MATH 676

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Finite element methods in scientific computing

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Lecture 26:

Time dependent problems:
A taxonomy

A taxonomy

Time dependent problems come in many forms:

- The heat equation, the prototype *parabolic equation*:

$$\frac{\partial u(x,t)}{\partial t} - \Delta u(x,t) = f(x,t) \qquad u(x,0) = u_0(x)$$

- The wave equation, the prototype *2nd order hyperbolic equation*:

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \Delta u(x,t) = f(x,t) \qquad u(x,0) = u_0(x), \quad \frac{\partial u(x,0)}{\partial t} = v_0(x)$$

- The transport equation, the prototype *1st order hyperbolic equation*:

$$\frac{\partial u(x,t)}{\partial t} + \vec{v} \cdot \nabla u(x,t) = f(x,t) \qquad u(x,0) = u_0(x)$$

A taxonomy

These equations

$$\frac{\partial u}{\partial t} - \Delta u = f \qquad \frac{\partial^2 u}{\partial t^2} - \Delta u = f \qquad \frac{\partial u}{\partial t} + \vec{v} \cdot \nabla u = f$$

follow certain patterns and are relatively easy to classify.

- We understand well how to analyze and discretize them
- They will serve as templates for “similar” equations

A taxonomy

However, there are many equations that don't quite fit the pattern.

Example 1: The Stokes equations:

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u + \nabla p &= f \qquad u(x,0) = u_0(x) \\ \nabla \cdot u &= 0 \end{aligned}$$

Notes:

- No time derivatives on the pressure, and consequently no initial values
- The *character* of the equations is *parabolic.*
- Conceptually: the heat equation on a subspace.

A taxonomy

However, there are many equations that don't quite fit the pattern.

Example 2: The Navier-Stokes equations:

$$
\begin{aligned}
\frac{\partial u}{\partial t} + u \cdot \nabla u - \nu \Delta u + \nabla p &= f \qquad\qquad u(x,0) = u_0(x) \\
\nabla \cdot u &= 0
\end{aligned}
$$

Notes:

- If ν is large, the *character* of the equations is *parabolic*
- If ν is small, the *character* of the equations is *hyperbolic.*

A taxonomy

However, there are many equations that don't quite fit the pattern.

Example 3: Two-phase flow in porous media

$$\begin{aligned} \frac{\partial S}{\partial t} + u \cdot \nabla S &= f \qquad S(x,0) = S_0(x) \\ K(S)^{-1} u + \nabla p &= 0 \\ \nabla \cdot u &= q \end{aligned}$$

Notes:

- First equation is certainly *hyperbolic*
- Other two equations have no time derivatives and have to hold at *all times*
- They can be thought of as *constraints*
- This is akin to Differential-Algebraic Equations (DAEs).

A taxonomy

However, there are many equations that don't quite fit the pattern.

Example 4: General conservation laws

$$\frac{\partial u}{\partial t} + \nabla \cdot F(u) = f \qquad u(x,0) = u_0(x)$$

Notes:

- Typically *hyperbolic*
- But can be degenerate, depending on *F(u)*

- Occasionally contain second order terms if $F=F(u,\nabla u)$, making it *formally parabolic* but *morally hyperbolic*

A taxonomy

Conclusion:

- Strict classification is not possible
- But equations often have properties that make them more similar to either hyperbolic or parabolic problems
- Spatial and temporal discretization techniques are very different for
  - 1^{st} order hyperbolic problems
  - 2^{nd} order hyperbolic problems
  - parabolic problems
- We often choose methods known to work for “similar” problems

Approaches to time dependent problems

Two ways for discretizing in space and time:

- *Method of lines*:
  - First discretize in space – get a (large) system of ordinary differential equations
  - Then use one of the ODE solvers to discretize in time

- *Rothe method*:
  - Discretize in time – obtain one PDE per time step
  - Discretize the spatial problem in each time step

Note 1: If you keep the mesh fixed, the two typically lead to the same solution.

Note 2: The Rothe method is conceptually more flexible because it allows changing the mesh between time steps!

Approaches to time dependent problems

Example for the Rothe method (time, then space):

- Take the (parabolic) heat equation:

$$\frac{\partial u(x,t)}{\partial t} - \Delta u(x,t) = f(x,t) \qquad \text{on } \Omega \times [0,T]$$

- Discretize in time using the backward Euler method:

$$\frac{u^n(x) - u^{n-1}(x)}{\Delta t} - \Delta u^n(x) = f(x,t^n) \qquad \text{on } \Omega$$

- This is a spatial PDE for the solution u^n approximating the solution $u(x,t^n)$

- We can then apply any spatial discretization method for this, including choosing a different grid at each time step

Approaches to time dependent problems

Example for the "method of lines" (space, then time):

- Take the (parabolic) heat equation:

$$\frac{\partial u(x,t)}{\partial t} - \Delta u(x,t) = f(x,t) \qquad \text{on } \Omega \times [0,T]$$

- Discretize in space using the finite element method:

$$\left(\phi_h, \frac{\partial u_h(x,t)}{\partial t} - \Delta u_h(x,t)\right) = \left(\phi_h, f(x,t)\right) \qquad \forall \phi_h$$

where $u_h(x,t) = \sum_j U_j(t) \phi_j(x)$

- Thus, receive a large system of equations:

$$M \frac{\partial U(t)}{\partial t} + A U(t) = F(t)$$

This can then be given to any ODE integrator.

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