MATH 676

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Finite element methods in scientific computing

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Lecture 27:

Time discretizations for parabolic systems

Parabolic problems

The prototype parabolic equation is the heat equation:

$$\frac{\partial u(x,t)}{\partial t} - \Delta u(x,t) = f(x,t) \qquad \text{on } \Omega \times [0,T]$$

$$u(x,0) = u_0(x) \qquad \text{on } \Omega$$

$$u(x,t) = g(x,t) \qquad \text{on } \partial\Omega \times [0,T]$$

- Character of the equation is diffusive, i.e., sharp features blur over time
- One can add lower order (advection-, reaction-) terms to the equation; it stays parabolic as long as they are not large.

Parabolic problems

For simplicity, discretize by the method of lines:

$$M \frac{\partial U(t)}{\partial t} + A U(t) = F(t)$$

We can use explicit or implicit methods:

Explicit time discretization: E.g., forward Euler method:

$$M \frac{U^n - U^{n-1}}{\Delta t} + A U^{n-1} = F(t^{n-1})$$

Implicit time discretization: E.g., backward Euler method:

$$M \frac{U^n - U^{n-1}}{\Delta t} + A U^n = F(t^n)$$

Explicit methods for parabolic problems

Let's see what happens for explicit methods:

Consider problems with *f(x,t)=0:*

$$M\frac{U^n - U^{n-1}}{\Delta t} + AU^{n-1} = 0$$

Remember that the matrix *A* corresponds to the Laplace operator. It is symmetric and positive definite (SPD).

Thus, there exists a *complete* set of eigenvectors/-values:

$$Aw_k = \lambda_k w_k \qquad k = 1 \dots \dim(A)$$

$$\mathrm{span}(w_k) = \mathbb{R}^{\dim(A)}$$

$$\lambda_k > 0$$

$$w_k^T w_{k'} = \delta_{kk'} \qquad \forall 1 \le k, k' \le \dim(A)$$

Explicit methods for parabolic problems

Let's see what happens for explicit methods:

The completeness implies that we can write

$$U^n = \sum_k \mu_k^n w_k$$

Then, for

$$M \frac{U^n - U^{n-1}}{\Delta t} + A U^{n-1} = 0 \quad \rightarrow \quad U^n = \left(I - \Delta t M^{-1} A\right) U^{n-1}$$

...we have the following:

$$\begin{aligned}
\sum_k \mu_k^n w_k &= \left(I - \Delta t M^{-1} A\right) \sum_k \mu_k^{n-1} w_k \\
&= \sum_k \mu_k^{n-1} \left(I - \Delta t M^{-1} A\right) w_k \\
&= \sum_k \mu_k^{n-1} \left(I - \Delta t \lambda_k M^{-1}\right) w_k
\end{aligned}$$

Explicit methods for parabolic problems

Let's see what happens for explicit methods:

About the mass matrix *M*:

$$M_{ij} = \int_\Omega \phi_i \phi_j$$

Because the support of every test function is only the cells surrounding its node, we have that

$$M_{ij} = O(h^d)$$

Furthermore:

- The matrix corresponds to the identity operator
- Its spectrum is therefore bounded from above and below independently of *h*
- We can approximate it (using quadrature) by

$$M \approx h^d I$$

Explicit methods for parabolic problems

Let's see what happens for explicit methods:

Thus, our expansion of coefficients

$$\sum_k \mu_k^n w_k = \sum_k \mu_k^{n-1} \left(I - \Delta t \lambda_k M^{-1}\right) w_k$$

can be written as follows:

$$\sum_k \mu_k^n w_k = \sum_k \mu_k^{n-1} \left(1 - \Delta t \lambda_k h^{-d}\right) w_k$$

Explicit methods for parabolic problems

Let's see what happens for explicit methods:

Given

$$\sum_k \mu_k^n w_k = \sum_k \mu_k^{n-1} (1 - \Delta t \lambda_k h^{-d}) w_k$$

we can use the orthogonality of eigenvectors w_k by multiplying by $w_{k'}$:

$$w_{k'}^T \sum_k \mu_k^n w_k = w_{k'}^T \sum_k \mu_k^{n-1} (1 - \Delta t \lambda_k h^{-d}) w_k$$

Consequently:

$$\mu_{k'}^n = (1 - \Delta t \lambda_{k'} h^{-d}) \mu_{k'}^{n-1}$$

Note 1: The method is stable if $|1 - \Delta t \lambda_k h^{-d}| \leq 1 \quad \forall k$

Note 2: The solution explodes if $|1 - \Delta t \lambda_k h^{-d}| > 1 \quad \text{for any } k$

Explicit methods for parabolic problems

What we know about the eigenvalues of *A*:

- In 1d on the unit interval, the continuous operator $-\Delta$ has the spectrum

$$\lambda_k = k^2\pi^2, \qquad k=1,2,\dots\infty$$

- This can be shown by observing that the eigenfunctions are

$$w_k(x) = \sin(k\pi x)$$

and that

$$-\Delta w_k(x) = -\Delta \sin(k\pi x) = k^2\pi^2 \sin(k\pi x) = k^2\pi^2 w_k(x)$$

Explicit methods for parabolic problems

What we know about the eigenvalues of *A*:

- In 1d on the unit interval, the discrete operator A has the spectrum

$$\lambda_k \approx h k^2 \pi^2, \qquad k=1,2,\ldots,\dim(A)$$

- This can be proven by showing that functions similar to the sine functions are eigenfunctions of the matrix *A*
- The factor *h* results from the integration of matrix entries
- We only need to consider oscillating functions that can be represented on the mesh, i.e.,

$$\sum_j (w_1)_j \phi_j(x) \approx \sin(\pi x)$$
$$\vdots$$
$$\sum_j (w_{\dim(A)})_j \phi_j(x) \approx \sin([\dim(A)]\pi x)$$

Explicit methods for parabolic problems

What we know about the eigenvalues of *A*:

- In 1d on the unit interval, the discrete operator A has the spectrum

$$\lambda_k \approx h k^2 \pi^2, \qquad k = 1, 2, \ldots, \dim(A)$$

- Since $\dim(A) = 1/h$, we have

$$\lambda_{\min} \approx h\pi^2$$
$$\lambda_{\max} \approx h \frac{\pi^2}{h^2}$$

Explicit methods for parabolic problems

What we know about the eigenvalues of *A*:

- In *d* space dimensions, on the unit hypercube, the discrete operator A has the spectrum

$$\lambda_k \approx h^d k^2 \pi^{2,} \quad k=1,2,\ldots,[\dim(A)]^{1/d}$$

- Since dim$(A)=1/h^d$, we have

$$\lambda_{\min} \approx h^d \pi^2$$
$$\lambda_{\max} \approx h^d \frac{\pi^2}{h^2}$$

Note 1: h^d is from integrating matrix entries.

Note 2: h^{-2} is from taking 2nd derivatives of functions that can at most oscillate at frequency $\omega=\frac{\pi}{h}$

Explicit methods for parabolic problems

Consequence:

- Remember that the method is only stable if

$$|1-\Delta t\,\lambda_k h^{-d}| \le 1 \qquad \forall k$$

- Also

$$\lambda_{\max} \approx h^d \frac{\pi^2}{h^2}$$

Thus, the explicit (forward) Euler time discretization of the heat equation is only stable if

$$\Delta t \le \frac{2h^2}{\pi^2}$$

Note: This is a *severe* restriction! Every mesh refinement forces us to divide the time step by 4!

Implicit methods for parabolic problems

Let's see what happens for implicit methods:

For this, it is sufficient to look at problems with *f(x,t)=0*:

$$M\frac{U^n-U^{n-1}}{\Delta t}+AU^n = 0 \quad \rightarrow \quad (I+\Delta t\, M^{-1}A)U^n = U^{n-1}$$

Using the eigenvectors/eigenvalues again, we can expand

$$U^n = \sum_k \mu_k^n w_k$$

to get

$$(I+\Delta t\, M^{-1}A)\sum_k \mu_k^n w_k = \sum_k \mu_k^{n-1} w_k$$

and consequently

$$\mu_{k'}^n = (1+\Delta t\,\lambda_{k'} h^{-d})^{-1}\mu_{k'}^{n-1}$$

Implicit methods for parabolic problems

Let's see what happens for implicit methods:

Since

$$\mu_k^n = \left(1+\Delta t \lambda_k h^{-d}\right)^{-1} \mu_k^{n-1}$$

and since

$$\lambda_k > 0$$

we have

$$|\mu_k^n| \leq |\mu_k^{n-1}|$$

In other words:

The implicit Euler method is stable for the heat equation for all time step sizes!

Summary for parabolic problems

All observations above can be generalized to other explicit or implicit time stepping methods.

For the heat equation, the following is true:

- Explicit methods are only stable for small enough time steps
- The limiting time step behaves as $O(h^2)$, making it *very small* on fine meshes
- Explicit methods are therefore prohibitively expensive for parabolic method

- Implicit methods are *unconditionally stable*

Thus, for parabolic problems, use implicit methods!

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