MATH 676

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Finite element methods in scientific computing

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Lecture 21.55:

Boundary conditions

Part 2: Neumann and Robin boundary conditions

Neumann boundary conditions

Consider the Laplace equation with non-zero Neumann boundary values on the boundary:

- Strong form:

$$
\begin{aligned}
-\Delta u &= f && \text{in } \Omega \\
\partial u / \partial n &= h && \text{on } \Gamma = \partial\Omega
\end{aligned}
$$

- This problem has no unique solution:
  – let *u* be a solution
  – then *u*+*c* is also a solution for any constant *c*

Neumann boundary conditions

Consider the Laplace equation with non-zero Neumann boundary values on the boundary:

- Strong form:

$$-\Delta u = f \qquad \text{in } \Omega$$
$$\partial u / \partial n = h \qquad \text{on } \Gamma = \partial \Omega$$

- This problem does not satify Hadamard's definition of well-posedness:
  – existence
  – uniqueness
  – stability of solutions, i.e. for solutions of two problems

$$\|u_1 - u_2\|_? \le C_f \|f_1 - f_2\|_? + C_h \|h_1 - h_2\|_? \qquad \text{as } \|f_1 - f_2\|, \|h_1 - h2_1\| \rightarrow 0$$

Neumann boundary conditions

Instead, consider the Helmholtz equation with non-zero Neumann boundary values:

- Strong form:

$$\begin{aligned} u - \Delta u &= f \qquad \text{in } \Omega \\ \partial u / \partial n &= h \qquad \text{on } \Gamma = \partial\Omega \end{aligned}$$

- The corresponding weak form is then

$$(v,u) + (\nabla v, \nabla u) - \left(v, \underbrace{n \cdot \nabla u}_{h}\right)_\Gamma = (v,f) \qquad \forall v$$

which yields this:

$$(v,u) + (\nabla v, \nabla u) = (v,f) + (v,h)_\Gamma \qquad \forall v \in V$$

Neumann boundary conditions

Derive the discrete formulation:

- Weak continuous form:

$$(v,u)+(\nabla v,\nabla u)=(v,f)+(v,h)_\Gamma \qquad \forall v\in V$$

- Substitute the discrete solution

$$u_h(x) = \sum_j U_j \phi_j(x)$$

and test with the shape functions:

$$\sum_j \underbrace{\left[(\phi_i,\phi_j)+(\nabla\phi_i,\nabla\phi_j)\right]}_{A_{ij}} U_j = \underbrace{(\phi_i,f)+(\phi_i,h)_\Gamma}_{F_i} \qquad \forall i=1\dots N$$

- This yields a linear system

$$AU = F$$

Neumann boundary conditions

Assembly of the matrix and right hand side:

- Start with the linear system

$$\sum_j \underbrace{\left[(\phi_i,\phi_j)+(\nabla\phi_i,\nabla\phi_j)\right]}_{A_{ij}} U_j = \underbrace{(\phi_i,f)+(\phi_i,h)_\Gamma}_{F_i} \qquad \forall i=1\ldots N$$

- Break integrals into cells and their faces:

$$\begin{aligned} A_{ij} &= (\phi_i,\phi_j)+(\nabla\phi_i,\nabla\phi_j) = \sum_K (\phi_i,\phi_j)_K+(\nabla\phi_i,\nabla\phi_j)_K \\ F_i &= (\phi_i,f)+(\phi_i,h)_\Gamma = \sum_K \left[(\phi_i,f)_K+\sum_{e\in\partial K\cap\Gamma}(\phi_i,h)_e\right] \end{aligned}$$

- Step-7 shows this
- But let's just do this based on step-6...

Robin boundary conditions

Now consider the Helmholtz equation with non-zero Robin boundary values:

- Strong form:

$$u - \Delta u = f \qquad \text{in } \Omega$$
$$u + \partial u / \partial n = h \qquad \text{on } \Gamma = \partial\Omega$$

- The corresponding weak form is then

$$(v,u) + (\nabla v, \nabla u) - \Big(v, \underbrace{n \cdot \nabla u}_{h-u}\Big)_\Gamma = (v,f) \qquad \forall v$$

which yields this:

$$(v,u) + (\nabla v, \nabla u) + (v,u)_\Gamma = (v,f) + (v,h)_\Gamma \qquad \forall v \in V$$

Robin boundary conditions

Derive the discrete formulation:

- Weak continuous form:

$$(v,u)+(\nabla v,\nabla u)+(v,u)_\Gamma=(v,f)+(v,h)_\Gamma \qquad \forall v\in V$$

- Substitute the discrete solution

$$u_h(x) = \sum_j U_j \phi_j(x)$$

and test with the shape functions:

$$\sum_j \underbrace{\left[(\phi_i,\phi_j)+(\nabla\phi_i,\nabla\phi_j)+(\phi_i,\phi_j)_\Gamma\right]}_{A_{ij}} U_j=\underbrace{(\phi_i,f)+(\phi_i,h)_\Gamma}_{F_i} \qquad \forall i=1\dots N$$

- This yields a linear system

$$AU = F$$

Robin boundary conditions

Assembly of the matrix and right hand side:

- Start with the linear system

$$\sum_j \underbrace{\left[(\phi_i,\phi_j)+(\nabla\phi_i,\nabla\phi_j)+(\phi_i,\phi_j)_\Gamma\right]}_{A_{ij}} U_j = \underbrace{(\phi_i,f)+(\phi_i,h)_\Gamma}_{F_i} \qquad \forall i=1\dots N$$

- Break integrals into cells and their faces:

$$\begin{aligned} A_{ij} &= (\phi_i,\phi_j)+(\nabla\phi_i,\nabla\phi_j)+(\phi_i,\phi_j)_\Gamma \\ &= \sum_K \left[(\phi_i,\phi_j)_K+(\nabla\phi_i,\nabla\phi_j)_K+\sum_{e\in\partial K\cap\Gamma}(\phi_i,\phi_j)_e\right] \end{aligned}$$

$$\begin{aligned} F_i &= (\phi_i,f)+(\phi_i,h)_\Gamma \\ &= \sum_K \left[(\phi_i,f)_K+\sum_{e\in\partial K\cap\Gamma}(\phi_i,h)_e\right] \end{aligned}$$

- Let's extend the program from before...

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