to numerical continuation methods*, Classics in Applied Mathematics 45, SIAM Press, Philadelphia (2003).
[2] P. Aluffi, *Computing characteristic classes of projective schemes*, Journal of Symbolic Computation 35 (2003), 3-19.
[3] D.J. Bates, J.D. Hauenstein, A.J. Sommese, C.W. Wampler, *Bertini: Software for Numerical Algebraic Geometry.* Available at http://www.nd.edu/∼sommese/bertini.
[4] W. Decker, L. Ein, F.-O. Schreyer, *Construction of Surfaces in* $\mathbb{P}^4$, Journal of Algebraic Geometry 2 (1993), 185-237.
[5] W. Decker, G.-M. Greuel, G. Pfister, H. Schönemann, *Singular 3-1-1 – A computer algebra system for polynomial computations.* Available at http://www.singular.uni-kl.de.
[6] S. Di Rocco, D. Eklund, C. Peterson, A. Sommese, *Chern numbers of smooth varieties via homotopy continuation and intersection theory.* To Appear in Journal of Symbolic Computation.
[7] D. Eisenbud, *Commutative Algebra. With a View Toward Algebraic Geometry*, Graduate Texts in Mathematics 150, Springer-Verlag, New York (1995).
[8] H. Flenner, L. O'Carroll, W. Vogel, *Joins and Intersections*, Springer Monographs in Mathematics, Springer-Verlag, Berlin (1999).
[9] W. Fulton, *Intersection Theory*, Second edition. Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, no. 2. Springer-Verlag, Berlin (1998).
[10] W. Fulton, S. Kleiman, R. Piene, H. Tai, *Some intrinsic and extrinsic characterizations of the projective space*, Bulletin de la Sociètè Mathèmatique de France, 113 (1985), 205-210.
[11] D. Grayson, M. Stillman, *Macaulay2: a software system for research in algebraic geometry*, Available at http://www.math.uiuc.edu/Macaulay2.
[12] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics 52, Springer-Verlag, New York-Heidelberg, (1977).

[13] J. Harris, *Algebraic Geometry, A First Course*, Graduate Texts in Mathematics 133, Springer-Verlag, New York, (1992).
[14] S. L. Kleiman, *The transversality of a general translate*, Compositio Math. 28 (1974), 287-297.
[15] J. Milnor, J. D. Stasheff, *Characteristic Classes*, Annals of Mathematics Studies No. 76, Princeton University Press (1974).
[16] S. Popescu, *On smooth surfaces of degree* ≥ 11 *in* $\mathbb{P}^4$, Dissertation, Saarbrücken (1992).
[17] F. Severi, *Sulle intersezioni delle varietá algebriche e sopra i loro caraterie singolaritá proiettive*, Mem. Accad. Sci. di Torino, S. II 52 (1902).

Department of Mathematics, Colorado State University, Fort Collins, CO 80523
E-mail address: bates@math.colostate.edu
URL: http://www.math.colostate.edu/∼bates

Department of Mathematics, KTH, 100 44 Stockholm, Sweden
E-mail address: daek@math.kth.se
URL: http://www.math.kth.se/∼daek

Department of Mathematics, Colorado State University, Fort Collins, CO 80523
E-mail address: peterson@math.colostate.edu
URL: http://www.math.colostate.edu/∼peterson