MATH 676

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Finite element methods in scientific computing

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Lecture 31.7:

Nonlinear problems

Part 5: Pseudo-time stepping for the minimal surface equation

The minimal surface equation

Consider the minimal surface equation:

$$
\begin{aligned}
-\nabla\cdot\left(\frac{A}{\sqrt{1+|\nabla u|^2}}\nabla u\right) &= f \qquad \text{in } \Omega \\
u &= g \qquad \text{on } \partial\Omega
\end{aligned}
$$

where we choose

$$\Omega = B_1(0) \subset \mathbb{R}^2, \qquad f = 0, \qquad g = \sin(2\pi(x+y))$$

Goal: Solve this numerically with via pseudo-time stepping.

Pseudo-time stepping

General approach: To solve

$$L(u) = f$$

by pseudo-time stepping, we seek the limit

$$u(x) = \lim_{\tau \to \infty} \bar{u}(x, \tau)$$

where $\bar{u}(x, \tau)$ solves

$$\frac{\partial \bar{u}(x, \tau)}{\partial \tau} \pm L(\bar{u}) = \pm f$$

Note: τ is an artificial “time-like” variable. We will call it *pseudo-time*.

Pseudo-time stepping

Requirements: To find a stationary limit of $\bar{u}(x,\tau)$ where

$$\frac{\partial \bar{u}(x,\tau)}{\partial \tau} \pm L(\bar{u}) = \pm f$$

we need that this time-dependent equation

- has a solution,
- the solution is unique
- the solution converges to a steady state as $\tau \rightarrow \infty$
- convergence is independent of the starting point
- the steady state is stable

Pseudo-time stepping

General guide: To find a stationary limit of $\bar{u}(x,\tau)$ where

$$\frac{\partial \bar{u}(x,\tau)}{\partial \tau} \pm L(\bar{u}) = \pm f$$

choose the sign so that

- the operator

  $$I \pm \epsilon G(\bar{u}) \qquad \text{(where } G(\bar{u})\bar{u} = L(\bar{u})\text{)}$$

  is a contraction for a sufficiently small $\epsilon > 0$

- the resulting equation is something that resembles a known "physical" equation

Pseudo-time stepping

Example: Solve $-\Delta u = f$ by finding the limit of

$$\frac{\partial \bar{u}(x,\tau)}{\partial \tau} \pm \left(-\Delta \bar{u}(x,\tau)\right) = \pm f(x)$$

We have two options:

- Plus sign:

$$\frac{\partial \bar{u}(x,\tau)}{\partial \tau} - \Delta \bar{u}(x,\tau) = f(x)$$

  This is the well-known heat equation: Unique solution!

- Minus sign:

$$\frac{\partial \bar{u}(x,\tau)}{\partial \tau} + \Delta \bar{u}(x,\tau) = -f(x)$$

  This is the “backward heat equation”: No unique solution!

Pseudo-time stepping

Boundary + initial values: To solve

$$\begin{aligned} L(u) &= f && \text{in } \Omega \\ u &= g && \text{on } \partial\Omega \end{aligned}$$

by pseudo-time stepping using the equation

$$\frac{\partial \bar{u}(x,\tau)}{\partial \tau} \pm L(\bar{u}) = \pm f(x)$$

we need boundary and initial values:

$$\begin{aligned} \bar{u}(x,\tau) &= g(x) && \text{on } \partial\Omega \times (0,\infty) \\ \bar{u}(x,0) &= \bar{u}_0(x) && \text{in } \Omega \end{aligned}$$

Note 1: We can (usually) choose initial conditions arbitrarily.
Note 2: But $\bar{u}_0(x) \approx u(x)$ means faster convergence!

Pseudo-time stepping

Pseudo-time discretization: Do time stepping scheme on

$$\begin{aligned} \frac{\partial \bar{u}(x,\tau)}{\partial \tau} \pm L(\bar{u}) &= \pm f(x) && \text{in } \Omega\times(0,\infty) \\ \bar{u}(x,\tau) &= g(x) && \text{on } \partial\Omega\times(0,\infty) \\ \bar{u}(x,0) &= \bar{u}_0(x) && \text{in } \Omega \end{aligned}$$

For example, try the implicit Euler method:

$$\begin{aligned} \frac{\bar{u}^n(x)-\bar{u}^{n-1}(x)}{\Delta\tau} \pm L(\bar{u}^n) &= \pm f(x) && \text{in } \Omega \\ \bar{u}^n(x,\tau) &= g(x) && \text{on } \partial\Omega \end{aligned}$$

Problem: If $L(u)$ is nonlinear, then this equation is still nonlinear in u^n – we wanted something linear!

Pseudo-time stepping

Pseudo-time discretization: Do time stepping scheme on

$$\begin{aligned} \frac{\partial \bar{u}(x,\tau)}{\partial \tau} \pm L(\bar{u}) &= \pm f(x) && \text{in } \Omega\times(0,\infty) \\ \bar{u}(x,\tau) &= g(x) && \text{on } \partial\Omega\times(0,\infty) \\ \bar{u}(x,0) &= \bar{u}_0(x) && \text{in } \Omega \end{aligned}$$

For example, try the explicit Euler method:

$$\begin{aligned} \frac{\bar{u}^n(x)-\bar{u}^{n-1}(x)}{\Delta\tau} \pm L(\bar{u}^{n-1}) &= \pm f(x) && \text{in } \Omega \\ \bar{u}^n(x,\tau) &= g(x) && \text{on } \partial\Omega \end{aligned}$$

Problem: If *L(u)* is a second order differential operator, we may have to take very small time steps! (See lecture 27.)

Pseudo-time stepping

Pseudo-time discretization: Do time stepping scheme on

$$\begin{aligned} \frac{\partial \bar{u}(x,\tau)}{\partial \tau} \pm L(\bar{u}) &= \pm f(x) && \text{in } \Omega \times (0,\infty) \\ \bar{u}(x,\tau) &= g(x) && \text{on } \partial\Omega \times (0,\infty) \\ \bar{u}(x,0) &= \bar{u}_0(x) && \text{in } \Omega \end{aligned}$$

For example, try a semi-implicit Euler method:

$$\begin{aligned} \frac{\bar{u}^n(x) - \bar{u}^{n-1}(x)}{\Delta\tau} \pm G(\bar{u}^{n-1})\bar{u}^n &= \pm f(x) && \text{in } \Omega \\ \bar{u}^n(x,\tau) &= g(x) && \text{on } \partial\Omega \end{aligned}$$

Here: Choose *G(u)u = L(u)* where *G(u)* is a linear operator. (See previous lecture.)

Pseudo-time stepping

Pseudo-time discretization: Do time stepping scheme on

$$\begin{aligned} \frac{\partial \bar{u}(x,\tau)}{\partial \tau} \pm L(\bar{u}) &= \pm f(x) && \text{in } \Omega\times(0,\infty) \\ \bar{u}(x,\tau) &= g(x) && \text{on } \partial\Omega\times(0,\infty) \\ \bar{u}(x,0) &= \bar{u}_0(x) && \text{in } \Omega \end{aligned}$$

For example, try a semi-implicit method + extrapolation:

$$\begin{aligned} \frac{\bar{u}^n(x)-\bar{u}^{n-1}(x)}{\Delta\tau} \pm G(\tilde{u}^n)\bar{u}^n &= \pm f(x) && \text{in } \Omega \\ \bar{u}^n(x,\tau) &= g(x) && \text{on } \partial\Omega \end{aligned}$$

Here: Extrapolate from previous time steps, e.g.

$$\tilde{u}^n = \bar{u}^{n-1} + \frac{\bar{u}^{n-1}-\bar{u}^{n-2}}{\Delta\tau}\Delta\tau = 2\bar{u}^{n-1}-\bar{u}^{n-2}$$

Pseudo-time stepping

Pseudo-time discretization: Do time stepping scheme on

$$\begin{aligned} \frac{\partial \bar{u}(x,\tau)}{\partial \tau} \pm L(\bar{u}) &= \pm f(x) && \text{in } \Omega\times(0,\infty) \\ \bar{u}(x,\tau) &= g(x) && \text{on } \partial\Omega\times(0,\infty) \\ \bar{u}(x,0) &= \bar{u}_0(x) && \text{in } \Omega \end{aligned}$$

Goal: Use a method that

- is stable
- *allows us to take large time steps*

- does not have to be particularly accurate
- does not necessarily have to follow a “physical” trajectory as long as the limit is correct!

Minimal surface equation

Concrete application: Solve the minimal surface equation

$$-\nabla\cdot\left(\frac{A}{\sqrt{1+|\nabla u|^2}}\nabla u\right) = f \quad \text{in } \Omega$$
$$u = g \quad \text{on } \partial\Omega$$

Step 1: Find the steady state limit of

$$\frac{\partial \bar{u}}{\partial \tau}-\nabla\cdot\left(\frac{A}{\sqrt{1+|\nabla \bar{u}|^2}}\nabla \bar{u}\right) = f \quad \text{in } \Omega$$
$$\bar{u} = g \quad \text{on } \partial\Omega$$

Note: Choose sign as in the heat equation.

Minimal surface equation

Step 2: For

$$\frac{\partial \bar{u}}{\partial \tau} - \nabla \cdot \left(\frac{A}{\sqrt{1+|\nabla \bar{u}|^2}} \nabla \bar{u} \right) = f \qquad \text{in } \Omega$$

$$\bar{u} = g \qquad \text{on } \partial\Omega$$

choose a semi-implicit discretization:

$$\frac{\bar{u}^n - \bar{u}^{n-1}}{\Delta \tau_n} - \nabla \cdot \left(\frac{A}{\sqrt{1+|\nabla \bar{u}^{n-1}|^2}} \nabla \bar{u}^n \right) = f \qquad \text{in } \Omega$$

$$\bar{u}^n = g \qquad \text{on } \partial\Omega$$

Note: This choice likely already implies a time step restriction.

Minimal surface equation

Step 3: For

$$\bar{u}^n - \Delta\tau_n \nabla\cdot\left(\frac{A}{\sqrt{1+|\nabla\bar{u}^{n-1}|^2}}\nabla\bar{u}^n\right) = \bar{u}^{n-1} + \Delta\tau_n f \quad \text{in } \Omega$$

$$\bar{u}^n = g \quad \text{on } \partial\Omega$$

choose a space discretization (here: finite elements):

$$\left(\phi_h, \bar{u}^n\right) + \Delta\tau_n\left(\nabla\phi_h, \left(\frac{A}{\sqrt{1+|\nabla\bar{u}^{n-1}|^2}}\nabla\bar{u}^n\right)\right) = \left(\phi_h, \bar{u}^{n-1} + \Delta\tau_n f\right) \quad \forall\phi_h \in V_h$$

Note: We need to also enforce the correct boundary conditions.

Minimal surface equation

Step 4: For

$$\bar{u}^n - \Delta\tau_n \nabla \cdot \left(\frac{A}{\sqrt{1+|\nabla \bar{u}^{n-1}|^2}} \nabla \bar{u}^n \right) = \bar{u}^{n-1} + \Delta\tau_n f \quad \text{in } \Omega$$

$$\bar{u}^n = g \quad \text{on } \partial\Omega$$

choose a suitable time step $\Delta\tau_n$:

- Small enough to be “reasonably accurate”
- Large enough to get to infinity “reasonably quickly”

- In practice: increase time step over tim

- Terminate iteration once solution “is converged”

Adapting step-26

Let's adapt step-26 for this purpose!

- If necessary:
  – read through step-26
  – watch lectures 26, 27, 29

- Change boundary values (previously: zero)
- Change right hand side (here: zero)
- Implement different stiffness matrix

- Left out:
  – time step size control
  – termination criterion

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