# MATH 676

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# Finite element methods in scientific computing

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# Lecture 33.25:

# Which element to use
# Part 2: Saddle point problems

# Stokes

**Consider the stationary Stokes equations:**

$$\begin{aligned} -\Delta u + \nabla p &= f \\ \nabla \cdot u &= 0 \end{aligned}$$

This can equivalently be considered as a minimization problem:

$$\begin{aligned} \min_{u \in H^1(\Omega)^d} \quad & \frac{1}{2}\|\nabla u\|^2 - (f, u) \\ \text{such that} \quad & \nabla \cdot u = 0 \end{aligned}$$

Let us consider the constraint in variational form:

$$\begin{aligned} \min_{u \in V = H^1(\Omega)^d} \quad & \frac{1}{2}\|\nabla u\|^2 - (f, u) \\ \text{such that} \quad & (q, \nabla \cdot u) = 0 \quad \forall q \in Q = L^2 \end{aligned}$$

# Stokes

**Consider the stationary Stokes equations:**

$$\begin{aligned} -\Delta u + \nabla p &= f \\ \nabla \cdot u &= 0 \end{aligned}$$

The discrete formulation for this seeks $u_h \in V_h \subset V, \ p_h \in Q_h \subset Q$ :

$$(\nabla v_h, \nabla u_h) - (\nabla \cdot v_h, p_h) - (q_h, \nabla \cdot u_h) = (v_h, f) \quad \forall v_h \in V_h, \ q_h \in Q_h$$

This corresponds to the finite dimensional minimization problem

$$\begin{aligned} &\min_{u_h \in V_h} \quad \frac{1}{2} \|\nabla u_h\|^2 - (f, u_h) \\ &\text{such that} \quad (q_h, \nabla \cdot u_h) = 0 \quad \forall q_h \in Q_h \end{aligned}$$

# Stokes

**Consider the discrete Stokes equations:**

$$\begin{aligned} \min_{u_h \in V_h} \quad & \frac{1}{2}\|\nabla u_h\|^2 - (f, u_h) \\ \text{such that} \quad & (q_h, \nabla \cdot u_h) = 0 \quad \forall q_h \in Q_h \end{aligned}$$

Here, we have *dim* $Q_h$ constraints on the velocity $u_h$.

Intuitively, if (asymptotically) $Q_h$ is "too large" compared to $V_h$, then:

- we have too many constraints on the velocity
- the velocity does not have enough degrees of freedom.

In this case the discrete solution may not converge.

# Stokes

**Consider the discrete Stokes equations:**

We only get convergence of discrete solutions of

$$(\nabla v_h, \nabla u_h) - (\nabla \cdot v_h, p_h) - (q_h, \nabla \cdot u_h) = (v_h, f) \qquad \forall v_h \in V_h,\ q_h \in Q_h$$

or equivalently

$$\min_{u_h \in V_h} \quad \frac{1}{2}\|\nabla u_h\|^2 - (f, u_h)$$

$$\text{such that} \quad (q_h, \nabla \cdot u_h) = 0 \quad \forall q_h \in Q_h$$

if the inf-sup/Babuska-Brezzi/LBB condition is satisfied:

*There exists a constant c independent of h so that*

$$\sup_{v_h \in V_h} \frac{(\nabla \cdot v_h, q_h)}{\|v_h\|_V} \geq c\|q_h\|_Q \qquad \forall q_h \in Q_h$$

**Note:** $V=H^1$, $Q=L_2$.

# Stokes

**The inf-sup condition:**

We can write the condition either as:

*There exists a constant c independent of h so that*

$$\sup_{v_h \in V_h} \frac{(\nabla \cdot v_h, q_h)}{\|v_h\|_V} \geq c \|q_h\|_Q \qquad \forall q_h \in Q_h$$

Or as:

*There exists a constant c independent of h so that*

$$\inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{(\nabla \cdot v_h, q_h)}{\|v_h\|_V \, \|q_h\|_Q} \geq c$$

# Stokes

**The inf-sup condition:**

The condition...

> *There exists a constant c independent of h so that*
>
> $$\sup_{v_h \in V_h} \frac{(\nabla \cdot v_h, q_h)}{\|v_h\|_V} \geq c\|q_h\|_Q \qquad \forall q_h \in Q_h$$

...can be satisfied by making

- the velocity space $V_h$ large enough
- the pressure space $Q_h$ small enough

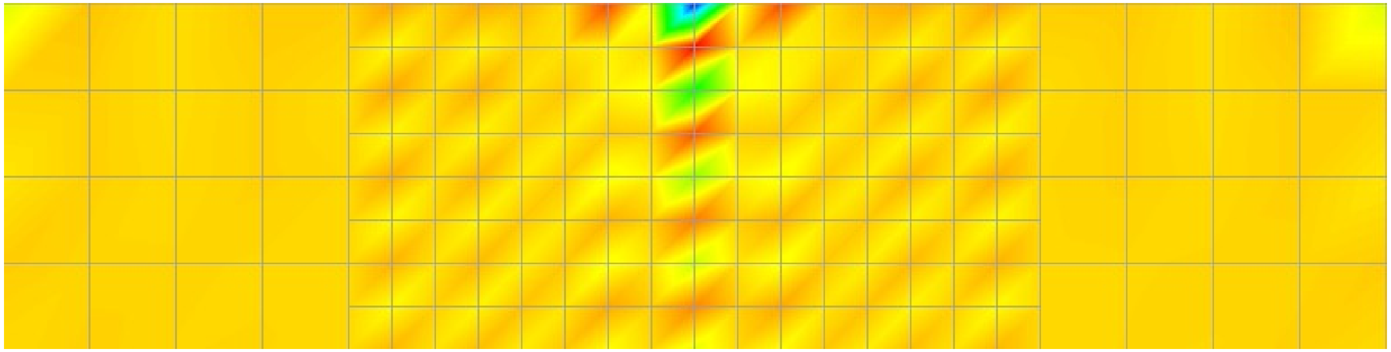
**Typical choices (the "*Taylor-Hood element*"):**

- $V_h = P_{k+1}$, $Q_h = P_k$ on triangles/tetrahedra
- $V_h = Q_{k+1}$, $Q_h = Q_k$ on quadrilaterals/hexahedra

# Stokes

**Why Taylor-Hood (*$P_{k+1}/P_k$* or *$Q_{k+1}/Q_k$*):**

- $P_k/P_k$ or $Q_k/Q_k$ is not stable:
  - the constant $c$ goes to zero as $h \to 0$
  - the matrix has a near-nullspace
  - the pressure develops a "checkerboard pattern":



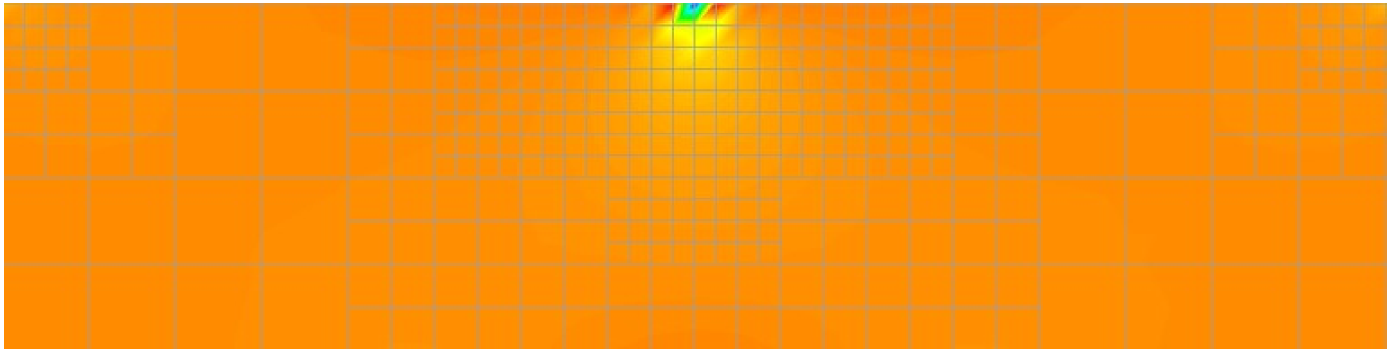
(using step-22 with equal order elements)

**Consequence:** We need to make the velocity space larger or the pressure space smaller!

# Stokes

**Why Taylor-Hood (*$P_{k+1}/P_k$* or *$Q_{k+1}/Q_k$*):**

- $P_{k+1}/P_k$ or $Q_{k+1}/Q_k$ is stable:
  - there is a constant $c>0$
  - the matrix remains regular
  - the pressure is stable



(using step-22 with non-equal order elements)

**Consequence:** This works!

# Stokes

**Why Taylor-Hood ($P_{k+1}/P_k$ or $Q_{k+1}/Q_k$):**

- $P_{k+2}/P_k$ or $Q_{k+2}/Q_k$ is stable:
  - there is a constant $c>0$
  - the matrix remains regular
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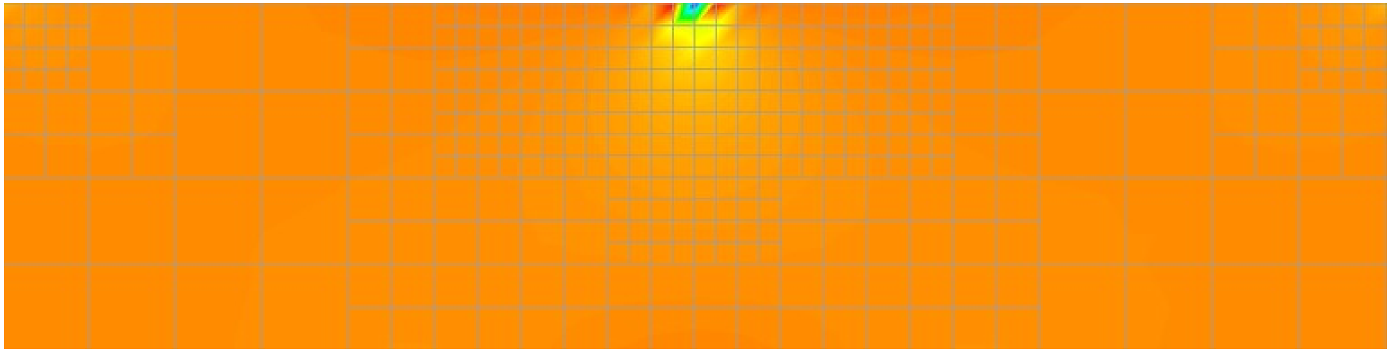


(using step-22 with non-equal order elements)

  - accuracy is now limited by the low-order pressure

**Consequence:** This choice is wasteful!

# Stokes

**Why Taylor-Hood ($P_{k+1}/P_k$ or $Q_{k+1}/Q_k$):**

- $P_k/P_k$ or $Q_k/Q_k$ doesn't work
- $P_{k+1}/P_k$ or $Q_{k+1}/Q_k$ works
- Can we find something in between?

**Option 1:** With a slightly smaller velocity space.

- We can't take away shape functions without either
  - violating unisolvency
  - making the shape functions discontinuous (which would make the element non-conforming)
- This option doesn't work

# Stokes

**Why Taylor-Hood ($P_{k+1}/P_k$ or $Q_{k+1}/Q_k$):**

- $P_k/P_k$ or $Q_k/Q_k$ doesn't work
- $P_{k+1}/P_k$ or $Q_{k+1}/Q_k$ works
- Can we find something in between?

**Option 2a:** With a slightly larger pressure space.

- Recall the bilinear form:

$$(\nabla v_h, \nabla u_h) - (\nabla \cdot v_h, p_h) - (q_h, \nabla \cdot u_h) = (v_h, f)$$

- We don't actually need continuity of the pressure
- We could try $Q_{k+1}/Q_k + DGQ_0$
- This actually works!

# Stokes

**Why Taylor-Hood ($P_{k+1}/P_k$ or $Q_{k+1}/Q_k$):**

- $P_k/P_k$ or $Q_k/Q_k$ doesn't work
- $P_{k+1}/P_k$ or $Q_{k+1}/Q_k$ works
- Can we find something in between?

**Option 2b:** With an even larger pressure space.

- Recall the bilinear form:

$$(\nabla v_h, \nabla u_h) - (\nabla \cdot v_h, p_h) - (q_h, \nabla \cdot u_h) = (v_h, f)$$

- We don't actually need continuity of the pressure
- We could try $Q_{k+1}/DGQ_k$
- This doesn't work, too many pressure degrees of freedom

# Stokes

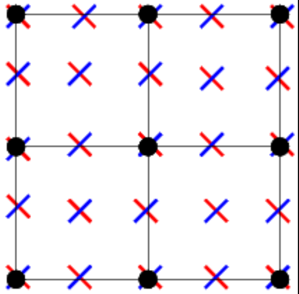
**Choosing a polynomial degree:**

We could choose either

- $P_{k+1}/P_k$ or $Q_{k+1}/Q_k$
- $Q_{k+1}/Q_k+DGQ_0$

**In practice one typically chooses *k=1*:**

- There are $2*3^2+2^2=22$ ($3*3^3+2^3=89$) degrees of freedom per cell in 2d (3d)
- On a uniform mesh, matrix rows may have up to
  – $2*5^2+3^2 = 59$
  – $3*5^3+3^3 = 402$
  entries



- *k>1* yields better accuracy, but matrix starts to get full

# Mixed Laplace

**Consider the mixed Laplace equations:**

$$\begin{aligned} u+\nabla p &= 0 \\ \nabla\cdot u &= f \end{aligned}$$

This can equivalently be considered as a minimization problem:

$$\begin{aligned} &\min_{u\in H^1(\Omega)^d} && \frac{1}{2}\|u\|^2 \\ &\text{such that} && \nabla\cdot u = f \end{aligned}$$

Let us consider the constraint in variational form:

$$\begin{aligned} &\min_{u\in V=H^1(\Omega)^d} && \frac{1}{2}\|u\|^2 \\ &\text{such that} && (q,\nabla\cdot u) = (q,f) \qquad \forall q\in Q=L^2 \end{aligned}$$

# Mixed Laplace

**Consider the mixed Laplace equations:**

$$\begin{aligned} u+\nabla p &= 0 \\ \nabla\cdot u &= f \end{aligned}$$

The discrete formulation for this seeks $u_h \in V_h \subset V,\ p_h \in Q_h \subset Q$ :

$$(v_h, u_h) - (\nabla\cdot v_h, p_h) - (q_h, \nabla\cdot u_h) = -(q_h, f) \qquad \forall v_h \in V_h,\ q_h \in Q_h$$

This corresponds to the finite dimensional minimization problem

$$\begin{aligned} &\min_{u_h \in V_h} && \frac{1}{2}\|u_h\|^2 \\ &\text{such that} && (q_h, \nabla\cdot u_h) = (q_h, f) \qquad \forall q_h \in Q_h \end{aligned}$$

# Mixed Laplace

**Consider the discrete mixed Laplace equations:**

We only get convergence of discrete solutions of

$$(v_h, u_h) - (\nabla \cdot v_h, p_h) - (q_h, \nabla \cdot u_h) = -(q_h, f) \quad \forall v_h \in V_h,\ q_h \in Q_h$$

or equivalently

$$\min_{u_h \in V_h} \quad \frac{1}{2}\|u_h\|^2$$

$$\text{such that} \quad (q_h, \nabla \cdot u_h) = (q_h, f) \qquad \forall q_h \in Q_h$$

if the inf-sup/Babuska-Brezzi/LBB condition is satisfied:

*There exists a constant c independent of h so that*

$$\sup_{v_h \in V_h} \frac{(\nabla \cdot v_h, q_h)}{\|v_h\|_V} \geq c\|q_h\|_Q \qquad \forall q_h \in Q_h$$

**Note:** *$V = H(div)$, $Q = L_2$.*

# Mixed Laplace

**We have the same situation as before:**

The condition...

*There exists a constant c independent of h so that*

$$\sup_{v_h \in V_h} \frac{(\nabla \cdot v_h, q_h)}{\|v_h\|_V} \geq c \|q_h\|_Q \qquad \forall q_h \in Q_h$$

...can be satisfied by making

- the velocity space $V_h$ large enough
- the pressure space $Q_h$ small enough

# Mixed Laplace

**One common choice:**

- $V_h = P_{k+1}, \quad Q_h = P_k$ on triangles/tetrahedra
- $V_h = Q_{k+1}, \quad Q_h = Q_k$ on quadrilaterals/hexahedra

This is again the Taylor-Hood element. It is stable

**We can play the same game:**

- Can we make the velocity space smaller?
  (Less numerical effort with essentially same accuracy.)
- Can we make the pressure space larger?
  (Better accuracy with only slightly more work.)

# Mixed Laplace

**Option 1: Make velocity space smaller.**

- $V_h = Q_{k+1}, \quad Q_h = Q_k$ works
- $V_h$ consists of continuous functions so that we can take the (weak) gradient, which we needed for Stokes:

$$(\nabla v_h, \nabla u_h) - (\nabla \cdot v_h, p_h) - (q_h, \nabla \cdot u_h) = (v_h, f) \qquad \forall v_h \in V_h, \; q_h \in Q_h$$

- But we don't need this for mixed Laplace:

$$(v_h, u_h) - (\nabla \cdot v_h, p_h) - (q_h, \nabla \cdot u_h) = -(q_h, f) \qquad \forall v_h \in V_h, \; q_h \in Q_h$$

- All we need is that the divergence is defined.

# Mixed Laplace

**Option 1: Make velocity space smaller.**

**We need:** An element with continuous normal vector component but possibly discontinuous tangential component.

This is the *Raviart-Thomas element.*

**This works:** Replace

- $V_h = Q_{k+1}, \quad Q_h = Q_k$

by

- $V_h = \text{Raviart-Thomas}(k), \quad Q_h = Q_k$ if $k>0$
- $V_h = \text{Raviart-Thomas}(0), \quad Q_h = DGQ_0$ if $k=0$

# Mixed Laplace

**Option 2: Make pressure space larger.**

Recall the bilinear form:

$$(v_h, u_h) - (\nabla \cdot v_h, p_h) - (q_h, \nabla \cdot u_h) = -(q_h, f) \qquad \forall v_h \in V_h,\ q_h \in Q_h$$

We don't need a continuous pressure.

**This works:** Replace

- $V_h = \text{Raviart-Thomas}(k),\ \ Q_h = Q_k$ if $k>0$
- $V_h = \text{Raviart-Thomas}(0),\ \ Q_h = DGQ_0$ if $k=0$

by

- $V_h = \text{Raviart-Thomas}(k),\ \ Q_h = DGQ_k$ (see step-20)

# Mixed Laplace

**Option 3: Alternatives**

There are any number of alternatives to the Raviart-Thomas element:

- Brezzi-Douglas-Marini (BDM)
- Arnold-Falk-Winther
- …

- Most of these use piecewise constant pressures at lowest order
- This leads to very slow convergence ($O(h)$)
- For practical applications: use higher orders
- Elements are relatively "sparse", i.e., not too many DoFs

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# Finite element methods in scientific computing

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