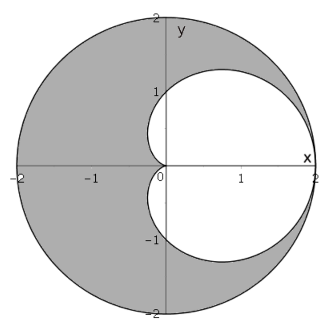
MATH 261 EXAM III PRACTICE PROBLEMS

These practice problems are pulled from actual midterms in previous semesters. Exam 3 typically has **5** (not 6!) problems on it, with no more than one problem of any given type (e.g., don't expect two problems about computing the centroid). Also, **please be aware** that this is not intended as a comprehensive list of all possible problem types! In other words, you are responsible for all topics covered during the period leading up to this exam, whether they are represented in this list or not. See your notes and the suggested homework for a comprehensive list.

1. Set up one double integral to integrate the function x^2+y^2 over the region in the plane between the cardioid $r = 1+\cos(\theta)$ and the circle of radius 2 centered at the origin, pictured below. Use polar coordinates, simplify the integrand as much as you can, and do not evaluate the integral.

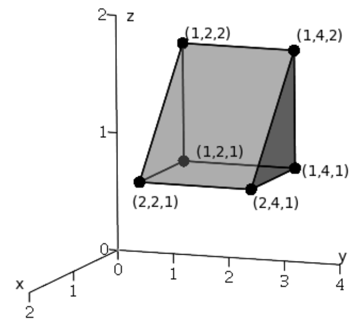


$$\int\int \_\_\_\_\_\_\_\_ d\_\_ d\_\_$$

2. Consider the triangular prism T at the top of the next page. The back wall lies on the plane $x = 1$, the side walls are triangles on the planes $y = 2$ and $y = 4$, the bottom lies on the plane $z = 1$, and the other face (the gray rectangle at the front) lies on the plane with equation $x + z = 3$.

   (a) Set up the triple integral to find the volume of T using the variable order $dz\,dy\,dx$:

$$\int\int\int 1\, dz\, dy\, dx$$



(b) Set up the triple integral to find the volume of T using the variable order $dy\ dx\ dz$:

$$\int \int \int \quad 1\ dy\ dx\ dz$$

(c) Actually find the volume of T, using either of the previous parts or any other method.

3. (a) Consider the substitution $\begin{cases} x = 6u + 2v, \\ y = 3u - 3v. \end{cases}$
Compute $|J|$, the absolute value of the determinant of the Jacobian.

(b) Suppose the bounds of an integral over solid V (top of the next page) are

$$\int_1^2 \int_0^{\frac{1}{x}} \int_0^2 \ldots dz\ dy\ dx.$$

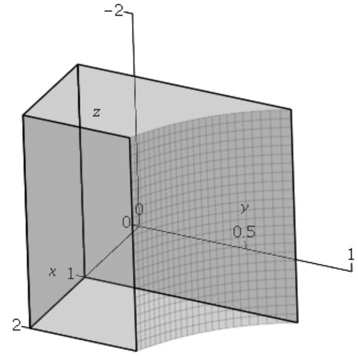
Consider the substitution $\begin{cases} \mathtt{x} = \mathtt{u}, \\ \mathtt{y} = \mathtt{v}/\mathtt{u}, \\ \mathtt{z} = \mathtt{w}/3 \end{cases}$

Use this substitution to transform the (x, y, z) bounds above into (u, v, w) bounds (ignoring the integrand):

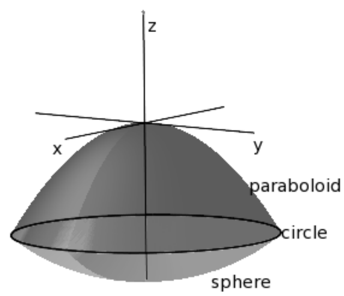
$$\int \int \int \ldots dw\ dv\ du.$$

4. One parameterization for the unit circle C is $\mathbf{r}(t) = \langle \cos 2t, \sin 2t \rangle$, for $0 \le t \le \pi$.

(a) Set up **but do not evaluate** the line integral of $f(x, y) = x^2 + y^2 + 3$ along C, using the parameterization above.



(b) Set up **<u>and evaluate</u>** the line integral to calculate the work done by moving through vector field $\mathbf{F}(x, y) = \langle -y, x\rangle$ while moving around C, using the parameterization above.



5. Let S be the solid above the sphere $x^2 + y^2 + z^2 = 2$ (light gray in the figure above) and below the paraboloid $z + x^2 + y^2 = 0$ (dark gray). Notice that the two surfaces intersect in the circle $x^2 + y^2 = 1$ on the plane $z = -1$.

   (a) Set up but **do not evaluate** a single triple integral to find the volume of S using *cylindrical* coordinates.

   (b) Set up but **do not evaluate** *two* triple integrals (added together) to find the volume of S using *spherical* coordinates.

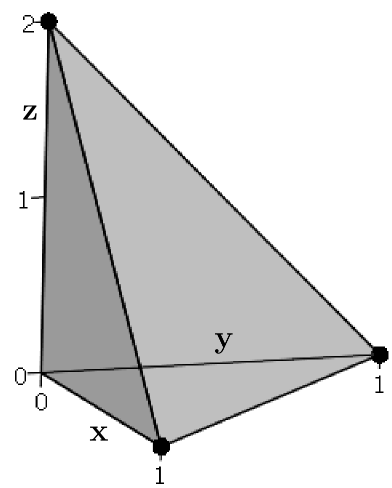
6. Let R be the tetrahedron in the first octant bounded by the coordinate planes and the plane passing through $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 2)$ with equation $2x + 2y + z = 2$, as shown below. Using rectangular coordinates, set up the triple integral to find the volume of R in each of the two following variable orders, but **DO NOT EVALUATE**.

   (a)

$$\int \int \int 1 \; dxdydz$$

(b)

$$\int \int \int 1 \; dzdydx$$



7. The two parts of this problem are both about substitution but do not depend on each other.

   (a) (10 pts) Suppose we want to replace x and y with u and v using the substitution

   $$\begin{cases} x = u^2 + v^2, \\ y = 3uv - 2v^2. \end{cases}$$

   Compute J, the determinant of the Jacobian for this substitution.

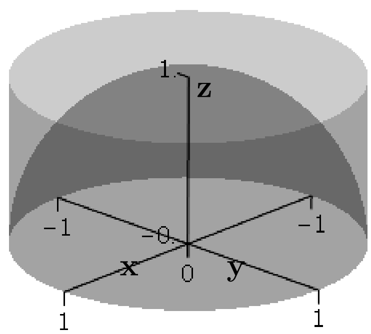
   (b) (8 pts) Suppose we want to integrate $f(x, y) = x + y$ over the diamond U with boundary lines $y = x + 1, y = x - 1, y = -x + 1,$ and $y = -x - 1$. Fill in the three blank spaces below (two bounds and the integrand) to write the integral of $f(x, y) = x + y$ as an integral in u and v, using substitution $x = u + v + 1$, $y = u - v + 2$. The first and third bounding lines above have already been converted into (u, v) bounds and included below. **DO NOT EVALUATE** this integral. (Hint: You may use the fact that $J = -2$ for this substitution.)

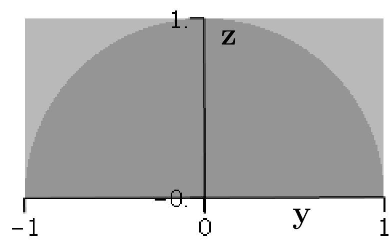
$$\int_0^{\_\_\_} \int_{\_\_\_}^{-1} \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ dudv$$

8. Consider the region S inside the cylinder $x^2+y^2=1$, outside the sphere $x^2+y^2+z^2=1$, and with z between 0 and 1. A view from above and a cross section (looking along the positive x axis) are shown below.

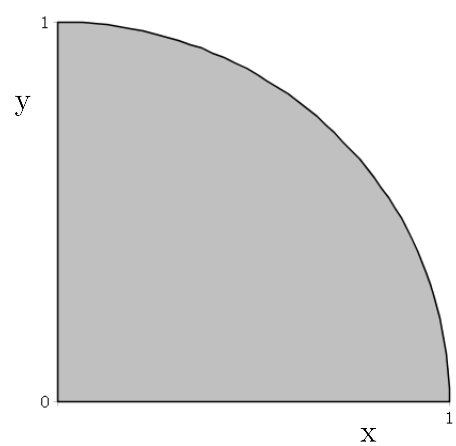
   (a) Set up one or more triple integrals to integrate a function $f(x, y, z)$ over S using **cylindrical** coordinates (using order $dz\ dr\ d\theta$), but **DO NOT EVALUATE**.

   (b) Set up one or more triple integrals to integrate a function $f(x, y, z)$ over S using **spherical** coordinates (using order $d\rho\ d\phi\ d\theta$), but **DO NOT EVALUATE**.





9. Consider the thin plate T consisting of the portion of the unit disk in the first quadrant of the (x,y)-plane (as shown below). Let $\delta(x, y) = \sqrt{x^2+y^2}$ be the density of T.



   Set up but **DO NOT EVALUATE** an integral to find the mass of T using **polar** coordinates.

10. Consider the region R within the cylinder $x^2+y^2 \leq 4$, bounded below by $z=0$ and above by $z=2-y$. Assume a mass density of $\delta = z$. [This is an older problem; any problem involving a 3D region like this will include a sketch on your exam.]

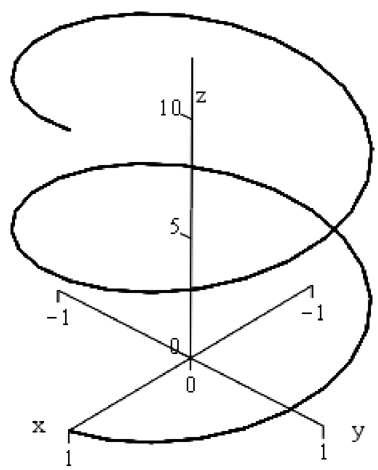
(a) Set up **and evaluate** the integral representing the mass of the solid, using cylindrical coordinates.

(b) Define, but **do not evaluate**, the integral representing M_{yz}, again with cylindrical coordinates.

11. A thin wire with density function $\delta(x, y, z) = x + y^2 + z^3$ can be represented by a helical curve C (shown below), given by the parameterization

$$\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle,$$

for $0 \leq t \leq 4\pi$. Set up a line integral for the mass of this wire. Please put a box around your answer. **DO NOT EVALUATE** the integral.



12. Consider the parameterization

$$\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$$

of a helical curve C. Set up **AND EVALUATE** the line integral to calculate the work done by moving through vector field

$$\mathbf{F}(x, y, z) = \langle 2x, 2y, 4z \rangle$$

while moving along C from $(1, 0, 0)$ to $(1, 0, 2\pi)$.

SOLUTIONS

WARNING: These solutions are not fully justified. Be sure to provide full justification with your solutions (especially where this is explicitly requested) so that we may provide partial credit, where applicable. If you are having trouble getting these answers, please come to office hours and/or exam review sessions. See the course website for details. Of course, if there seems to be an error with these solutions (which is very possible!), please let an instructor or the coordinator know.

1.
$$\int_0^{2\pi} \int_{1+\cos(\theta)}^2 r^3 dr d\theta$$

2. (a)
$$\int_1^2 \int_2^4 \int_1^{-x+3} 1 \; dz \; dy \; dx$$

(b)
$$\int_1^2 \int_1^{3-z} \int_2^4 1 \; dy \; dx \; dz$$

(c) 1

3. (a) 24

(b)
$$\int_1^2 \int_0^1 \int_0^6 \ldots dw \; dv \; du.$$

4. (a)
$$\int_0^{\pi} 8dt.$$

(b)
$$\int_0^{\pi} 2dt = 2\pi.$$

5. (a)
$$\int_0^{2\pi} \int_0^1 \int_{-\sqrt{2-r^2}}^{-r^2} dz \; r \; dr \; d\theta.$$

(b)

$$\int_0^{2\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}}\int_0^{-\frac{\cos\phi}{\sin^2\phi}}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta+\int_0^{2\pi}\int_{\frac{3\pi}{4}}^{\pi}\int_0^{\sqrt{2}}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

6. (a)

$$\int_0^2\int_0^{\frac{2-z}{2}}\int_0^{\frac{2-2y-z}{2}} 1\ dxdydz$$

(b)

$$\int_0^1\int_0^{1-x}\int_0^{2-2x-2y} 1\ dzdydx$$

7. (a) $6u^2-8uv-6v^2$

(b) $\displaystyle\int_0^1\int_{-2}^{-1} 2(2u+3)dudv$

8. (a)

$$\int_0^{2\pi}\int_0^1\int_{\sqrt{1-r^2}}^1 f\cdot rdzdrd\theta$$

(b)

$$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_1^{\sec\phi} f\cdot\rho^2\sin\phi d\rho d\phi d\theta+\int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_1^{\csc\phi} f\cdot\rho^2\sin\phi d\rho d\phi d\theta.$$

9.

$$\int_0^{\frac{\pi}{2}}\int_0^1 r^2drd\theta.$$

10. (a)

$$M=\int_0^{2\pi}\int_0^2\int_0^{2-r\sin\theta} zr\ dz\ dr\ d\theta=10\pi.$$

(b)

$$M_{yz} = \int_0^{2\pi} \int_0^2 \int_0^{2-r\sin\theta} zr^2 \cos\theta \ dz \ dr \ d\theta.$$

11.

$$\int_0^{4\pi} (\cos(t) + \sin^2(t) + t^3)\sqrt{2}dt.$$

12.

$$\int_0^{2\pi} 4tdt = 8\pi^2.$$