MATH 676

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Finite element methods in scientific computing

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Lecture 28:

Time discretizations for second-order hyperbolic systems

Hyperbolic problems

The prototype hyperbolic equation is the wave equation:

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \Delta u(x,t) = f(x,t) \quad \text{on } \Omega\times[0,T]$$
$$u(x,0) = u_0(x) \quad \text{on } \Omega$$
$$\frac{\partial u(x,0)}{\partial t} = u_1(x) \quad \text{on } \Omega$$
$$u(x,t) = g(x,t) \quad \text{on } \partial\Omega\times[0,T]$$

- Character of the equation is “transport”
- Sharp features are transported along as sharp features
- No smoothing

Hyperbolic problems

For simplicity, discretize by the method of lines:

$$M \frac{\partial^2 U(t)}{\partial t^2} + A U(t) = F(t)$$

We can use explicit or implicit methods:

Explicit time discretization: For example

$$M \frac{U^n - 2U^{n-1} + U^{n-2}}{\Delta t^2} + A U^{n-1} = F(t^{n-1})$$

Implicit time discretization: For example

$$M \frac{U^n - 2U^{n-1} + U^{n-2}}{\Delta t^2} + A U^n = F(t^n)$$

Explicit methods for hyperbolic problems

Let's see what happens for explicit methods:

Consider problems with *f(x,t)=0:*

$$M\frac{U^n - 2U^{n-1} + U^{n-2}}{\Delta t^2} + AU^{n-1} = 0$$

That is:

$$U^n = 2U^{n-1} - U^{n-2} - \Delta t^2 M^{-1} A U^{n-1}$$

We can now use all the same techniques already used for the heat equation.

Explicit methods for hyperbolic problems

Let's see what happens for explicit methods:

From

$$U^n = 2U^{n-1} - U^{n-2} - \Delta t^2 M^{-1} A U^{n-1}$$

and using the eigenvector/value expansion we get for the expansion coefficients:

$$\begin{aligned}\mu_k^n &= 2\mu_k^{n-1} - \mu_k^{n-2} - \Delta t^2 h^{-d} \lambda_k \mu_k^{n-1} \\ &= \left(2 - \Delta t^2 h^{-d} \lambda_k\right) \mu_k^{n-1} - \mu_k^{n-2}\end{aligned}$$

Explicit methods for hyperbolic problems

Thus:

Given

$$\mu_k^n = \left(2 - \Delta t^2 h^{-d} \lambda_k\right) \mu_k^{n-1} - \mu_k^{n-2}$$

we can roughly expect the discretization to be stable if expansion coefficients decay by magnitude, i.e., if

$$\left|2 - \Delta t^2 h^{-d} \lambda_k - 1\right| \leq 1 \qquad \forall k$$

Note: This can be made more formal (but we only want to show the *idea* here).

Explicit methods for hyperbolic problems

Thus:

Given

$$|2-\Delta t^2 h^{-d}\lambda_k - 1| \leq 1 \qquad \forall k$$

and what we know about the eigenvalues of *A*, we obtain:

This explicit discretization of the wave equation is stable if $\Delta t \leq 2\frac{h}{\pi}$!

Note 1: This is an entirely reasonable restriction!

Note 2: Similar statements hold for all explicit methods.

The CFL number

The physically correct form of the wave equation is

$$\frac{\partial^2 u(x,t)}{\partial t^2} - c^2 \Delta u(x,t) = f(x,t)$$

where *c* is called the *wave speed.*

For this equation:

- Our explicit method is stable if $\Delta t \leq 2\frac{h}{c\pi}$
- Introduce the Courant-Friedrichs-Lewy number $CFL = \frac{\Delta t \; c}{h}$
- The method is stable if $CFL \leq \frac{2}{\pi}$
- Other explicit discretizations have different constants on the right hand side, but the idea remains the same!

Implicit methods for hyperbolic problems

Let's see what happens for implicit methods:

Consider problems with *f(x,t)=0:*

$$M\frac{U^n - 2U^{n-1} + U^{n-2}}{\Delta t^2} + AU^n = 0$$

That is:

$$\left(M + \Delta t^2 A\right)U^n = M\left(2U^{n-1} - U^{n-2}\right)$$

Implicit methods for hyperbolic problems

Let's see what happens for implicit methods:

From

$$(M+\Delta t^2 A)U^n = M(2U^{n-1}-U^{n-2})$$

and using the eigenvector/value expansion we get for the expansion coefficients:

$$\mu_k^n = \frac{h^d}{h^d+\Delta t^2\lambda_k}(2\mu_k^{n-1}-\mu_k^{n-2})$$

$$= \frac{1}{1+\Delta t^2 h^{-d}\lambda_k}(2\mu_k^{n-1}-\mu_k^{n-2})$$

Implicit methods for hyperbolic problems

Thus:

Given

$$\mu_k^n = \frac{1}{1 + \Delta t^2 h^{-d} \lambda_k} \left(2\mu_k^{n-1} - \mu_k^{n-2}\right)$$

we find:

The implicit discretization of the wave equation is unconditionally stable for any time step size!

Note 1: One may therefore choose the time step size arbitrarily large.

Note 2: But, this only affects *stability*, not *accuracy*!

On the CFL number

Recall the character of the wave equation:

- Information is transported undamped
- Transport speed is *c*
- Time to transport information by one cell is $\frac{h}{c}$

- The explicit time stepping method with

$$U^n = 2U^{n-1} - U^{n-2} - \Delta t^2 M^{-1} A U^{n-1} \qquad \text{where } M \approx h^d I$$

  discretized by the FEM only couples neighboring cells
- Thus, it transports information by one cell per time step

- We can't expect to choose $\Delta t \geq \frac{h}{c}$
- This matches the stability limit: $\Delta t \leq \frac{2h}{\pi c}$

On the CFL number

Recall the character of the wave equation:

- Information is transported undamped
- Transport speed is *c*
- Time to transport information by one cell is $\frac{h}{c}$

- The implicit time stepping method with

$$U^n = \left(M + \Delta t^2 A\right)^{-1} \left(2 \mathrm{U}^{n-1} - U^{n-2}\right)$$

does couple beyond neighboring cells, but not strongly

- For accuracy, we can then not expect to be able to choose the time step much larger than

$$\Delta t \approx \frac{h}{c}$$

Hyperbolic equations

Summary

- Time step limit for explicit methods is *not* unreasonable
- Explicit methods are cheap

- One can choose time step unconditionally large for implicit methods
- However, for accuracy it has to be chosen similar to that in explicit methods
- Implicit methods are expensive

For hyperbolic problems, one frequently chooses explicit methods for efficiency.

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