MATH 261 FINAL EXAM PRACTICE PROBLEMS

These practice problems are pulled from the final exams in previous semesters. The 2-hour final exam typically has **8-9** problems on it, with 4-5 coming from the post-Exam 3 material and 4-5 coming from previous material. Only problems from the post-Exam 3 material are included here; see the old practice midterms to prepare for the questions over the older material.

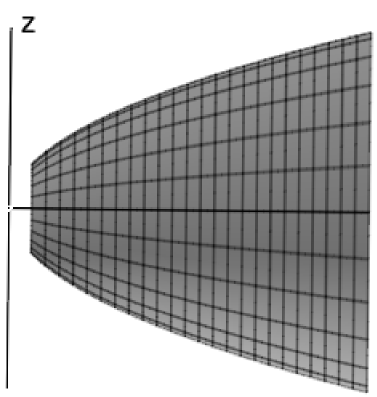
Please be aware that this is not intended as a comprehensive list of all possible problem types! In other words, you are responsible for all topics covered throughout the course, whether they are represented on this exam or not. See your notes, the suggested homework, and the four exam review sheets for a comprehensive list.

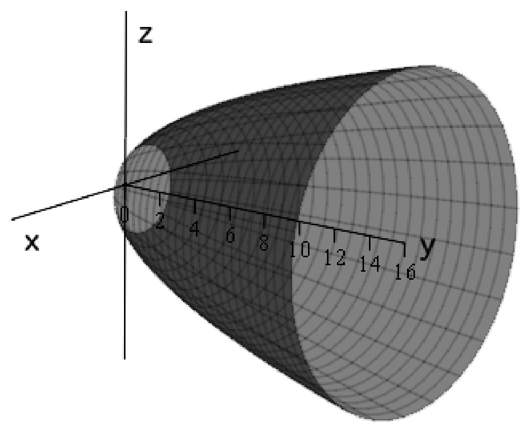
1. (a) Is the vector field $\langle yz - \sin(x)\sin(z), xz + \cos(x)\sin(z), xy + \cos(x)\cos(z)\rangle$ conservative? Please answer "yes" or "no" and provide adequate justification.

   (b) $\mathbf{F} = \langle 2x, -3z, -3y + 3z^2\rangle$ is a conservative vector field with potential function $f = x^2 - 3yz + z^3$. Find the work done when moving along a straight line segment from $(1, 2, 0)$ to $(1, 0, 1)$ through $\mathbf{F}$.

   (c) $\mathbf{G} = \langle e^x + y\cos(xy), 2yz + x\cos(xy), y^2\rangle$ is a conservative vector field. Find the potential function g so that $g(0, 0, 0) = 0$.

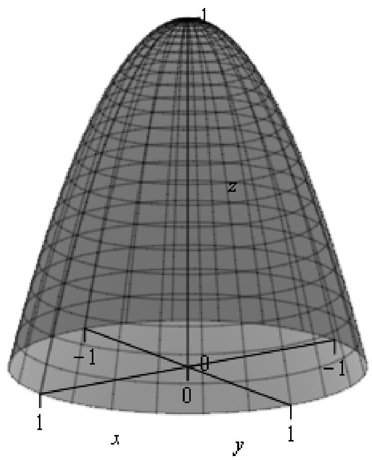
2. The paraboloid $y = x^2 + z^2$ is parameterized by $\mathbf{r}(r, \theta) = \langle r\cos\theta, r^2, r\sin\theta\rangle$. Let $\delta(x, y, z) = x^2 + y^2 + z^2$ be the density of the paraboloid. Set up (**but do not evaluate**) the surface integral below for finding the first moment M_{xz} of the paraboloid, sliced by the planes $y = 1$ and $y = 16$.





$$M_{xz} = \int\int \underline{\hspace{8cm}} d\_\_d\_\_$$

3. Let S be the portion of the paraboloid $z = 1 - x^2 - y^2$ above the plane $z = 0$ and C be the boundary of S, i.e., the unit circle in the plane $z = 0$ (with counter-clockwise orientation). According to Stokes' Theorem, circulation around C of vector field $\mathbf{F} = \langle xy, -y^2, 0\rangle$ can be computed as a line integral or a surface integral. <u>Be sure to do both parts below.</u>



(a) Circle the letter of the option below that correctly computes the circulation around C as a line integral.

(i) $\displaystyle\int_0^{2\pi} 1\ dt$

(ii) $\displaystyle\int_0^{2\pi} \cos(t)\sin(t)\ dt$

(iii) $\displaystyle\int_0^{2\pi} -\sin^2(t) - 2\sin(t)\cos(t)dt$

(iv) $\displaystyle\int_0^{2\pi} -2\cos(t)\sin^2(t)\ dt$

(v) $\displaystyle\int_0^{2\pi} \cos(t) - \sin(t)\ dt$

(vi) None of the above.

(b) Circle the letter of the option below that correctly computes the circulation around C as a surface integral over S (with outward pointing normal).

(i) $\displaystyle\int_0^{2\pi}\int_0^1 -r\cos(\theta)\ drd\theta$

(ii) $\displaystyle\int_0^{2\pi}\int_0^1 -r^2\cos(\theta)\ drd\theta$

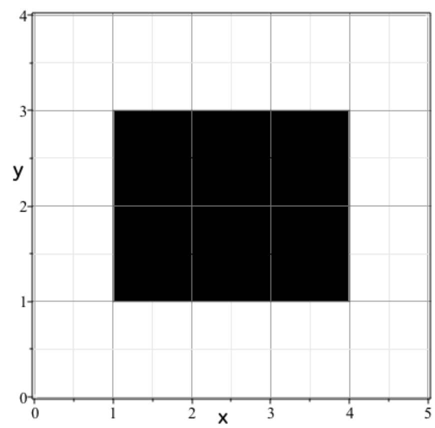
(iii) $\displaystyle\int_0^{2\pi}\int_0^1 r^2\sin(\theta)\ drd\theta$

(iv) $\displaystyle\int_0^{2\pi}\int_0^1 r^3\cos(\theta)\ drd\theta$

(v) $\displaystyle\int_0^{2\pi}\int_0^1 r\ drd\theta$

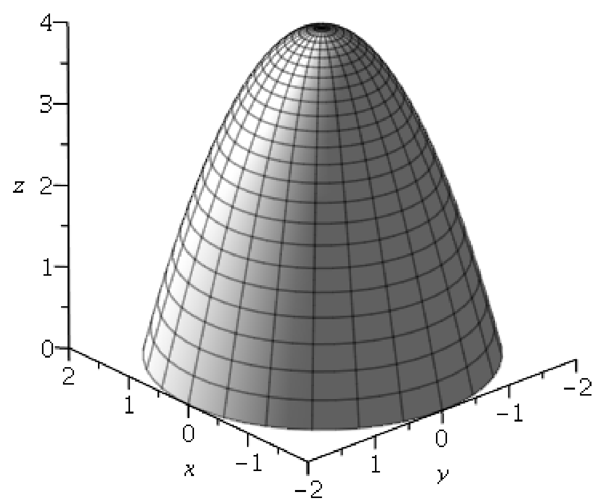
(vi) None of the above.

4. **Compute** $\oint_C y^2dx + x^2dy$, where C is the border of the black rectangle below (with counterclockwise orientation).



5. Setup (**but do not evaluate**) *one* integral (of any type) to find the flux of vector field $\mathbf{F}$ through surface S, where S is the unit cube given by $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$, $\mathbf{F} = \nabla f$, and $f = x^3 + xy + y^3 + yz + z^3$.

6. Consider the portion of the paraboloid $z = 4 - x^2 - y^2$ above the (x, y)-plane. One parameterization for this surface is $\mathbf{r}(r, \theta) = \langle r\cos(\theta), r\sin(\theta), 4 - r^2\rangle$, with $0 \le r \le 2$ and $0 \le \theta \le 2\pi$. Write down **BUT DO NOT EVALUATE** a double integral in r and θ that computes the integral of $g(x, y, z) = z^2$ over this surface.



7. Use either version of Green's Theorem to **EVALUATE** $\oint_C y^3 dx - x^3 dy$ as a double integral, where C is the circle of radius 3 centered at the origin and with counterclockwise orientation.

8. Let S be the surface of the cylinder given by $x^2 + y^2 = 4$ with $0 \leq z \leq 3$, including the top and the bottom. Set up a triple integral *in cylindrical coordinates* to compute the flux of $\mathbf{F} = \langle xy, yz, xz \rangle$ across surface S **BUT DO NOT EVALUATE IT.**

9. Using Green's Theorem and the appropriate choice of variables (i.e., NOT CARTESIAN), evaluate $\oint_C y^2 dx + 3xy dy$ where C is the boundary of the region in the upper half plane bounded by $x^2 + y^2 = 1$, $x^2 + y^2 = 4$ and $y = 0$. Be sure to clearly identify the integral you evaluate.

10. Consider the surface S which is the portion of the cone $y^2 = x^2 + z^2$ between $y = 1$ and $y = 3$ and the function $G(x, y, z) = z$.

    (a) Parametrize the surface using parameters r and θ, including lower and upper bounds on both parameters.

    (b) Set up the integral
    $$\iint_S G(x, y, z) d\sigma,$$
    clearly indicating the bounds of integration and all parts of the integrand, but DO NOT EVALUATE.

    [NOTE: #10 is an older problem, from before my time. I would typically include a plot of something like this.]

11. This problem is all about vector fields. The three parts are independent.

    (a) Determine (without searching for a potential function) whether the following vector field is conservative. Your answer should be *yes* or *no*, followed by your (brief) reasoning.
    $$\mathbf{F} = \langle x^2 - yx, y^2 - xy, z^2 - xy \rangle$$

    (b) $\mathbf{G} = \langle 2xy^3z^4, 3x^2y^2z^4, 4x^2y^3z^3 + 2 \rangle$ is a conservative vector field. Find the potential function $g(x, y, z)$ such that $g(0, 0, 0) = 0$.

    (c) $h(x, y, z) = x^2 + y^2 + z^2$ is a potential function for conservative vector field $\mathbf{H} = \langle 2x, 2y, 2z \rangle$. C is an unknown curve joining $(0, 0, 0)$ to $(1, 2, 3)$. Compute the work done in moving along C in the presence of $\mathbf{H}$.

12. Consider the vector field $\mathbf{F} = \langle M, N, P \rangle$ with
$$M = yz^2 \cos(xy) + \ln(z + 1),$$
$$N = xz^2 \cos(xy) + 3\sin(z)e^{3y\sin(z)},$$
$$P = 2z\sin(xy) + \frac{x}{z + 1} + 3y\cos(z)e^{3y\sin(z)}.$$

(a) Without finding the potential function, show that $\mathbf{F}$ is conservative.

(b) Find the potential function, ϕ, for the force.

(c) Evaluate $\int_{(\pi,1,0)}^{(0,\pi/2,\pi)} \mathbf{F} \cdot d\mathbf{r}$.

[NOTE: #12 is an older problem, from before my time. I wouldn't usually have (b) depend on (a) or (c) depend on (b).]

SOLUTIONS

WARNING: These solutions are not fully justified. Be sure to provide full justification with your solutions (especially where this is explicitly requested) so that we may provide partial credit, where applicable. If you are having trouble getting these answers, please come to office hours and/or exam review session. See the course website for details. Of course, if there seems to be an error with these solutions (which is very possible!), please let an instructor or the coordinator know.

1. (a) No! $P_x = y - \sin(x)\cos(z) = M_z$, but the other two component test equalities fail.
   (b) $f(1,0,1) - f(1,2,0) = 1$.
   (c) $g(x,y,z) = e^x + sin(xy) + y^2 z - 1$. (The constant term makes $g(0,0,0) = 0$, as requested.)
2. $M_{xz} = \int_0^{2\pi} \int_1^4 \ r^2(r^2 + r^4)\sqrt{4r^4 + r^2} dr d\theta$
3. (a) (iv)
   (b) (ii)
4. $\displaystyle\oint_C y^2 dx + x^2 dy = \int_1^3 \int_1^4 -2y + 2x \ dx dy = 6$ (using either version of Green's theorem).
5. $\displaystyle\iint_S \mathbf{F} \cdot \mathbf{n} \ d\sigma = \iiint_D \nabla \cdot \mathbf{F} \ dV = 6 \int_0^1 \int_0^1 \int_0^1 x + y + z \ dx dy dz$ (or any other order of the variables).
6. $\int_0 \int^{2\pi} \int_0^2 (4 - r^2)^2 \sqrt{4r^4 + r^2} \ dr d\theta$.
7. $\oint_C y^3 \ dx - x^3 \ dy = \int_0^{2\pi} \int_0^3 -3r^3 \ dr d\theta = \frac{-243\pi}{2}$.
8. Flux $= \iiint_D \nabla \cdot \mathbf{F} dV = \int_0^{2\pi} \int_0^2 \int_0^3 (r\cos(\theta) + r\sin(\theta) + z) r \ dz dr d\theta$
9. $\oint_C y^2 \ dx + 3xy \ dy = \int_0^{\pi} \int_1^2 r^2 \sin(\theta) \ dr d\theta = \frac{14}{3}$.
10. (a) $\mathbf{r}(r,\theta) = \langle r\cos(\theta), r, r\sin(\theta) \rangle$, $1 \le r \le 3$, $0 \le \theta \le 2\pi$.
    (b) $\int_0^{2\pi} \int_1^3 \sqrt{2} \ r^2 \sin(\theta) \ dr d\theta$.

11. (a) No, via the component test: $M_y = -x, but N_x = -y$. (It's false if at least one of the three fails!)
    (b) $g = x^2y^3z^4 + 2z$. ($C = 0$ since the problem required that $g(0,0,0) = 0$.)
    (c) $h(1,2,3) - h(0,0,0) = 14$.
12. (a) Just check the component test. (Tedious, but feasible.)
    (b) $\phi = z^2 \sin(xy) + x \ln(z+1) + e^{3y \sin(z)}$. (Again, tedious, but feasible.)
    (c) 0. Yep, after all that work....

Best of luck on the final exam!!