Math 261 Course Outline

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(with minor revisions by Dan Bates)

Contents

Course goals and basic outline

Students will be exposed to vectors (in $\mathbb{R}^2$ and $\mathbb{R}^3$), vector-valued functions $f : \mathbb{R} \to \mathbb{R}^n$, for $n \in \{2, 3\}$, the **TNB**-frame, real-valued functions $f : \mathbb{R}^n \to \mathbb{R}$, gradients of real-valued functions, unconstrained optimization, constrained optimization via Lagrange multipliers, double integrals, triple integrals, polar, cylindrical, and spherical coordinates, vector fields, line integrals, potential functions, Green's theorem in the plane, surface integrals, Stokes' theorem, and Divergence theorem.

In general, only include proofs if (a) they are quick or (b) you are ahead. Quizzes and exams will be primarily computational (sometimes conceptual) in nature.

You may assume the students remember Calculus 1 and 2. You are welcome to remind them of bits as you see fit, but it is not your job to reteach them material they should remember. They can go to TILT (or office hours) for help.

CHAPTER 12

Vectors and the geometry of space

1. Three-Dimensional Coordinate Systems

Goals:

- Plot points in $\mathbb{R}^3$,
- Interpret solutions to equations and inequalities geometrically,
- Compute (Euclidean) distance between two points,
- Plot spheres, and algebraically determine their radius and center.

Summary:

- Most of these students have never done any sort of graphing in $\mathbb{R}^3$, so make sure you are defining everything you need. Discuss the Cartesian coordinate system and how to plot points given a triple of real numbers. Identify things like the origin, the coordinate planes, the first octant.
- Interpret solutions to equations and inequalities geometrically. Discuss how to interpret multiple equations and/or inequalities geometrically.
- State the distance formula.
- Discuss spheres. Talk about how to plot them in $\mathbb{R}^3$, and use algebra to find the center and radius of the sphere from it's equation.

Suggested Examples:

(1) Plot a few points.
(2) Plot a few systems of equations and/or inequalities.
(3) Compute the distance between two points.
(4) Plot a sphere given by an equation which is not in its "nicest" form.

Tips:

This is quite a short section, so do not spend too much time. However, you do not want to go too fast since this builds the foundation for everything we do. You want to make sure everyone is on the same page.

2. Vectors

Goals:

- Conceptually understanding of vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$,
- Compute the vector from point P to point Q,
- Compute the magnitude of a vector,
- Add and scale vectors,
- Interpret vectors as magnitude with direction.

Summary:

- Introduce the concept of a vector. Talk about initial and terminal points, and how vectors in standard form can be identified by their terminal point.

Define the component form of a vector, and discuss how we can determine a vector's magnitude.
- State the basic algebraic operations for vectors: addition and scalar multiplication. Talk about what this looks like geometrically.
- Define unit vectors and state the standard unit vectors **i**, **j**, and **k**. Describe how we can interpret vectors as recording magnitude and direction.
- Feel free to mention the midpoint of a line segment, but I do not see how this comes up in the future.
- SKIP the subsection on applications, but encourage the students to work through the examples.

Suggested Examples:

(1) Draw vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ and write them in component form.
(2) Compute the magnitude of various vectors.
(3) Solve various algebraic equations with vectors.
(4) Given a vector **v**, determine the unit vector in the same direction as **v**.
(5) Given a vector **v**, rewrite **v** as its magnitude times its direction.

Tips:

Use angle bracket notation or **i**, **j**, **k** notation as you see fit. Both are used in the book and either could appear on exams.

3. The Dot Product

Goals:

- Compute the angle between two vectors,
- Understand the connection between the dot product and the angle between two vectors,
- Algebraically determine if two vectors are orthogonal,
- Given vectors **u** and **v**, compute the projection of **u** onto **v**.

Summary:

- Discuss the basic idea for computing the angle between two vectors, and state the theorem. Do not prove the theorem, but convince the students that they could prove it if they needed.
- Define the dot product, and discuss how the dot product is related to angles between vectors.
- Talk about what it means for two vectors to be orthogonal. Furthermore, discuss how we can determine if two vectors are orthogonal using algebra.
- Introduce the idea of projection. Projection is an important concept used throughout the coming chapters. I believe it is important for students to really understand this. You can discuss work as an application of projection, but this is not necessary.

Suggested Examples:

(1) Compute the angles between various pairs of vectors and check your answers geometrically.
(2) Compute various projections and check your answers geometrically.
(3) Compute the work done by a constant force (optional).

Tips:

None so far!

4. The Cross Product

Goals:

- Conceptual understanding of the cross product,
- Compute the cross product of two vectors,
- Compute the area of a parallelogram determined by two vectors,
- Compute the volume of a parallelepiped determined by three vectors.

Summary:

- Introduce the notion of the cross product between two vectors. Give them an idea for why we might care and how the cross product relates to the other two vectors. At this point, students do not know how to compute a cross product.
- Discuss the properties of the cross product. The cross product behaves differently than anything the students have seen so far, so make sure the students are aware.
- Talk about how we can interpret the area of the parallelogram determined by $\mathbf{u}$ nd $\mathbf{v}$ as $|\mathbf{u} \times \mathbf{v}|$. This comes up in section 16.5 for a derivation of surface area, but is otherwise not too important.
- Teach the students how to caluclate the cross product of two vectors as a determinant of a 3 by 3 matrix. There are other approaches, but 3 by 3 determinants come up later (for substitutions when integrating), so please introduce the 3 by 3 determinant technique here.
- SKIP the subsection on torque.
- Discuss how we can compute the volume of a parallelepiped using the cross product and dot product. This is not terribly important, so do not spend much time here.

Suggested Examples:

(1) Do various computations of cross products.
(2) Given two vectors, determine the area of the parallelogram they determine.
(3) Given two vectors, determine the area of the triangle they determine.
(4) Given three vectors, determine the volume of the parallelepiped they determine.

Tips:

Somewhere in this section, the text writes that two vectors are parallel if, and only if, their cross product is the zero vector. Students think this is the way one determines if two vectors are parallel, and so they waste their time on exams. Of course the text is not wrong, but the students assume this is the only way. Remind them that they can see if one vector is a multiple of the other.

5. Lines and Planes in Space

Goals:

- Determine the vector equation or parametric equations of the line determined by either two points or a point and a vector,
- Determine the equation of the plane determined by either three points or a point and a vector,

- Given a point S and a line L, compute the (shortest) distance from S to L,
- Given a point S and a plane P, compute the (shortest) distance from S to P,
- Given two planes, determine their line of intersection and their angle of intersection.

Summary:

- This is a hefty section with a lot of material, and it is likely the first hard section in this course. Discuss the vector equation and parametric equations for the line passing through the point P and in the direction of $\mathbf{v}$. Even though the output of a vector equation is a vector, we graph it by plotting the terminal points of each of the vectors (through this is obvious to us, it is not so obvious to some students).
- Discuss how we could compute the shortest distance between a line and another point off the line.
- Introduce planes, and discuss how we can determine equations for planes given a point and a vector. Define a normal vector to the plane.
- Talk about how we can determine whether or not two planes intersect, and if they intersect, determine their line of intersection. Furthermore, determine the angle between the two planes.
- Derive a formula that computes the shortest distance from a plane to a point off the plane.

Suggested Examples:

(1) Determine both vector and parametric equations for lines fitting certain parameters.
(2) Determine equations for planes fitting certain parameters.
(3) Compute the distance from a line to a point off the line.
(4) Compute the distance from a plane to a point off the plane.
(5) Determine if two planes intersect, and if they do, then determine the line and angle of intersection.

Tips:

This section is all about geometry. Students have a lot of assumptions about lines, but they are built in $\mathbb{R}^2$. I find it helpful to bring these assumptions to light, and discuss their validity in $\mathbb{R}^3$. I encourage students to transfer their intuition about lines in $\mathbb{R}^2$ to planes in $\mathbb{R}^3$.

6. Cylinders and Quadric Surfaces

Goals:

- Graph cylinders (book's definition) in $\mathbb{R}^3$,
- Graph and identify various quadric surfaces.

Summary:

- Define a cylinder (book's definition). These are "easy" surfaces to graph, provided it is easy to graph in the plane. Note the key feature of cylinders is that their equations contain exactly two of the three variables.
- Before discussing quadric surfaces, remind them of conic sections (if you need a reminder as well, check 11.6). Do not worry about discussing all the geomteric properties, but mention the generic equations.

- Talk about the various quadric surfaces. Discourage students from just memorizing the equations. Instead teach them how to determine the surface based on slicing the surface with various planes. For each quadric surface, identify each of the three cross sections by its conic section (the one exception is the cone).

Suggested Examples:

(1) Given a quadratic equation in x, y, z, identify the quadric surface.
(2) Match equations with their quadric surfaces.

Tips:

At this stage, we only know how to graph lines, planes, and spheres. This section we will add more shapes to our arsenal.

This section starts to test the imagination and visualization skills of the students. Make sure you know how to draw your surfaces before you lecture – your students will learn how to draw and think in three dimensions from you, so take this seriously!

If you have not been using different colors, I strongly suggest you start using different colors as you draw. This will help the students distinguish different shapes on one set of axes.

Urge them not to commit the table to memory as it will not be directly tested. However, they might be given an equation and expected to plot it as part of the solution of a problem. It is highly unlikely that such plots would be graded, though, as grading sketches is very messy business.

7. Review

12.1
- Compute the distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2)
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$
- Determine the radius and center of a sphere
$$r^2 = (x - a)^2 + (y - b)^2 + (z - c)^2.$$

12.2
- Compute the vector from (x_1, y_1, z_1) to (x_2, y_2, z_2)
$$\mathbf{v} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle.$$
- Compute the magnitude of $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$
$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$
- Given a vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, find the unit vector in the direction of $\mathbf{v}$
$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}.$$

12.3
- Compute the dot product of two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$
$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta = u_1 v_1 + u_2 v_2 + u_3 v_3.$$
- Given vectors $\mathbf{u}$ and $\mathbf{v}$ compute the angle between them
$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right).$$
- Determine if two vectors are orthogonal.

- Compute the projection of $\mathbf{u}$ onto $\mathbf{v}$

$$\text{proj}_{\mathbf{v}}\,\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}\right)\frac{\mathbf{v}}{|\mathbf{v}|}.$$

12.4 • Compute the cross product of two vectors $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$

$$\mathbf{u}\times\mathbf{v} = (|\mathbf{u}||\mathbf{v}|\sin\theta)\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

- Compute the area of the parallelogram determined by two vectors $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$

$$A = |\mathbf{u}\times\mathbf{v}|.$$

- Compute the volume of the parallelepiped determined by three vectors $\mathbf{u} = \langle u_1, u_2, u_3\rangle$, $\mathbf{v} = \langle v_1, v_2, v_3\rangle$, and $\mathbf{w} = \langle w_1, w_2, w_3\rangle$

$$V = |(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}|.$$

12.5 • Determine the equation of the line passing through (a, b, c) in the direction of $\mathbf{v} = \langle v_1, v_2, v_3\rangle$

$$\ell(t) = \langle a, b, c\rangle + t\langle v_1, v_2, v_3\rangle$$

$$x = a + v_1 t, \quad y = b + v_2 t, \quad z = c + v_3 t.$$

- Given a line $\ell(t) = \mathbf{P} + \mathbf{v}t$ and a point S off the line, determine the distance from S to ℓ

$$d = \frac{|\overrightarrow{PS}\times\mathbf{v}|}{|\mathbf{v}|}.$$

- Determine the equation of the plane containing the point (a, b, c) with normal $\mathbf{n} = \langle n_1, n_2, n_3\rangle$

$$n_1(x-a) + n_2(y-b) + n_3(z-c) = 0.$$

- For a plane with normal $\mathbf{n}$ containing the point P and a point S off the plane, determine the distance from S to the plane

$$d = \frac{|\overrightarrow{PS}\cdot\mathbf{n}|}{|\mathbf{n}|}.$$

- For planes with normals $\mathbf{n}_1$ and $\mathbf{n}_2$, determine the angle between them

$$\theta = \cos^{-1}\left(\frac{\mathbf{n}_1\cdot\mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right).$$

- For intersecting planes with normals $\mathbf{n}_1$ and $\mathbf{n}_2$ and common point $\mathbf{P}$, determine the line of intersection

$$\ell(t) = \mathbf{P} + t(\mathbf{n}_1\times\mathbf{n}_2).$$

12.6 • Graph and identify cylinders (book's definition).
- Graph and identify quadric surfaces

Ellipsoid: $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Elliptical Paraboloid: $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$

Elliptical Cone: $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

Hyperboloid of one sheet: $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Hyperboloid of two sheets: $$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Hyperboloic paraboloid: $$\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}$$

CHAPTER 13

Vector-valued functions and motion in space

1. Curves in Space and Their Tangents

Goals:

- Understanding of vector-valued functions,
- Compute limits of vector functions,
- Understand continuity and differentiability in the context of vector functions,
- Compute derivatives of vector functions,
- If a vector function is interpreted as position, know how to compute velocity, speed, and acceleration.

Summary:

- Introduce the concept of a vector equation. Right now, students only know these things as vector equations for lines. Get them to start thinking about these functions as position functions since this is the interpretation used throughout the course.
- Discuss the limit definition for vector equations. This should look familiar to the definition for functions of one variable. Give them the conclusion that the limit of the vector equation is the limit of each of the component functions. Use this to quickly define all the calculus gadgets for vector equations: continuity, differentiability. Don't waste much time on these bits.
- Define what it means to be smooth and piecewise smooth. These will be used in this chapter and in chapter 16.
- Interpreting $\mathbf{r}(t)$ as a position function, define velocity, speed, and acceleration. Illustrate how $\mathbf{r}'(t)$ points in the direction of motion, i.e., along the tangent.
- Even though it is used in a later section, you can skip "Vector Functions of Constant Length."

Suggested Examples:

(1) Plot various vector functions in $\mathbb{R}^3$.
(2) Compute a few limits and derivatives of vector functions.

Tips:

This section is actually pretty quick and easy. You can easily do this in about 30 minutes.

2. Integrals of Vector Functions; Projectile Motion

Goals:

- Compute definite and indefinite integrals for vector functions.

Summary:

- Using the fact that limits of vector functions are limits of the component functions, quickly discuss how to compute indefinite integrals of vector functions. This, therefore, applies to definite integrals.
- SKIP projectile motion.

Suggested Examples:

(1) Do an example of a definite/ indefinite integral.

Tips:

This should take 15 minutes.

3. Arc Length in Space

Goals:

- For a curve parametrized by $\mathbf{r}(t)$, compute the arc length from $t = a$ to $t = b$.
- For a curve parametrized by $\mathbf{r}(t)$, compute the unit tangent vector.

Summary:

- Start by talking about arc length of a curve. They have seen arc length for planar curves from the second semester of calculus, so the formula for a curve in $\mathbb{R}^3$ should be believable.
- Define the arc length parameter. From here on out, the variable s is fixed to this meaning. It is used very heavily in chapter 16. Discuss interpretations of ds/dt.
- Define the unit tangent vector. It will be helpful to also use

$$\mathbf{T} = \frac{d\mathbf{r}}{ds}$$

for the next section.

Suggested Examples:

(1) Compute the arc length between two t values.
(2) Compute the arc length between two points on the curve.
(3) Compute $\mathbf{T}$ for various vector functions.

Tips:

This section builds part of the foundation for 16.1.

4. Curvature and Normal Vectors of a Curve

Goals:

- Understand the idea of curvature as it relates to curves in $\mathbb{R}^3$,
- Understand and compute the principal unit normal vector for a vector function.

Summary:

- Introduce the notion of curvature and state how to compute it. We want to emphasize intuition here. Students will not need to compute curvature on exams, but it is still fundamental to the principal unit normal vector.
- Define the principal unit normal vector, and explain how it relates to curvature. Then derive a formula that one can use for computing $\mathbf{N}$.

- You can skip "Circle of Curvature for Plane Curves," but Figure 13.20 might be worth drawing and explaining.
- If you only defined curvature and **N** for plane curves, generalize it for curves in $\mathbb{R}^3$.

Suggested Examples:

(1) Do an example or two where you compute **N** for a given $\mathbf{r}(t)$.

Tips:

Most of the formulas in this section go unused. In the next section, we come up with an easier way for computing **N** in terms of **T** and **a**. Students should not bother committing this stuff to memory.

5. Tangential and Normal Components of Acceleration

Goals:

- A decent understanding of the **TNB** frame,
- Understand how **v** and **a** fit into the **TNB** frame,
- Given $\mathbf{r}(t)$, compute the tangential and normal scalar components of acceleration.

Summary:

- Define the binormal vector, and discuss how these three unit vectors fit together to form the **TNB** frame (or Frenet frame). This animation might help students understand `http://tinyurl.com/qexokbv`. Unfortunately, students do not know the meaning of a basis. Communicate to them that everything we care about can be built from **T**, **N**, and **B**, much like **i**, **j**, and **k**. In fact, the **TNB** frame might be better, especially for moving objects.
- Viewing **TNB** as a desirable basis, discuss how to rewrite **v** and **a** in terms of **TNB**. Make sure to note that **a** is always in the plane determined by **T** and **N**.
- Discuss all formulas that go into rewriting **a** in terms of **T** and **N**. There are two formulas not found in the text which are incredibly useful

$$a_T = \mathbf{a} \cdot \mathbf{T}, \qquad a_N = \mathbf{a} \cdot \mathbf{N}.$$

  The above equations are obtained when we take the magnitude of the projection of **a** onto either **T** or **N**.
- Skip Torsion.

Suggested Examples:

(1) Rewrite **v** and **a** in terms of **T** and **N** and check that they make sense geometrically.
(2) Given **r**, **v**, and **a** at a specific t, determine **T**, **N**, a_T and a_N.

Tips:

None so far!

6. Velocity and Acceleration in Polar Coordinates

SKIP!

7. Review

13.1
- Compute limits of vector functions
$$\lim_{t\to a}\langle f(t), g(t), h(t)\rangle = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\right\rangle.$$
- Compute derivatives of vector functions
$$\frac{d}{dt}\langle f(t), g(t), h(t)\rangle = \left\langle \frac{df}{dt}, \frac{dg}{dt}, \frac{dh}{dt}\right\rangle.$$
- Viewing $\mathbf{r}(t)$ as position at time t, compute its velocity $\mathbf{v} = d\mathbf{r}/dt$, its speed $|\mathbf{v}|$, and its acceleration $\mathbf{a} = d\mathbf{v}/dt = d^2\mathbf{r}/dt^2$.

13.2
- Compute indefinite integrals of vector functions
$$\begin{aligned}\int \langle f(t), g(t), h(t)\rangle\, dt &= \left\langle \int f(t)\, dt, \int g(t)\, dt, \int h(t)\, dt\right\rangle \\ &= \langle F(t) + C_1, G(t) + C_2, H(t) + C_3\rangle.\end{aligned}$$
- Compute definite integrals of vector functions
$$\int_a^b \langle f(t), g(t), h(t)\rangle\, dt = \left\langle \int_a^b f(t)\, dt, \int_a^b g(t)\, dt, \int_a^b h(t)\, dt\right\rangle.$$

13.3
- Compute the arc length of $\mathbf{r}(t)$ from $t = a$ to $t = b$
$$L = \int_a^b |\mathbf{v}|\, dt.$$
- Compute the unit tangent vector of $\mathbf{r}(t)$
$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}.$$

13.4
- Given $\mathbf{r}(t)$, compute $\mathbf{N}$
$$\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}.$$

13.5
- Rewrite acceleration in terms of $\mathbf{T}$ and $\mathbf{N}$
$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}.$$
- Compute the tangential and normal scalar components of acceleration
$$a_T = \frac{d^2s}{dt} = \frac{d}{dt}|\mathbf{v}| = \mathbf{a}\cdot\mathbf{T}$$
$$a_N = \mathbf{a}\cdot\mathbf{N}.$$
- Understand that the lengths of $\mathbf{a}$, $\mathbf{T}$, $\mathbf{N}$ form a right triangle, so
$$|\mathbf{a}|^2 = a_T^2 + a_N^2.$$

CHAPTER 14

Partial derivatives

1. Functions of Several Variables

Goals:

- Proficient understanding of what it means to be a function of multiple variables,
- Compute domains and ranges of functions with several variables,
- Decent understanding of open and closed sets in $\mathbb{R}^2$,
- Graph (reasonable) functions of two variables,
- Proficient understanding of level sets and how they relate to the graph of a function.

Summary:

- Define what it means to be a function $f : \mathbb{R}^n \to \mathbb{R}$, but, of course, use the book's definition. Then discuss domain and range. Most students have a good understanding of this based off what they learned for 1 variable functions.
- Define interior, exterior, and boundary points for sets in $\mathbb{R}^2$, and then talk about open, closed, and bounded sets (not necessarily using all adjectives). It might be fun to talk about sets which are neither open nor closed and sets which are both open and closed – not really necessary though. The goal is not for them to prove stuff with these definitions. We are just trying to give them vocabulary for future theorems. Often, we will need an open set D in $\mathbb{R}^2$ for the theorem.
- Graph some functions and define level curves. Explain how we can use level curves to get an understanding of the shape of the graph, and draw some contour maps.
- Mention that for real-valued functions of 3 variables, one way to attempt to visualize the graph is to look at level surfaces. The idea of level surfaces comes up a few times throughout this chapter.

Suggested Examples:

(1) Compute the domains and ranges of several functions.
(2) Draw some examples of sets that are open and not open, closed and not closed, bounded and not bounded.
(3) Draw some graphs with level curves.

Tips:

There is a lot to discuss, but the students already have some decent intuition. Although you do not want to spend too much time in this section, you still want to make sure everyone has a good foundation.

2. Limits and Continuity in Higher Dimensions

Goals:

- Conceptual understanding of limits for functions of 2 variables,
- Proficient understanding of continuity,
- Prove limits do not exist by exhibiting two paths which approach different values,
- Use algebra to compute a limit.

Summary:

- Define the limit of $f(x, y)$ as (x, y) approaches (a, b) using ε and δ. It is not required that they use this definition. Many of these students had to use that definition in the first semester of calculus, so it will look familar. One reason for presenting this definition is to show students that limits for functions of 2 variables are really no different from 1 variable. This will allow them to transfer all the (hopefully correct) intuition from previous caluclus courses.
- Define continuity for functions of 2 variables. Again, this is the same as it was for 1 variable.
- Discuss what we expect of them when it comes to proving a limit exists. If the function is not defined at the point P the limit is approaching, then we expect them to use algebra to manipulate the expression and eventually get a function that is continuous at P. On the other hand, if the limit does not exist, then we expect them to provide at least two different paths resulting in two different limits.

Suggested Examples:

(1) Do some trivial limit computations.
(2) For some function f that is not continuous at a point P, try to find some value a such that if $f(P) = a$, then f is continuous. Or show no such a exists.
(3) Do various limits that do not exist by using nontrivial paths.

Tips:

I have found that if in all your examples your limits go to $(0, 0)$, then your students will have a hard time if a limit goes to, say, $(1, -2)$. Make sure to approach different points besides the origin. A common mistake is for students to use paths that do not actually approach the point you care about. Make a note of this either by an example or just telling them.

The book provides some decent pictures of graphs which do not have limits as $(x, y) \to (0, 0)$. This may be helpful.

3. Partial Derivatives

Goals:

- Compute partial derivatives of functions of multiple variables,
- Compute partial derivatives of higher order,
- Working understanding of differentiability.

Summary:

- Introduce the idea of a partial derivative, define it as a limit. The students have already suffered through these limit definitions, so this should look

familiar to students. Use the limit definition to convince the students that all other variables are held as constants. Do not limit yourself to only 2 variables.
- Discuss how all the derivative rules from the first semester of calculus still apply (e.g., product rule, chain rule, etc.).
- Take more derivatives! Do higher order derivatives, and state the Mixed Derivative Theorem.
- Discuss differentiability. It is important that they realize differentiability is a bit more delicate for functions with more than 1 variable. The mere existence of a partial derivative is not enough. You can briefly go through example 8 to make this point.
- Theorem 3 and the definition of differentiability are a mess, and the students will not retain it. Let alone get any intuition from it. Do what you want with these, but make sure you state the corollary which ties differentiability to the continuity of the partials and the original function. Theorem 3 will not appear on the exams.

Suggested Examples:

(1) Do various partial derivatives which require the (possibly forgotten) rules from the first semester calculus.
(2) Do implicit differentiation.
(3) Compute higher order derivatives, and verify the Mixed Derivative Theorem.
(4) Check that functions are (or are not) differentiable.

Tips:

At this stage, students still believe there's one derivative (to rule them all). I pose the problem of finding rates of change, which the students readily parse as "derivatives." Just as we did in the first semester of calculus, I introduce the idea of a partial derivative using tangent lines. For a specific point on a smooth surface, there are many tangent lines going in all different directions, but in this section, we focus on two specific directions. In 14.5, we see how to use these two directions to "build" all the other ones.

4. The Chain Rule

Goals:

- Take the derivative of functions by means of the chain rule,
- Use the chain rule to simplify implicit differentiation.

Summary:

- This section is generally pretty quick. Discuss the chain rule for functions of multiple variables. The book has multiple theorems for the different number of variables. This seems a bit silly, so make sure you explain the general idea.
- Introduce the diagram for determining the chain rule formula. This gives students a clear and organized way to get the correct formula.
- Explain how the chain rule helps with implicit differentiation.

Suggested Examples:

(1) Use the chain rule to compute the derivatives of functions with several variables which also depend on several variables.

(2) Do some implicit differentiation using the chain rule.

Tips:

None so far!

5. Directional Derivatives and Gradient Vectors

Goals:

- Compute and interpret directional derivatives,
- Compute gradients of real-valued functions of several variables,
- Understand how gradients connect with rates of change,
- Understand some geometric facts dealing with level curves, tangent lines, and gradients.

Summary:

- Define the directional derivation in the direction of $\mathbf{u}$ as a limit. Then use the chain rule to derive the formula for directional derivatives in terms of the gradient. Discuss how we can interpret the numerical value of the directional derivative.
- Discuss that the gradient points in the direction of most rapid increase, and that the rate of change in the direction of the gradient is just the magnitude of the gradient. Morevoer, talk about the most rapid decrease, and the directions of no change.
- Finally explain how this all fits together with respect to level curves in $\mathbb{R}^2$.
- There is no pressing need to list the algebra rules for the gradient.

Suggested Examples:

(1) Compute various directional derivatives where the given vector is not a unit vector.
(2) Compute various gradients and verify they point in the direction of most rapid ascent.
(3) Draw the level curves of an elliptical paraboloid. For a point on one of the curves, draw ∇f, $-\nabla f$, and the tangent line.

Tips:

Students sometimes have trouble understanding that $\mathbf{u}$ and $\nabla \mathbf{f}$ live "downstairs." A good drawing will help.

6. Tangent Planes and Differentials

Goals:

- Given a point P on a smooth surface, determine the equation of the tangent plane and normal line at P,
- Given a function f, compute its linearization, and determine an upper bound for its error in approximating f.

Summary:

- This section seems a bit out of place. At this point, the students should understand that smooth surfaces have tangent planes. There are two cases here: We can find the equation of the tangent plane of the *level surface*

$f(x, y, z) = c$ with normal ∇f or the tangent plane to the graph of a function given by $z = f(x, y)$. These are related, but different. Furthermore, define the normal line.
- SKIP "Estimating Change in a Specific Direction."
- Define the linearization of a function, and discuss that this is a form of approximating the function. In fact, this is the first order Taylor series approximation, but we will touch on this in 14.9.
- State the formula for an upper bound for the error in approximating with linearization. Generally with M, we do not need the sharpest upper bound, and therefore, we encourage students to choose integers. However, when the algebra is simple, students should be expected to find the best M. Give an example where they need to use a little algebra (e.g., where one partial is linear) and ram home the point that they must consider all points in the box, not just the center.
- SKIP "Differentials."

Suggested Examples:

(1) Find the equation of the tangent plane for surfaces given as $z = f(x, y)$ and $f(x, y, z) = c$.
(2) Two surfaces intersect at a curve. For a point on the curve, find the line tangent to the curve.
(3) Compute various linearizations and find an upper bound for the error.

Tips:

None so far!

7. Extreme Values and Saddle Points

Goals:

- Proficient knowledge of first and second derivative tests,
- Given a function, find all critical points,
- Given a function and a critical point, use the second derivative test to classify it as either a local max, local min, or saddle point, provided the test is conclusive.
- Given a function on a closed and bounded region, find all absolute extrema.

Summary:

- Define local extrema and critical points. State the first derivative test, and give some idea why this is true. This is a nice place for geometry!
- Of course we want to be able to classify our critical points, so state the second derivative test. Define the discriminant (or Hessian) and saddle points.
- Now consider absolute extrema on closed and bounded regions. Discuss strategies for finding such points (do an unconstrained search, then look along each boundary as a one-variable problem).

Suggested Examples:

(1) Find the critical points of various functions and classify them. Make sure to provide examples with local mins and maxs along with saddle points (not necessarily all in the same example).

(2) Find all absolute extrema of a function on a closed and bounded region. Choose a simple one as it takes time to cover the interior and all portions of the boundary!

Tips:

None so far!

8. Lagrange Multipliers

Goals:

- Use Lagrange multipliers to solve constrained optimization problems.

Summary:

- Consider Example 1 from the text as motivation for Lagrange multipliers.
- State and (optionally) sketch a proof of the Orthogonal Gradient Theorem. It is not important that the students know the proof, but it nice to see some of our recent ideas come together.
- With the Orthogonal Gradient Theorem in mind, discuss how we could find mins and maxs with constrained variables. Then introduce Lagrange multipliers.
- Mention that Lagrange multipliers works in higher dimension, and the ideas are not too different.

Suggested Examples:

(1) Consider doing Example 1 from the text to show that unconstrained approaches can be faulty.
(2) Do several examples using Lagrange multipliers.

Tips:

Often, when doing a Lagrange multiplier problem, we have to consider different cases. One of the most common examples is whether a variable is zero or nonzero. Students have difficulty identifying when this needs to happen (e.g., just divide by a variable without consider if it could be zero). Furthermore, they have trouble branching off into all the cases when this does occur. I have no solutions other than to work them out in front of them and hope they learn how to solve them.

9. Taylor's Formula for Two Variables

Goals:

- Working understanding of Taylor's formula for two variables.

Summary:

- Do not spend much time here. Write Taylor's formula up to the second or third term. After this, students can see the pattern. They do not need to memorize this formula. Remind them that the first order Taylor polynomial is just the linearization.
- Discuss the error approximation. This should look similar to the linearization error.

Suggested Examples:

(1) Write out a second order Taylor polynomial for a two variable function.

Tips:

I have skipped all examples in the past – especially if I am short on time.

10. Partial Derivatives with Constrained Variables

SKIP!

11. Review

14.1
- Compute domains and ranges of functions of several variables.
- Graph contour curves for functions $f(x, y)$.

14.2
- Use algebra and arguments involving continuity to compute limits of functions with two variables.
- Use two different paths which evaluate to two different limits to prove a limit does not exists.

$$\lim_{(x,g(x))\to(a,b)} f(x, g(x)) = c \neq d = \lim_{(x,h(x))\to(a,b)} f(x, h(x)).$$

14.3
- Compute partial derivatives of functions with several variables.
- Compute higher order partial derivatives.
- Use implicit differentiation to compute derivatives.
- Test whether a function is differentiable or not.

14.4
- Use the chain rule to compute the partial derivative of a function with respect to t

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t} + \cdots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial t}.$$

- Use the chain rule to simplify implicit differentiation

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}.$$

14.5
- Given a function f and a vector $\mathbf{u}$, compute the derivative of f in the direction of $\mathbf{u}$

$$D_{\mathbf{u}} f = \nabla f \cdot \frac{\mathbf{u}}{|\mathbf{u}|}.$$

- Determine whether or not a function is increasing or decreasing in a specified direction.
- Compute the gradient of a real-valued function

$$\nabla f = \left\langle \frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n} \right\rangle.$$

- Given a function f and a point P in its domain, determine the direction of most rapid increase, most rapid decrease, and directions of no change.

14.6
- Given a smooth surface $f(x, y, z) = c$ and a point (a, b, c) on the surface, find the equation of the tangent plane at (a, b, c)

$$f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c) = 0.$$

- Given a smooth surface $f(x, y, z) = c$ and a point (a, b, c) on the surface find the equation of the normal line at (a, b, c)

$$\ell(t) = \langle a + f_x(a, b, c)t, b + f_y(a, b, c)t, c + f_z(a, b, c)t \rangle.$$

- Given a function $f(x, y)$ and a point (a, b) in its domain, compute the linearization of $f(x, y)$ at (a, b)

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

- Determine a reasonable upper bound for the linearization of $f(x,y)$ at (a,b) on a region $R = \{(x,y) \mid |x-a| \le c, |y-b| \le d\}$
$$|E(x,y)| \le \frac{1}{2}M(|x-a|+|y-b|)^2.$$

14.7
- Use the first derivative test to find all the critical points of $f(x,y)$.
- Use the second derivative test to classify all the critical points as local mins, local maxes, or saddle points, provided the test is conclusive.
- Given a function f and a closed and bounded region R, find all absolute extrema of f on R.

14.8
- Use Lagrange multipliers to minimize or maximize a fucntion $f(x,y,z)$ subject to the constraint of $g(x,y,z) = 0$
$$\nabla f = \lambda \nabla g.$$
- In the context of Lagrange multipliers, determine which function we want to minimize/ maximize and which function is our constraint function.

14.9
- Given a function $f(x,y)$, compute its second order Taylor polynomial at (a,b)

$$\begin{aligned} Q(x,y) = f(a,b) \\ &+ f_x(a,b)(x-a) + f_y(a,b)(y-b) \\ &+ \frac{1}{2}\left(f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2\right). \end{aligned}$$

CHAPTER 15

Multiple integrals

1. Double and Iterated Integrals over Rectangles

Goals:

- Evaluate double integrals over rectangles,
- For a surface above the xy-plane, compute the volume underneath the surface.

Summary:

- Introduce the idea of computing volume under a surface (similar to area under a curve). It will be beneficial to go through the motions of partitioning R into rectangles and approximating the volume as a kind of Riemann sum. These ideas of approximating by partitioning come up over and over again in the coming sections.
- Discuss how we can compute the volume by looking at either horizontal or vertical cross sections. Thinking in terms of cross sections will come up several times in this chapter.
- State Fubini's Theorem, but do not prove it.

Suggested Examples:

(1) Various integrals over rectangles in different orders.

Tips:

This is the first major section requiring integration. In the later sections, I usually do not go through all the steps of integration, but I find that, at this stage, students could use a reminder.

2. Double Integrals over General Regions

Goals:

- Evaluate double integrals over more general regions,
- Given a region, find the limits of integration for that region,
- Given a double integral, determine the region of integration,
- Setup and evaluate (if possible) integrals two different ways.

Summary:

- The discussion of double integrals over general regions should look like it did in 15.1. You should obtain the formula by approximating the volume by Riemann sums.
- State Fubini's Theorem, and do not prove it. It is worth spending time discussing how to find the limits of integration. You should be using words like "horizontal cross-sections" and "vertical cross-sections."

- It's not necessary to write all the properties of double integrals (the table on page 846). These are very reasonable, and follow directly from single integrals.

Suggested Examples:

(1) Do a simple double integral, but solve it both ways (ie. $dxdy$ and $dydx$).
(2) Compute a double integral over a region which favors one order over the other.
(3) Compute a double integral whose integrand favors one order over the other.

Tips:

None so far!

3. Area by Double Integration

Goals:

- Set up and interpret the area of a region in the xy-plane as a double integral,
- Compute average values of functions over regions.

Summary:

- This is a baby section, and it really feels like it should be part of 15.2. Nevertheless, illustrate how double integrals could be interpreted as area.
- State the formula for average value of f over R.

Suggested Examples:

(1) Set up a double integral over R with integrand $f = 1$, and then set up the definite integral for the area of R (between two curves).
(2) Compute the average value of f over R.

Tips:

None so far!

4. Double Integrals in Polar Form

Goals:

- Set up double integrals with polar coordinates,
- Convert a double integral in the Cartesian coordinate system into a double integral in the polar coordinate system.

Summary:

- Because of the derivations in the previous sections, the essence of the discussion for double integrals in polar coordinates is about dA. You can state or derive the formula $dA = r\,drd\theta$; later in section 15.8, we will have an easy way to compute it.
- Explain how we can find the limits of integration in polar space, and compare with Cartesian coordinates.

Suggested Examples:

(1) Set up a double integral over R using polar coordinates.
(2) Convert double integrals in Cartesian coordinates to polar coordinates.
(3) Convert a double integral in Cartesian coordinates whose region is more easily expressed in polar coordinates.

(4) Convert a double integral in Cartesian coordinates whose integrand is more easily expressed in polar coordinates.

Tips:

Students have forgotten the majority of what they knew about polar coordinates. You will have to remind them of some basic facts if you want them to follow along; many do not know how to graph polar functions. Maybe take a minute to plot some cardioid by evaluating at a few θ values.

5. Triple Integrals in Rectangular Coordinates

Goals:

- Set up and evaluate triple integrals over solids,
- Given a triple integral over a solid, use the information in the limits of integration to reconstruct the solid,
- Given a triple integral over a solid, change the order of integration.

Summary:

- The beginning of this section should feel the same as the beginning of the previous sections in this chapter. Briefly derive the formula for volume of a solid, and then expand this for triple integrals with different integrands.
- Discuss how we find the limits of integrations for triple integrals. This should feel exactly the same as it did in section 15.2.
- State the formula for the average value of f over D.

Suggested Examples:

(1) Various triple integrals in different orders.
(2) Change the order of integration based on a challenging integrand.
(3) Decide a reasonable order of integration based on the shape of the solid.

Tips:

Generally, students feel the need to be able to completely visualize the solid they need to integrate over. While it is nice to be able to do this, it is generally not entirely needed. It might be worth your time doing an example over a solid which you cannot draw, at least not easily.

6. Moments and Centers of Mass

Goals:

- To compute mass, first moments, and center of mass as double and triple integrals.

Summary:

- This is another small section; do not spend too much time on this. Discuss the formulas for mass, first moments, and center of mass as double and triple integrals. Then do some examples.
- SKIP second moments.

Suggested Examples:

(1) Compute the center of mass of a thin sheet.
(2) Use symmetry to deduce coordinates of center of mass without computing any integrals.

Tips:

Don't spend too much time here.

7. Triple Integrals in Cylindrical and Spherical Coordinates

Goals:

- Intuitive understanding of cylindrical and spherical coordinate systems.
- Set up and evaluate triple integrals in cylindrical and spherical coordinate systems.
- Convert triple integrals from Cartesian coordinate system to either cylindrical or spherical coordinates.

Summary:

- Introduce the cylindrical coordinate system. Graph the "basic shapes" given by $r = a$, $\theta = b$, and $z = c$. At some point, you want the students to encounter different orientations (e.g., r, θ, y).
- Discuss integration in the cylindrical system and the formula $dV = r\, dzdrd\theta$. Talk about how we can determine the limits of integration, and at this point, this should be familar to students.
- Like with cylindrical coordinates, introduce spherical coordinates. We assume that $\rho \geq 0$ and $0 \leq \phi \leq \pi$. Students sometimes struggle with the idea that ϕ measures "down" from the positive z axis, so emphasize this. Graph the "basic shapes" given by $\rho = a$, $\phi = b$, and $\theta = c$. Talk about integration and state that $dV = \rho^2 \sin\phi\, d\rho d\phi d\theta$. Students can see a proof of this in the next section if they desire.
- We do not stress switching the order of integration for cylindrical or spherical integrals. Usually, it is either trivial or unintuitive.

Suggested Examples:

(1) Set up triple integrals for some basic shapes in both cylindrical and spherical coordinates.
(2) Convert a triple integral in Cartesian coordinates into either cylindrical or spherical coordinates because of the solid.
(3) Convert a triple integral in Cartesian coordinates into either cylindrical or spherical coordinates because of the integrand.

Tips:

Students need a lot of time to digest this section, but they generally get it after a few days.

8. Substitutions in Multiple Integrals

Goals:

- Given a transformation from the xy-plane to the uv-plane, compute the Jacobian of the coordinate transformation.
- Given a transformation from the xy-plane to the uv-plane, apply the substitution to a double integral in x and y to solve an equivalent double integral in u and v.

Summary:

- Introduce substitution with multiple integrals. There are different kinds of motivation you can use. Some examples are u-sub for definite integrals; another is to think about the transformations into cylindrical or spherical coordinates.

- Define the Jacobian. Emphasize that the Jacobian is necessary (they often forget it), and give some intuition why it is required.
- Create a recipe for these kinds of problems.
  (1) Graph the region given and the region after applying the transformation.
  (2) Compute the Jacobian.
  (3) Apply the substitution to the integrand.
- Mention we can do this for multiple variables (e.g., 3). We actually have been doing this with cylindrical and spherical coordinates. These kinds of problems are rather tedious, so it does not make sense to do one for 3 variables unless it is quite simple.

Suggested Examples:

(1) Do a linear transformation with a region whose edges are straight lines. Applying the transformation to the region is equivalent to applying it on the vertices.
(2) Do another one where the region is not a polygon.
(3) Do an elliptical transformation, say, $x = ar\cos\theta$ and $y = br\cos\theta$.

Tips:

None so far!

9. Review

15.1

- Set up and evaluate a double integral of $f(x, y)$ over rectangles in the xy-plane.

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dxdy.$$

- For a surface above the xy-plane given by $f(x, y)$, compute the volume underneath the surface over a rectangle in the xy-plane.

$$V = \iint_R f(x,y)\,dA.$$

- Know how to switch the order of integration for integrals over rectangles in the xy-plane

$$\int_c^d \int_a^b f(x,y)\,dxdy = \int_a^b \int_c^d f(x,y)\,dydx.$$

15.2

- Set up and evaluate double integrals over a given region in the xy-plane.

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g(y)}^{h(y)} f(x,y)\,dxdy.$$

- For a surface above the xy-plane, compute the volume underneath the surface over a given region in the xy-plane.
- Know how to switch the order of integration for integrals over regions in the xy-plane

$$\int_a^b \int_{g_1(y)}^{g_2(y)} f(x,y)\,dxdy = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x,y)\,dydx.$$

15.3 • Set up the area of a region in the xy-plane as a double integral

$$A = \iint_R dA.$$

• Compute the average value of a function $f(x, y)$ over a region in the xy-plane

$$\text{Average value of } f \text{ over } R = \frac{\iint_R f\, dA}{\iint_R dA}.$$

15.4 • Set up and evaluate double integrals using polar coordinates

$$\iint_R f(r, \theta)\, dA = \int_a^b \int_{g(\theta)}^{h(\theta)} f(r, \theta)\, r\, dr d\theta.$$

• Convert a double integral in Cartesian coordinates into a double integral in polar coordinates

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy dx = \int_c^d \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr d\theta.$$

15.5 • Set up and evaluate a triple integral of $f(x, y, z)$ over a solid D in $\mathbb{R}^3$

$$\iiint_D f(x, y, z)\, dV = \int_a^b \int_{g_1(y)}^{g_2(y)} \int_{h_1(y,z)}^{h_2(y,z)} f(x, y, z)\, dx dz dy.$$

• Compute the volume of a solid as a triple integral

$$V = \iiint_D dV.$$

• Change the order of integration

$$\int_a^b \int_{g_1(y)}^{g_2(y)} \int_{h_1(y,z)}^{h_2(y,z)} f(x, y, z)\, dx dz dy = \int_c^d \int_{k_1(x)}^{k_2(x)} \int_{\ell_1(x,y)}^{\ell_2(x,y)} f(x, y, z)\, dz dy dx.$$

• Compute the average value of $f(x, y, z)$ over D

$$\text{Average value of } f \text{ over } D = \frac{\iiint_D f\, dV}{\iiint_D dV}.$$

15.6 • Given a density function δ, compute the mass of R or D (for D change to triple integrals)

$$M = \iint_R \delta\, dA,$$

and its center of mass $(M_{yz}/M,\ M_{xz}/M,\ M_{xy}/M)$, where

$$M_{yz} = \iint_R x\delta\, dA, \quad M_{xz} = \iint_R y\delta\, dA, \quad M_{xy} = \iint_R z\delta\, dA.$$

15.7
- Given a solid D, set up and evaluate a triple integral in cylindrical coordinates over D

$$\iiint_D f(r,\theta,z)\,dV = \int_a^b \int_{g_1(\theta)}^{g_2(\theta)} \int_{h_1(r,\theta)}^{h_2(r,\theta)} f(r,\theta,z)\,r\,dzdrd\theta.$$

- Given a solid D, set up and evaluate a triple integral in spherical coordinates over D

$$\iiint_D f(\rho,\phi,\theta)\,dV = \int_a^b \int_{g_1(\theta)}^{g_2(\theta)} \int_{h_1(\phi,\theta)}^{h_2(\phi,\theta)} f(\rho,\phi,\theta)\,\rho^2\sin\phi\,d\rho d\phi d\theta.$$

- Convert a triple integral in Cartesian coordinates into a triple integral in cylindrical coordinates

$$\iiint_D f(x,y,z)\,dV = \int_a^b \int_{g_1(\theta)}^{g_2(\theta)} \int_{h_1(r,\theta)}^{h_2(r,\theta)} f(r\cos\theta, r\sin\theta, z)\,r\,dzdrd\theta.$$

- Convert a triple integral in Cartesian coordinates into a triple integral in spherical coordinates

$$\iiint_D f(x,y,z)\,dV = \int_a^b \int_{g_1(\theta)}^{g_2(\theta)} \int_{h_1(\phi,\theta)}^{h_2(\phi,\theta)} f(\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho d\phi d\theta.$$

15.8
- Given a transformation from the xy-plane to the uv-plane, compute the Jacobian of the coordinate transformation

$$J(u,v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}.$$

- Given a transformation from the xy-plane to the uv-plane and a region R in the xy-plane, graph the region R' in the uv-plane after applying the transformation to R.
- Given a transformation from the xy-plane to the uv-plane and a double integral in x and y, apply the substitution to the integral to set up and evaluate a double integral in u and v

$$\iint_R f(x,y)\,dA = \iint_{R'} f(g(u,v),h(u,v))\,|J(u,v)|\,dA.$$

CHAPTER 16

Integration in Vector Fields

1. Line Integrals

Goals:

- Parametrize piecewise smooth curves in $\mathbb{R}^2$ and $\mathbb{R}^3$,
- Compute line integrals of real-valued functions over curves.

Summary:

- We want to emphasize that line integrals are not too different from definite integrals. In a sense, smooth curves C represent 'twisted' intervals in space. The following animation might be helpful for students: `http://tinyurl.com/k63c6fg`. It is worth spending a small amount of time explaining that we can approximate line integrals by partitioning the curve into little segments.
- Remind them of the arc length function
$$s(t) = \int_a^t |\mathbf{v}(\tau)|\, d\tau,$$
so that $ds = |\mathbf{v}(t)|\, dt$. In order to compute a line integral, we need a parametrization, $\mathbf{r}(t)$ of C, and thus,
$$\int_C f\, ds = \int_a^b f(\mathbf{r})|\mathbf{v}|\, dt.$$
- It's worth mentioning that arc length (or like '1-dimensional area') of C is just $\int_C ds$. An analog of this comes up when we look at surface area of surfaces in 16.5: $\iint_S d\sigma$.

Suggested Examples:

(1) Line integrals over a piecewise smooth C made up of straight lines.
(2) Line integrals over a nonplanar smooth C.
(3) Line integrals from points A to B with two different paths.

Tips:

Students struggle with parametrizations of all kinds, so it's worth doing a wide spectrum of examples. For a straight line parametrization from P to Q, I just give them
$$\mathbf{r}(t) = \mathbf{P}(1-t) + \mathbf{Q}t$$
for $0 \le t \le 1$. Another strategy I use is to start with, say,
$$\mathbf{r}(x, y) = \langle x, y \rangle,$$
and figure out how to eliminate a variable. If the curve was on $y = x^2$ from $(0, 0)$ to $(2, 4)$, one could replace the y above with x^2, so you end with
$$\mathbf{r}(x, y) = \langle x, x^2 \rangle$$

where $0 \leq x \leq 2$. The calculus of line integrals is quite simple, and after some acclimation, parametrization becomes the hardest part.

2. Vector Fields and Line Integrals: Work, Circulation, and Flux

Goals:

- Graph and identify vector fields in $\mathbb{R}^2$ and $\mathbb{R}^3$,
- Compute line and flux integrals of vector fields over curves.

Summary:

- Introduce vector fields and some basic vocabulary (e.g., continuous). No need to get rigorous with the definitions. Make sure to explain how to graph and interpret these things. The book provides some interesting pictures on pages 908 and 909.
- Gradients provide an interesting example of vector fields. In the next section, we study their properties, so it's good to at least give some examples of gradient fields.
- Dicuss how we can compute the work of a vector field $\mathbf{F}$ along a curve. This amounts to adding up all the scalar components of the projection of $\mathbf{F}$ onto $\mathbf{T}$, which can be done with line integrals. During this time, expose the students to the various kinds of notion:
$$\int_C \mathbf{F} \cdot \mathbf{T} ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M dx + N dy + P dz.$$
Emphasize that the ds form is rarely (if ever) used for computation.
- Discuss how we can compute the flux of $\mathbf{F}$ across a piecewise smooth curve in the plane (the book states only smooth for some reason; ex. #36). This should be very similar to the work integral, so its derivation is typically much quicker.

Suggested Examples:

(1) Graphing and matching various vector fields.
(2) Line integrals over piecewise smooth curves in both $\mathbb{R}^2$ and $\mathbb{R}^3$.
(3) Flux integrals over piecewise smooth curves in $\mathbb{R}^2$.

Tips:

It's worthwhile to make clear to the students that there are two fundamentally different physical topics in this chapter – flux (only ever called flux) and work (also called circulation and flow).

3. Path Independence, Conservative Fields, and Potential Functions

Goals:

- Proficient understanding of the Fundamental Theorem of Line Integrals,
- Test if a vector field is conservative or not (component test),
- Find a potential function of a conservative field.

Summary:

- State the definitions for path independence, conservative field, and potential function.

- Explain that the Fundamental Theorem of Line Integrals is just the Fundamental Theorem of Calculus but with different objects, which is why it is often written
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_A^B \mathbf{F} \cdot d\mathbf{r}.$$
It's nice to give an idea of the proof, but not necessary to prove the theorem. Thus line integrals of gradient fields are "easy" to compute.
- State the other theorems: Conservative fields are gradient fields, Loop property of conservative fields. While the latter is not really used in this course, it seems to add intuition for the students. Since conservative fields are gradient fields, we can use the Fundamental Theorem of Line Integrals to "easily" compute their line integrals.
- At this stage, we need two algorithms. The first should decide whether or not a vector field is conservative, and the second should determine a potential function for a conservative field. The component test will address the first issue which requires
$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$
The algorithm for determining a potential function is not quite as simple nor as easy to state as the component test.
- SKIP the subsection on Exact Differential Forms.

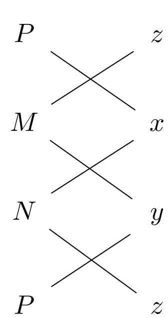
Suggested Examples:

(1) Use the component test to decide if a vector field is conservative.
(2) Find a potential function for a conservaitve field.
(3) Verify the Fundamental Theorem of Line Integrals with a simple conservative field.
(4) Use the Fundamental Theorem of Line Integrals to evaluate line integrals over complicated conservative fields.

Tips:

Students are generally engaged with this material because many of them have encountered conservative forces in other classes. Some examples of conservative fields are gravitional, spring, magnetic, and electric fields. Some non-examples are friction and air drag forces since energy is "lost" to heat.

I do not mention that the component test comes from computing the curl of $\mathbf{F}$ since there is so much to cover in this section. They will see this soon enough. Instead, I give them a neat mnemonic for remembering the component test is the following diagram.



$$\Longrightarrow \quad \frac{\partial P}{\partial x} = \frac{\partial M}{\partial z}$$
$$\Longrightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$
$$\Longrightarrow \quad \frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}$$

4. Green's Theorem in the Plane

Goals:

- Intuitive understanding of divergence and circulation density,
- Apply Green's theorem to a line or flux integral without explicit instruction.

Summary:

- Introduce the ideas of divergence and circulation density. It is worth spending some time doing the derivation of one of them, and then, at least, discussing the differences of the other. While they do not need to know the derivation, the ideas provide a lot of intuition and physical meaning to the formulas. Go through some examples of some vector fields and compute the divergence and circulation density, and check that the answers match with our understanding.
- State both forms of Green's Theorem, but do not prove them. It is worth spending some time talking about the ideas of the proof. In fact, the ideas are already familiar. First, tessellate the region into a bunch of rectangles, and then look at the divergence of each of the edges of each of the rectangles. The only edges that contribute to the sum are the edges on the boundary.

Suggested Examples:

(1) Given a vector field function, compute divergence and circulation density.
(2) Given a picture of a vector field, decide if the divergence and circulation density are negative, zero, or positive.
(3) Use Green's Theorem to compute line and flux integrals over closed curves in the xy-plane.
(4) Use both forms of Green's Theorem to compute a line integral in the form

$$\int_C M dx + N dy.$$

(5) For specific examples, use Green's Theorem to show that either flux or circluation depends only on the area of the region.

Tips:

None so far!

5. Surfaces and Area

Goals:

- Parametrize piecewise smooth surfaces in $\mathbb{R}^3$,
- Compute the surface area of a smooth surface.

Summary:

- We are essentially starting over again. Here we learn how to parametrize smooth surfaces, which should feel like parametrizing curves from 16.1, and then we learn how to compute surface areas, which sould feel like computing arc length from 13.3.
- Explain how to parametrize a surface with two variables. Emphasize that *surfaces* are 2 dimensional objects, and *solids* are 3 dimensional objects.

- The basic ideas of getting $d\sigma$ in terms of u and v is quite simple. Partitioning the region in the uv-plane into rectangles yields parallelograms on the surface. The area of each parallelogram is $|\mathbf{r}_u \times \mathbf{r}_v|$.
- SKIP the subsection on Implicit surfaces. It is a useful tool, but not necessary. Students can use this method if they want, but all surfaces will either be explicit or parametrizable with a cylindrical or spherical type of parametrization.

Suggested Examples:

(1) Parametrize all kinds of surfaces: explicit, cylindrical, spherical.
(2) Compute various surface areas.

Tips:

Again, parametrization is very difficult for students. Using the same strategy presented in section 16.1, I often start with something like

$$\mathbf{r}(x, y, z) = \langle x, y, z \rangle,$$

and figure out how to replace one of the variables using what we know. For example, if I need to parametrize the cylinder with radius 2 centered on the y-axis cut by the planes $y = 0$ and $y = 3$, I would first set up

$$\mathbf{r}(r, \theta, y) = \langle r\cos\theta, y, r\sin\theta \rangle,$$

and then use $r = 2$ to eliminate r.

Another common problem is not knowing what to parametrize. Student will have problems which ask the following. Parametrize the portion of the plane cut by the paraboloid, and they will try to parametrize the paraboloid and get confused. I believe it is worth talking about a general strategy for these kinds of problems.

6. Surfaces Integrals

Goals:

- Compute surface integrals of real-valued functions over smooth surfaces,
- Setup and evaluate the surface integral which computes the (specified direction for) flux of a vector field across a surface.

Summary:

- This section should feel like 16.1 and 16.2 all over again. We start with computing surface integrals over real-valued functions. A physical interpretation of this is very similar to that of line integrals. In fact, you could use the same example but with a surface instead.
- Define an orientable surface. Pictures and examples will probably be the easiest way for them to understand it. All surfaces will be orientable in this class.
- State how we can use surface integrals to compute the flux of $\mathbf{F}$ across an orientable surface. This should look the same as it did with line integrals.
- Briefly state that we can compute mass and moments of surfaces much like we could with line integrals.

Suggested Examples:

(1) Surface integrals of real-valued functions over surfaces defined by $z = f(x, y)$ (or whatever permutation of x, y, z).

(2) Surface integrals of real-valued function over cylindrical and spherical surfaces.
(3) Surface integrals which compute flux across the same surfaces as above.

Tips:
None so far!

7. Stokes' Theorem

Goals:

- Intuitive understanding of curl,
- Apply Stokes' theorem to line integrals over curves and to surface integrals which compute flux of the curl.

Summary:

- Introduce the curl of $\mathbf{F}$ and give some intuition for it. Observe that $(\nabla \times \mathbf{F}) \cdot \mathbf{k}$ is circulation density back from section 16.4. Moreover, the component test for $\mathbf{F}$ is equivalent to $\nabla \times \mathbf{F} = \mathbf{0}$. So what does that mean for $\nabla \times \nabla f$?
- State Stokes' Theorem. Again, do not prove it, but give the basic ideas – similar to what was done for Green's Theorem. What does this idea say about surfaces without a boundary curve C?
- Explain how
$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$$
can be interpreted as a flux computation.

Suggested Examples:

(1) Compute the curl of a variety of vector fields.
(2) Use Stokes' Theorem to convert line integrals into surface integrals.
(3) Use Stokes' Theorem to convert surface integrals into line integrals.
(4) Use Stokes' Theorem to convert a surface integral into an easier surface integral.

Tips:
None so far!

8. The Divergence Theorem and a Unified Theory

Goals:

- Intuitive understanding of divergence,
- Apply the divergence theorem to surface integrals which compute flux over closed surfaces.

Summary:

- Introduce divergence in $\mathbb{R}^3$.
- State the Divergence Theorem, and again, do not prove it. It is worth discussing the ideas, and at this point, the ideas are exactly the same as before, so it should be quick.
- Discuss how the Divergence Theorem applies to the solid region between two concentric spheres. This is rather minor, but it does provide an interesting example.

- SKIP the subsections on the applications to electromagnetic theory and hydrodynamics.

Suggested Examples:

(1) Check that divergence matches our intuition from 16.4.
(2) Apply the Divergence Theorem to various surfaces.
(3) Apply the Divergence Theorem to concentric spheres (Ex. 4 page 977).
(4) Discuss the shortcomings of applying the Divergence Theorem to surfaces with boundary curves; for example, a hemisphere.

Tips:

Another interpretation of Divergence Theorem is to relate the notion of divergence with density. For example, if $\operatorname{div} \mathbf{F} > 0$ at a point P, then the denity near P is decreasing. The net change in density of a solid (ie. $\iiint_D \operatorname{div} \mathbf{F}\, dV$) depends only on what happens at the boundary.

9. Review

16.1

- Parametrize curves to compute line integrals of real-valued functions

$$\int_C f\, ds = \int_a^b f(\mathbf{r})|\mathbf{v}|\, dt.$$

- For a piecewise smooth curve C, distinguish its smooth segments to evaluate line integrals over C

$$\int_C f\, ds = \int_{C_1} f\, ds + \cdots + \int_{C_n} f\, ds.$$

- Given a density function δ, compute the mass of C

$$M = \int_C \delta\, ds,$$

and its center of mass $(M_{yz}/M,\, M_{xz}/M,\, M_{xy}/M)$, where

$$M_{yz} = \int_C x\delta\, ds, \quad M_{xz} = \int_C y\delta\, ds, \quad M_{xy} = \int_C z\delta\, ds.$$

16.2

- Graph vector fields, and match equations of vector fields to pictures.
- Compute the gradient field of a real-valued function.
- Compute the line integral of $\mathbf{F}$ over a curve C

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F}(\mathbf{r}) \cdot \mathbf{v}\, dt.$$

- Compute the work done by $\mathbf{F}$ along C

$$\text{Work} = \int_C \mathbf{F} \cdot d\mathbf{r}.$$

- Compute the flux of $\mathbf{F}$ across C (all in xy-plane)

$$\text{Flux} = \int_C \mathbf{F} \cdot \mathbf{n}\, ds.$$

- Be able to work with different notations

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M dx + N dy + P dz, \quad \int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_C M dy - N dx.$$

16.3
- Given a vector field, use the component test to determine if it is conservative

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

- Given a conservative field $\mathbf{F}$, find a potential function for $\mathbf{F}$.
- Compute line integrals of conservative fields using the Fundamental Theorem of Line Integrals

$$\int_A^B \nabla f \cdot d\mathbf{r} = f(B) - f(A).$$

16.4
- Compute divergence and circulation density of $\mathbf{F}$

$$\operatorname{div} \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}, \quad \operatorname{curl} \mathbf{F} \cdot \mathbf{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}.$$

- Evaluate line integrals for circulation using Green's Theorem

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C M\, dx + N\, dy = \iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} dA.$$

- Evaluate line integrals for flux using Green's Theorem

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_C M\, dy - N\, dx = \iint_R \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} dA.$$

- Use either form of Green's Theorem to compute line integrals

$$\int_C f(x,y)dx + g(x,y)dy = \iint_R \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} dA.$$

16.5
- Parametrize smooth surfaces.
- Compute the surface area differential

$$d\sigma = |\mathbf{r}_u \times \mathbf{r}_v| dA.$$

- Compute the surface area of a piecewise smooth surface

$$SA = \iint_S d\sigma = \iint_R |\mathbf{r}_u \times \mathbf{r}_v| dA.$$

16.6
- Compute surface integrals of real-valued functions over piecewise smooth surfaces S

$$\iint_S G\, d\sigma = \iint_R G(\mathbf{r}) |\mathbf{r}_u \times \mathbf{r}_v|\, dA.$$

- Compute the flux of $\mathbf{F}$ across a piecewise smooth surface

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_R \mathbf{F}(\mathbf{r}) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA.$$

- Use surface integrals to compute mass, moments, and center of mass of a surface

$$M = \iint_S \delta\, d\sigma,$$

$$M_{yz} = \iint_S x\delta\, d\sigma, \quad M_{xz} = \iint_S y\delta\, d\sigma, \quad M_{xy} = \iint_S z\delta\, d\sigma.$$

16.7
- Compute the curl of a vector field: $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F}$.

- Use Stokes' Theorem to convert a line integral of $\mathbf{F}$ over C into a surface integral for flux of the curl of $\mathbf{F}$ over S (with $\partial S = C$)
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma.$$
- Given a surface integral for flux of the curl of $\mathbf{F}$ over S_1, use Stokes' Theorem to change to a more manageable surface S_2
$$\iint_{S_1} \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_{S_2} \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma.$$
- Given a surface integral for flux of the curl of $\mathbf{F}$ over S, where $\partial S = C$, use Stokes' Theorem to instead evaluate a line integral over C.
$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \int_C \mathbf{F} \cdot d\mathbf{r}.$$

16.8

- Compute the divergence of a vector field: $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F}$.
- Use the Divergence Theorem to compute outward flux of $\mathbf{F}$ across the piecewise smooth closed surface S
$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D \nabla \cdot \mathbf{F}\, dV.$$
- Use the Divergence Theorem to compute the flux of $\mathbf{F}$ across a not simple surface S. For example, D is the region between two concentric spheres, so ∂D is the two concentric spheres.