Applications of Numerical Terracini's Lemma

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Abstract

This paper illustrates how methods such as homotopy continuation and monodromy, when combined with a numerical version of Terracini's lemma, can be used to produce a high probability algorithm for computing the dimensions of secant and join varieties. The use of numerical methods allows applications to problems that are difficult to handle by purely symbolic algorithms.

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1 Introduction

In this paper we study the dimension of secant and join varieties from a numerical point of view. In particular, we show how methods from numerical algebraic geometry can combine with a numerical version of Terracini's lemma to produce a high reliability algorithm for computing the dimension of such varieties. There are at least four situations where the utilization of numerical methods may be advantageous to purely symbolic methods. First, the method can be applied to any group of irreducible components of an algebraic set. In particular, it is not necessary to perform a partial or full primary decomposition of the radical of the ideal. Second, the ideals of the varieties involved in the computation can be generated by non-sparse polynomials with arbitrary (but bounded) coefficients. In other words, the coefficients can be any complex number; they are not restricted to be rational, algebraic, etc. Third, the varieties involved can have high codimension and are not required to have rational points. Finally, information

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can be extracted from ideals whose generators have small inaccuracies in their coefficients.

Interest in secant and join varieties spans several fields including algebraic geometry, combinatorics, complexity theory, numerical analysis, statistics and physics [6, 8, 13, 22]. Much of this interest derives from the connection between secant varieties and problems involving tensor rank [5]. Rather than focus on the connection with tensor rank and optimizing in this direction, this paper will consider the general problem of computing the dimension of secant and join varieties. Examples are purposefully chosen to illustrate situations where a numeric-symbolic approach may be more natural than a symbolic approach alone. First we recall some basic definitions that will be used throughout the paper. If $Q_1, \ldots, Q_p$ are points in $\mathbb{P}^m$ then we let $< Q_1, \ldots, Q_p >$ denote their linear span. If $X_1, \ldots, X_p$ are projective varieties in $\mathbb{P}^m$ then the *join* of the varieties, $J(X_1, \ldots, X_p)$, is defined to be the Zariski closure of the union of the linear span of p-tuples of points $(Q_1, \ldots, Q_p)$ where $Q_i \in X_i$. In other words

$$J(X_1, \ldots, X_p) = \overline{\bigcup_{Q_1 \in X_1, \ldots, Q_p \in X_p} < Q_1, \ldots, Q_p >}.$$

If $X_1, \ldots, X_p$ have dimensions $d_1, \ldots, d_p$ then the expected dimension (and the maximum possible dimension) of $J(X_1, \ldots, X_p)$ is $\min\{m, p - 1 + \sum d_i\}$. The *p-secant* variety of X is defined to be the join of p copies of X. We will denote this by $\sigma_p(X)$. Hence $\sigma_2(X) = J(X, X)$ is the variety of secant lines to X. The expected dimension (and the maximum possible dimension) of $\sigma_p(X)$ is $\min\{m, pr+(p-1)\}$. X is said to have a *defective* p-secant variety if $\dim \sigma_p(X) < \min\{m, pr + (p - 1)\}$ while X is called *defective* if there exists a p such that $\dim \sigma_p(X) < \min\{m, pr + (p - 1)\}$.

Terracini's lemma is perhaps the most useful direct computational tool for computing the dimension of secant and join varieties. The lemma asserts that to compute the tangent space to a join variety, $J(X_1, \ldots, X_s)$, at a generic point, P, it is enough to compute the span of the tangent spaces at generic points on each of the X_i's. Terracini's lemma was originally stated in the situation where $X_1 = \cdots = X_s$. This allowed one to compute the dimension of $\sigma_s(X_1)$ from s generic points on X_1. If, for points Q_i on X_i, the tangent spaces $T_{Q_1}, \ldots, T_{Q_s}$ are independent then $J(X_1, \ldots, X_s)$ has dimension $s - 1 + \sum_{i=1}^{s} \dim X_i$. This is precisely the dimension that $J(X_1, \ldots, X_s)$ is expected to have in the situation where $m \geq s - 1 + \sum_{i=1}^{s} \dim X_i$. By upper semicontinuity, if there exist smooth points (not necessarily generic) such that the tangent spaces $T_{Q_1}, \ldots, T_{Q_s}$ are independent (or else span the ambient space) then $J(X_1, \ldots, X_s)$ has the expected dimension. Thus, to show that X does not have a *defective* p-secant variety, it is enough to find p smooth points on X such that the tangent spaces at these points are either linearly independent or else span the ambient space. As a consequence, to check that X is not defective, it is enough to check that $\sigma_\alpha(X)$ and $\sigma_\beta(X)$ have the correct dimension when $\alpha = \max\{p \,|\, pr+p-1 \leq m\}$ and $\beta = \min\{p \,|\, pr + p - 1 \geq m\}$.

Let $\mathbf{I}$ denote the homogeneous ideal of a variety V. One can apply Terracini's lemma to compute $\dim \sigma_p(V)$ by evaluating the Jacobian matrix of $\mathbf{I}$ at p general points on V and then utilizing each evaluated Jacobian matrix to construct a basis for the tangent spaces at these p general points. Since in practice one does not pick true general points, Terracini's lemma can be used to prove non-deficiency but it cannot be used to prove deficiency. Proving non-deficiency works extremely well for varieties where one can produce many points [2, 15]. However, the requirement of Terracini's lemma to understand the independence of tangent spaces at points on a variety poses, in general, an obstacle since one cannot expect to produce points on a typical variety. Since numerical techniques can be used to produce points *arbitrarily close* to a variety, one can hope to use a version of Terraccini's lemma in the numerical setting.

The weakness of numerical methods, it is sometimes argued, is the loss of exactness. On the other hand, it is precisely this loss of exactness that allows numerical techniques to apply to problems that are unreachable by a purely symbolic algorithm. By combining numerical methods with ideas from symbolic computation, algebraic relationships can be made relatively stable under small perturbations and their presence can be detected. The main goal of this paper is to demonstrate a relatively elementary use of numerical-symbolic methods on a particular set of problems arising naturally in algebraic geometry. The method relies mainly on known theory, so the main contribution is the application of that theory rather than the theory itself.

2 Background

2.1 Homotopy Continuation

Given a polynomial system $F : \mathbb{C}^n \to \mathbb{C}^n$, one may compute accurate numerical approximations for all isolated solutions of F via homotopy continuation. This is a well-established numerical method in which F is cast as a member of a parameterized family of polynomial systems one of which, say G, has known isolated solutions. Given a homotopy such as $H = (1-t) \cdot F + \gamma \cdot t \cdot G$ where γ is a random complex number and G is a compatibly chosen polynomial system, there are paths leading from the isolated solutions of G at $t = 1$ to solutions of F at $t = 0$. These paths may be numerically tracked using standard predictor/corrector methods such as Euler's method and Newton's method. A collection of techniques, known as endgames, are utilized to reduce the computational cost of computing singular endpoints with a given reliability. The set of path endpoints computed at $t = 0$ contains all isolated solutions of F as well as points lying on higher-dimensional irreducible components of the algebraic set defined by F.

Let $Dim(V)$ denote the dimension of the top dimensional component of the algebraic set, V, determined by F. The basic algorithms of numerical algebraic geometry produce, for each dimension d with $0 \leq d \leq Dim(V)$, discrete data called a witness point set [17, 19]. This consists of a set of points W_d, which we by abuse of language often call a *witness point set* for dimension d, and a generic codimension d linear space L_d with the basic property:

- Within a user-specified tolerance, the points of W_d are the intersections of L_d with the union of the d-dimensional components of V.

Since a general linear space meets each d-dimensional irreducible component W of V in exactly $Deg(W)$ points, each d-dimensional irreducible component W of V has at least one witness point in W_d. This cascade algorithm involves repeated applications of homotopy continuation to polynomial systems constructed from F. Each of the polynomial systems constructed from F is obtained by adding extra linear equations (corresponding to slices by generic hyperplane sections).

Using techniques such as monodromy, it is possible to partition W_d into subsets, which are in one-to-one correspondence with the d-dimensional irreducible components of V. In particular, by tracking points in a witness set W_d around various loops in the parameter space, one can organize the points in W_d into sets such that all points of a set lie on the same irreducible component. Although a stopping criterion for monodromy by itself is unknown, linear traces provide a means to certify, and sometimes even carry out, such partitions with certainty.

Thus, given an ideal $\mathbf{I}$, it is possible to produce by numerical methods a collection of subsets of points such that the subsets are in one to one correspondence with the irreducible components of the algebraic set determined by $\mathbf{I}$. Furthermore, the points within a given subset can be chosen to be within a pre-specified tolerance of the irreducible component represented by the subset. A classic introduction to the subject can be found in [3]. For an overview of newer algorithms and techniques within this field, see [19, 20]. For details on the cascade algorithm, see [16, 19].

2.2 Singular Value Decomposition

Every $m \times n$ matrix M may be decomposed as $M = U\Sigma V^*$ where U and V are square, unitary matrices and Σ is an $m \times n$ diagonal matrix with real non-negative entries. This factorization is known as the singular value decomposition (SVD) of M. The diagonal entries of Σ are the singular values of M and the columns of U and V are the left and right singular vectors of M, respectively. A real number σ is a singular value if and only if there exist unit length vectors $\mathbf{u}, \mathbf{v}$ such that $M\mathbf{v} = \sigma\mathbf{u}$ and $M^*\mathbf{u} = \sigma\mathbf{v}$.

The singular value decomposition of a matrix is a key tool in determining both the numerical rank of a matrix and a basis for the nullspace of the matrix. In particular, for a matrix with exact entries, the number of nonzero singular values is exactly the rank of the matrix. In a floating point setting, one may compute the numerical rank of M by counting the number of singular values larger than some tolerance ϵ. Thus the numerical rank is a function of ϵ. This raises the natural question: *How do you choose ϵ?* A general precise answer to this question is both unclear and application-dependent. However, in the setting of numerical algebraic geometry it is often possible to increase certainty by increasing precision. Singular values which would be zero in an exact setting but which are non-zero due to imprecision and round off errors can be made to shrink towards zero by recomputing with increased precision. This will be illustrated in examples appearing later in the paper. Another fact that will be used is that the right singular vectors corresponding to the (numerically) zero singular values form a basis for the (numerical) nullspace of the matrix, as described in [24]. The SVD is also known for providing a method for producing the nearest rank k matrix to a given matrix (in the Frobenius norm), simply by setting to 0 the appropriate number of smallest entries of Σ and then re-multiplying.

The computation of the SVD of a matrix is more costly than the computation of the QR decomposition. However, the SVD may be computed more stably and is a good choice for ill-conditioned (i.e., nearly singular) matrices. The computation of the SVD relies on Householder reflectors or Givens rotations to reduce the matrix to bidiagonal form. From there, a number of methods related to the QR decomposition may be used to extract the full decomposition. A good general reference is provided by [25], while [21] gives a particularly direct description of the decomposition algorithm. It should be noted that many of the computations that are made in the following examples could also be computed using the rank revealing method of Li and Zeng [14]. However, for the purposes of this paper, the full singular value decomposition yielded more detailed information and could be efficiently computed. Currently, the SVD algorithm is the only rank revealing method implemented in Bertini.

2.3 Terracini's Lemma

Terracini's lemma, as originally formulated, provides an effective method for computing the dimension of a secant variety [23]. The applicability of the lemma was later extended to join varieties and higher secant varieties [1]. The lemma asserts that the tangent space to a join variety, $J(X_1, \ldots, X_s)$, at a generic point, P, is equal to the span of the tangent spaces at related generic points on each of the X_i's. In particular, it states:

Lemma 1 **(Terracini's Lemma).** *Let $X_1, \ldots, X_s$ be irreducible varieties in $\mathbb{P}^n$ and let $Q_1, \ldots, Q_s$ be distinct generic points (with Q_i on X_i for each i). Let T_{Q_i} be the tangent space to X_i as an affine variety at (a representative of) the point Q_i and let $\mathbf{P}(T_{Q_i})$ be the projectivized tangent space to X_i at Q_i. Then* $\dim J(X_1, \ldots, X_s) = \dim < \mathbf{P}(T_{Q_1}), \ldots, \mathbf{P}(T_{Q_s}) > = \dim < T_{Q_1}, \ldots, T_{Q_s} > -1$.

By upper semicontinuity, it follows as an immediate corollary of Terracini's lemma that if there exist smooth points (not necessarily generic) such that the tangent spaces $T_{Q_1}, \ldots, T_{Q_s}$ are independent (or else span the ambient space) then $J(X_1, \ldots, X_s)$ has the expected dimension. Since one can never be sure of choosing true generic points, if the tangent spaces $T_{Q_1}, \ldots, T_{Q_s}$ are not independent, one cannot conclude that $J(X_1, \ldots, X_s)$ does not have the expected dimension. Thus, while Terracini's lemma can be used to furnish proofs in the nondefective case, it can only be used as a *guide* in the defective case.

2.4 Bertini software package

All examples of this paper were run using Bertini and Maple$^{\text{TM}}$. Bertini [4] is a software package under continuing development for computation in numerical algebraic geometry. Bertini is written in the C programming language and makes use of the GMP-based library MPFR for multiple precision floating point numbers. It also makes use of adaptive multiprecision; automatic differentiation; and straight line programs for efficient evaluation. Bertini will accept as input a set of homogeneous polynomials over any product of projective spaces or a set of inhomogeneous polynomials over any product of complex spaces.

Bertini is capable of finding all complex solutions of a polynomial system with complex coefficients via methods rooted in homotopy continuation. By using automatic adaptive precision techniques and special methods known as endgames, Bertini is capable of providing these solutions to several hundred digits.

3 Four illustrative examples

In the first subsection below, we illustrate an exact method and two approximate methods in conjunction with Terracini's lemma applied to the Veronese surface in $\mathbb{P}^5$. The second subsection contains two examples chosen to illustrate situations well-suited to a numeric approach but that may be difficult via Gröbner basis techniques. There are other situations where a symbolic approach is better suited than a numerical approach. As a consequence, the methods illustrated in this paper should be seen as complementary to the exact methods that one finds in programs such as [7, 11, 12]. Perhaps in the future, symbolic based programs will be able to exchange information in a meaningful way with numeric based programs.

3.1 Secant variety of the Veronese Surface in $\mathbb{P}^5$

In this section we show, in some detail, three methods utilizing Terracini's Lemma to compute the dimension of the secant variety to a variety. For simplicity we use the well-studied Veronese surface in $\mathbb{P}^5$ as our object of study. In each of the three methods, we compute an (approximate) basis for the tangent space to the affine cone of the variety at two distinct (approximate) points P_1, P_2. We next look at the dimension of the span of the two tangent spaces and then apply Terracini's Lemma to interpret the results. In example 2, we review the standard method to apply Terracini's Lemma. In example 3, we modify the standard approach to fit a numerical setting. Finally, in example 4 we use a numerical version of Terracini's lemma in the setting of a highly sampled variety.

Example 2 (Secant variety by the exact Jacobian method). Let $I = (F_1, \ldots, F_t) \subseteq \mathbb{C}[x_1 \ldots, x_n]$. The associated Jacobian matrix of I is the $t \times n$ matrix whose $(i,j)^{th}$ entry is $\frac{\partial F_i}{\partial x_j}$. The Veronese surface $V \subseteq \mathbb{P}^5$ is defined as the image of the map $[x : y : z] \to [x^2 : xy : xz : y^2 : yz : z^2]$. The homogeneous ideal of V is $I = (e^2 - df, ce - bf, cd - be, c^2 - af, bc - ae, b^2 - ad)$. The associated Jacobian matrix of I is therefore

$$Jac(a,b,c,d,e,f) = \begin{pmatrix} 0 & 0 & 0 & -f & 2e & -d \\ 0 & -f & e & 0 & c & -b \\ 0 & -e & d & c & -b & 0 \\ -f & 0 & 2c & 0 & 0 & -a \\ -e & c & b & 0 & -a & 0 \\ -d & 2b & 0 & -a & 0 & 0 \end{pmatrix}$$

Consider the points $[1 : 3 : 7]$ and $[1 : 5 : 11]$ in $\mathbb{P}^2$. They map to the points $P = [1 : 3 : 7 : 9 : 21 : 49]$ and $Q = [1 : 5 : 11 : 25 : 55 : 121]$ in $\mathbb{P}^5$. Evaluating the Jacobian matrix at P yields the matrix

$$\begin{pmatrix} 0 & 0 & 0 & -49 & 42 & -9 \\ 0 & -49 & 21 & 0 & 7 & -3 \\ 0 & -21 & 9 & 7 & -3 & 0 \\ -49 & 0 & 14 & 0 & 0 & -1 \\ -21 & 7 & 3 & 0 & -1 & 0 \\ -9 & 6 & 0 & -1 & 0 & 0 \end{pmatrix}$$

This matrix has rank 3 and spans the normal space to V at P. The tangent space T_P to V at P (as an affine variety) is spanned by the right null vectors to this matrix, thus T_P is the row space of the matrix

$$A = \begin{pmatrix} 2 & 3 & 7 & 0 & 0 & 0 \\ 0 & 1 & 0 & 6 & 7 & 0 \\ -1 & -3 & 0 & -9 & 0 & 49 \end{pmatrix}$$

Similarly, we determine that T_Q is the row space of the matrix

$$B = \begin{pmatrix} 2 & 5 & 11 & 0 & 0 & 0 \\ 0 & 1 & 0 & 10 & 11 & 0 \\ -1 & -5 & 0 & -25 & 0 & 121 \end{pmatrix}$$

The dimension of the space spanned by the row spaces of A and B is, of course, the rank of the matrix

$$C = \begin{pmatrix} 2 & 3 & 7 & 0 & 0 & 0 \\ 0 & 1 & 0 & 6 & 7 & 0 \\ -1 & -3 & 0 & -9 & 0 & 49 \\ 2 & 5 & 11 & 0 & 0 & 0 \\ 0 & 1 & 0 & 10 & 11 & 0 \\ -1 & -5 & 0 & -25 & 0 & 121 \end{pmatrix}$$

A quick computation (in exact arithmetic) shows the rank of C to be 5. As a consequence, the secant variety to the Veronese surface is (most likely) four dimensional as a projective variety.

It is important to note that the example above does not give a proof that the dimension of the secant variety is four since it is possible (though unlikely) that the points we chose were special points for this particular example. Terracini's Lemma requires the points to be generic. If the rank of the matrix had been 6 then we would have a proof that the secant variety is five dimensional since the points were smooth points on the surface, the dimension achieved the maximum possible and we could apply upper semicontinuity. In the next two examples, we illustrate two numerical methods for "determining" the dimension of the secant variety to the Veronese.

Example 3 (Secant variety by the approximate Jacobian method). Consider the points $\left[\sqrt{2} : \frac{22}{7} : 3^{(1/7)}\right]$ and $\left[7^{(1/5)} : \frac{15}{2} : \sqrt{5}\right]$. As in the example above, let P and Q be their images in $\mathbb{P}^5$. Let P' and Q' denote the points obtained from P, Q by rounding their coordinates to 40 digits of accuracy. The Jacobian matrix evaluated at P' has 3 singular values between 12 and 15 and 3 singular values between 10^{-42} and 10^{-38}. The Jacobian matrix evaluated at Q' has 3 singular values between 61 and 68 and 3 between 10^{-42} and 10^{-38}. The numerical rank of each matrix is 3 provided we round "appropriately small" singular values to zero. We use the singular value decomposition to find the closest rank 3 matrices to the Jacobians of P' and Q' and to find 3 linearly independent vectors in the right null space of each of these rank 3 matrices. We use these 2 sets of three vectors to build a 6×6 matrix, C. Finally, we use the singular value decomposition to determine that C has five singular values between 0.1 and 1.5 and one small singular value that was approximately 2.8×10^{-40}. Thus we have obtained a matrix that, in an appropriate sense, has a numerical rank equal to 5. Again this suggest that the secant variety to the Veronese surface is (most likely) four dimensional as a projective variety.

Example 4 (Secant variety by a sampling method). In this example we start by constructing two small clusters of 5 points each on the Veronese surface. We use these clusters to approximate the tangent space to the cone of the surface at the points corresponding to the centers of the clusters. In particular we chose the points $[0.2 : 0.3 : 0.4]$ and $[0.8 : 0.6 : 0.4]$ and then varied each of them by 0.0001 or 0.0002 in 5 (pseudo)random directions. By looking at their images in $\mathbb{C}^6$ we obtained two different 5×6 matrices, A and B. The 5 singular values of each of these matrices were on the order of $1, 10^{-4}, 10^{-4}, 10^{-8}, 10^{-8}$. Knowing ahead of time that we were trying to approximate the tangent space to the cone of a surface, we took the best rank 3 matrix approximations of A and B and interpreted the row spaces of these matrices as representing the tangent spaces. Stacking these rank 3 matrices yielded a 10×6 matrix which had 5 singular values between 0.3 and 1.5 and one singular value in the range of 4.9×10^{-6}. Thus we obtained a matrix that, within an appropriate tolerance, has a numerical rank equal to 5. This suggest, once again, that the secant variety to the Veronese surface is (most likely) four dimensional as a projective variety. Repeating the experiment with perturbations on the order of 0.0000015 yielded a matrix whose smallest singular value was roughly 5×10^{-9}.

Each of the previous three examples gives consistent and strong evidence for the dimension of a certain secant variety. None of the computations *prove* that this dimension is correct. If a variety is nondefective then exact arithmetic can typically be used to prove this fact *provided sufficiently general points can be found which lie on the variety.* Unfortunately, it is usually impossible to find any points which lie on a given variety. This necessitates either a different symbolic approach or else a non-symbolic approach. The numeric approach presented in this paper offers one possibility.

3.2 Two more examples

The following two examples present situations that may well frustrate a symbolic algebra system due to computational complexity. It is meant to illustrate a situation well suited to a numerical approach but not well suited to a symbolic approach.

Example 5 (Three secant variety of a curve in $\mathbb{P}^8$). In this example, we let C be a curve in $\mathbb{P}^8$ defined as the complete intersection of 6 quadrics and a cubic. The coefficients of the defining equations were chosen randomly in the interval $[-3, 3]$ and included 16 digits past the decimal point. We used the homotopy continuation implementation in *Bertini* to produce three points whose coordinates are within 10^{-8} of an exact point on C. Performing computations with 16 digits of accuracy, we used the SVD to compute approximate bases for the tangent spaces to C (as an affine cone) for each of the three points. We combined the approximated basis vectors into a single 6×9 matrix. The singular value decomposition determined that the matrix had 6 positive singular values ranging from 0.35 to 1.37. The computation was repeated at 50 digits of accuracy and the smallest singular value was again roughly 0.35. The higher accuracy computation caused the smallest singular value to move by less than 10^{-11}. As a result, we conclude that $\sigma_3(C)$ is (very likely) 5 dimensional as a projective variety.

Example 6 (Jumbled Segre Variety). In this example, we took the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ into $\mathbb{P}^8$. Next a random linear change of variables was performed in order to produce a Segre variety, S, whose homogeneous ideal had a scrambled set of generators. The linear change of variables that we used involved various complicated algebraic and transcendental numbers. We used homotopy continuation in Bertini to produce two random points with coordinates within 10^{-8} of the coordinates of an exact point on S as an affine variety. The SVD decomposition was applied to find bases for the tangent space to S at the two points. The basis vectors for the tangent spaces were combined to form a 9×10 matrix. The SVD was applied to determine that the matrix had 8 singular values between 0.36 and 1.42 and 1 singular value that was approximately 0.91×10^{-11}. As a result we conclude that the dimension of the secant variety to the Segre variety is 7. In other words, the Segre variety is 2-deficient.

4 Further computational experiments

4.1 Computations on systems which are close together

Many symbolic algorithms require a computation of a Gröbner basis. If the defining equations of an ideal involve many different prime numbers, then the rational operations involved in the computation of the Gröbner basis can lead to fractions whose numerator and denominator are extremely large. In other words, symbolic techniques over $\mathbb{Q}$ often lead to *coefficient blowup*. In numerical methods, coefficient blowup of this type is avoided by employing finite precision arithmetic at various parts of a calculation. However, the small inaccuracies that occur when using finite precision can accumulate. As a consequence, it is crucial that the nature of these errors is understood in order to ensure that meaning can be attached to the output of the computation. A symbolic version of roundoff that is frequently utilized, to prevent coefficient blowup, is to carry out computations with exact arithmetic but over a field with finite characteristic. This can lead to problems. To increase the chance of a proper interpretation of the output, the computation is typically made over a field with a sufficiently large characteristic. An alternative method is to repeat a computation over two fields with differing characteristic. In the computer package Bertini, adaptive multiprecision is employed in a similar manner to help avoid potential pitfalls of low accuracy computations. The following example describes how computations will vary on two slightly different systems of equations. The first part of the example is carried out using an exact system of equations. The second part of the example is carried out after the system has been slightly perturbed.

Example 7 (The twisted cubic and the perturbed twisted cubic). In this example, we start with the twisted cubic defined by the ideal $\mathbf{I} = (x^2 - wy, xy - wz, y^2 - xz)$. We can compute basis vectors for the tangent space to the twisted cubic at two exact points. Upon combining these vectors into a single 4×4 matrix we find that the matrix has rank 4 using exact arithmetic. This leads us to conclude that the secant variety of the twisted cubic fills the ambient space. If we repeat the computation with inexact points that lie very close to the twisted cubic (as an affine variety) then we obtain a 4×4 matrix which has no minuscule singular values. This leads us to suspect that the secant variety fills the ambient space. Now suppose we perturb each of the 3 quadric generators of $\mathbf{I}$ via a random homogeneous quadric with very small coefficients. The perturbed ideal defines 8 distinct points. Assuming the perturbation is very small, the homotopy continuation algorithm and the cascade algorithm will conclude that the perturbed ideal defines a curve if one chooses a certain range of tolerances but will conclude that the ideal defines 8 points if the tolerance is chosen sufficiently small.

4.2 Determining equations from a generic point

Let $\mathbf{I} \subseteq \mathbb{C}[x_0, \ldots, x_n]$ be the homogenous ideal of a projective variety V. Let $\mathbf{J}$ be a homogeneous ideal whose saturation with respect to the maximal ideal is $\mathbf{I}$. Suppose there exists a set of generators for $\mathbf{J}$ lying in $\mathbb{Q}[x_0, \ldots, x_n]$. Let $Sec(\mathbf{I})$ denote the homogenous ideal of the secant variety of V. Since Gröbner basis algorithms can be used to determine $\mathbf{I}$ from $\mathbf{J}$ and $Sec(\mathbf{I})$ from $\mathbf{I}$, there must exist a set of generators for $Sec(\mathbf{I})$ which lie in $\mathbb{Q}[x_0, \ldots, x_n]$.

Using the methods described in the previous sections, one can produce points arbitrarily close to generic points on V. By taking a general point on the secant line through two such points, one can produce points arbitrarily close to generic points on $Sec(V)$. The next example illustrates that from a numerical approximation to a single generic point on a variety, one *may* be able to obtain exact equations in the homogeneous ideal of the variety.

Example 8. In this example we consider the Veronese surface $S \subset \mathbb{P}^5$ in its standard embedding. The homogeneous ideal of S has a generating set lying in $\mathbb{Q}[a, \ldots, f]$. Thus the homogeneous ideal of the secant variety to the Veronese surface must lie in $\mathbb{Q}[a, \ldots, f]$. We took two points on the Veronese surface involving a combination of algebraic and transcendental numbers. We then took a weighted sum of the points to obtain a point, P, on the secant variety of the Veronese surface. We next took a decimal approximation of the coordinates accurate to 75 digits to obtain a point Q which lies very close to the secant variety. The LLL and PSLQ algorithms provide a means for finding vectors, with relatively small integral entries, whose dot product with a given vector is very close to zero. We decided to use the PSLQ algorithm (as implemented in Maple$^{\text{TM}}$) due to its increased numerical stability [9, 10]. We applied the algorithm to the 3-uple embedding of Q (a vector in $\mathbb{R}^{56}$). The outcome was a vector (almost) in the null space of the 3-uple embedding which corresponded to the equation $adf - ae^2 + 2bce - b^2 f - c^2 d \in \mathbb{Q}[a, \ldots, f]$. The vanishing locus of this equation is well known as the hypersurface corresponding to the secant variety of S.

5 Conclusions

An application of a numeric-symbolic algorithm has been presented which allows estimates of the dimension of join and secant varieties to be made *with high reliability*. The examples were run in the Bertini software package [4] with some additional computations run in Maple$^{\text{TM}}$. The Veronese surface in $\mathbb{P}^5$ was shown (with high reliability) to be defective via three different methods (all using Terracini's lemma). Other examples were presented which demonstrate situations well suited to numerical methods but not well suited to exact methods. An example of a perturbed ideal was included to illustrate that care must be used when applying numerical techniques and that increased precision can be used to increase reliability of output. The example of the perturbed twisted cubic simultaneously demonstrates a strength and a limitation. The perturbed system is, in a numerical sense, very close to the original system. However, in an exact sense, it describes a variety with completely different characteristics. A symbolic algorithm would certainly distinguish between these two examples while a numerical-symbolic system only differentiates between the two varieties when very high precision is utilized. With a numerical approach, the closeness of the perturbed ideal to an ideal of a curve can be detected. Thus the loss of exactness, that comes with a numerical approach, is accompanied by a new collection of questions that can be answered and new regions of problems from which meaningful information may be extracted.

One could improve the approach of this paper by developing an array of criteria for certifying that a small singular value would indeed converge to zero if infinite precision could be utilized. The ultimate cost of such a certification would likely be very high. When determining which singular values of a matrix are zero via numerical methods, we were forced to choose a threshold below which a singular value is deemed to be zero. If one is dealing with inexact equations then there is a limit on the accuracy of the information obtained. If one can quantify the inaccuracy then reasonable thresholds can be determined. However, if one is dealing with exact equations, increased confidence in the result of a computation follows from performing the computation at a higher level of precision. This approach was used in example 5 and is satisfactory in many, perhaps even most, situations. However, there will always be problems that can be constructed which can fool a numerically based system. By starting with exact equations and then using numerical methods one is necessarily losing information. It is an important research problem to quantify this in such a way as to yield improved confidence levels.

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