MATH 676

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Finite element methods in scientific computing

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Lecture 31.6:

Nonlinear problems

Part 3: Newton's method for the minimal surface equation, step-15

The minimal surface equation

Consider the minimal surface equation:

$$
\begin{aligned}
-\nabla\cdot\left(\frac{A}{\sqrt{1+|\nabla u|^2}}\nabla u\right) &= f \qquad \text{in } \Omega \\
u &= g \qquad \text{on } \partial\Omega
\end{aligned}
$$

where we choose

$$\Omega = B_1(0) \subset \mathbb{R}^{2,} \qquad f=0, \qquad g=\sin(2\pi(x+y))$$

Goal: Solve this numerically with Newton's method.

Approach

Newton's method requires us to iterate these two steps (in weak form):

$$\left(\nabla\phi, \frac{A}{\sqrt{1+|\nabla u_k|^2}}\nabla\delta u_k\right) - \left(\nabla\phi, \frac{A(\nabla u_k\cdot\nabla\delta u_k)}{\left(1+|\nabla u_k|^2\right)^{3/2}}\nabla u_k\right)$$

$$= -\left(\nabla\phi, \frac{A}{\sqrt{1+|\nabla u_k|^2}}\nabla u_k\right) \qquad \forall\phi\in H_0^1$$

$$u_{k+1} = u_k + \delta u_k$$

Approach

Discrete form:

$$\sum_j \left[\left(\nabla \phi_i, \frac{A}{\sqrt{1+|\nabla u_{k,h}|^2}} \nabla \phi_j \right) - \left(\nabla \phi_i, \frac{A(\nabla u_{k,h} \cdot \nabla \phi_j)}{\left(1+|\nabla u_{k,h}|^2\right)^{3/2}} \nabla u_{k,h} \right) \right] \delta U_{k,j}$$

$$= -\left(\nabla \phi_i, \frac{A}{\sqrt{1+|\nabla u_{k,h}|^2}} \nabla u_{k,h} \right) \qquad \forall i=1 \ldots N$$

$$U_{k+1} = U_k + \delta U_k$$

Approach

Discrete form (in matrix form):

$$\begin{aligned} A_k \, \delta U_k &= F_k \\ U_{k+1} &= U_k + \delta U_k \end{aligned}$$

where

$$A_{k,ij} = \left(\nabla \phi_i, \frac{A}{\sqrt{1+|\nabla u_{k,h}|^2}} \nabla \phi_j \right) - \left(\nabla \phi_i, \frac{A \left(\nabla u_{k,h} \cdot \nabla \phi_j \right)}{\left(1+|\nabla u_{k,h}|^2 \right)^{3/2}} \nabla u_{k,h} \right)$$

$$F_{k,i} = -\left(\nabla \phi_i, \frac{A}{\sqrt{1+|\nabla u_{k,h}|^2}} \nabla u_{k,h} \right)$$

Practical considerations

Consideration 1: To solve

$$A_k \, \delta U_k = F_k$$

with

$$A_{k,ij} = \left(\nabla \phi_i, \frac{A}{\sqrt{1+|\nabla u_{k,h}|^2}} \nabla \phi_j \right) - \left(\nabla \phi, \frac{A (\nabla u_{k,h} \cdot \nabla \phi_j)}{\left(1+|\nabla u_{k,h}|^2\right)^{3/2}} \nabla u_k \right)$$

what solver is appropriate?

Answer: The matrix is symmetric and positive definite, so CG is ok.

Note: *A* is SPD because the minimal surface equation corresponds to minimizing a convex energy functional *E(u)*.

Practical considerations

Consideration 2: What boundary values does δu_k have?

Answer: Let us choose u_0 so that it already has the correct boundary values:

$$u_0|_{\partial\Omega} = g$$

Then all of the δu_k have zero boundary values because we want that

$$(u_k + \alpha_k \delta u_k)|_{\partial\Omega} = g$$

Practical considerations

Consideration 3: Newton's method does not always converge if we choose

$$U_{k+1} = U_k + \delta U_k$$

But, we can often make it converge by relaxing the iteration:

$$U_{k+1} = U_k + \alpha_k \delta U_k$$

Algorithms for choosing α_k are typically called “line search”.

Practical considerations

Idea of line search: Using the damped iteration:

$$U_{k+1} = U_k + \alpha_k \delta U_k$$

- We can prove quadratic convergence if $\alpha_k = 1$
- We get slower convergence if $\alpha_k < 1$

- We may not converge if $\alpha_k = 1$
- We frequently converge if $\alpha_k < 1$

Practical considerations

Idea of line search: We want to achieve that

$$R(u) = L(u) - f = 0$$

So try this in iteration k:

- Set $\alpha_k = 1$
- If

$$\|R(u_k + \alpha_k \delta u_k)\| \leq c \|R(u_k)\|$$

  then use this α_k; otherwise set $\alpha_k := \alpha_k/2$ and try again
- Compute $U_{k+1} = U_k + \alpha_k \delta U_k$

Practical considerations

Practice of line search:

- We need to say which norm we mean in

$$\|R(u_k)\|$$

- There are other criteria in addition to

$$\|R(u_k+\alpha_k \delta u_k)\| \leq c\|R(u_k)\|$$

These additional criteria are often called Wolfe and Armijo-Goldstein conditions

Practical considerations

In step-15:

- For simplicity, just always choose

$$\alpha_k = 0.1$$

- The “Results” section of step-15 has more details on how to implement an actual line search

Practical considerations

Consideration 4: We want to use a sequence of meshes.

Practical implementation:

1. Start with a coarse mesh
2. Do 5 Newton iterations
3. If $\|R(u_k)\| \leq \mathrm{tol}$ then stop
4. Save the solution on the current mesh
5. Refine the mesh
6. Go to step 2

The minimal surface equation

Let's look at all of this in real code!

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