MATH 676

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Finite element methods in scientific computing

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Lecture 21.5:

Boundary conditions

Part 1: Theory

General considerations

There are many kinds of boundary conditions:

- Dirichlet conditions
- Neumann conditions
- Robin conditions

- Strong conditions
- Natural conditions

- Force conditions
- Tractions conditions
- …

Question: What do they all mean, where do they enter the picture, and how do we deal with them?

General considerations

There are many kinds of boundary conditions:

- Dirichlet conditions $u = g$ on $\partial\Omega$
- Neumann conditions $a \partial u / \partial n = g$ on $\partial\Omega$
- Robin conditions $bu \pm a \partial u / \partial n = g$ on $\partial\Omega$

- Strong conditions
- Natural conditions

- Force conditions
- Tractions conditions
- …

Question: What do they all mean, where do they enter the picture, and how do we deal with them?

Laplace equation

Laplace equation with zero boundary values:

- Strong form:

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

- Weak form: *find* $u \in V_0 = H^1_0$ *so that*

$$(\nabla v, \nabla u) = (v, f) \qquad \forall v \in V_0$$

- This is equivalent to finding the minimizer of

$$\min_{u \in V_0} J(u) = \frac{1}{2}\|\nabla u\|^2 - (f, u)$$

Laplace equation

Laplace equation with non-zero boundary values:

- Strong form:

$$-\Delta u = f \qquad \text{in } \Omega$$
$$u = g \qquad \text{on } \partial\Omega$$

- Weak form: *find* $u \in V_g = \{v \in H^1 : v = g \text{ on the boundary}\}$

$$(\nabla v, \nabla u) = (v, f) \qquad \forall v \in ??$$

Question: What is the appropriate test space here?

Laplace equation

Laplace equation with non-zero boundary values:

- Strong form:

$$-\Delta u = f \qquad \text{in } \Omega$$
$$u = g \qquad \text{on } \partial\Omega$$

- Equivalent to finding the minimizer of

$$\min_{u \in V_g} J(u) = \frac{1}{2}\|\nabla u\|^2 - (f, u)$$

- The minimizer needs to satisfy the optimality condition:

$$J(u) \leq J(u + \varepsilon v) \qquad \forall u + \varepsilon v \in V_g$$

- Since $u = g \text{ on } \partial\Omega$ we need $v = 0 \text{ on } \partial\Omega$ so that $u + \varepsilon v \in V_g$

Laplace equation

Laplace equation with non-zero boundary values:

- Strong form:

$$\begin{aligned} -\Delta u = f \qquad & \text{in } \Omega \\ u = g \qquad & \text{on } \partial\Omega \end{aligned}$$

- Equivalent to finding the minimizer of

$$\min_{u \in V_g} J(u) = \frac{1}{2}\|\nabla u\|^2 - (f, u)$$

- For differentiable *J* the optimality condition is:

$$J'(u)(v) = \lim_{\varepsilon \to 0} \frac{J(u+\varepsilon v) - J(u)}{\varepsilon} = (\nabla v, \nabla u) - (v, f) = 0 \qquad \forall v$$

- The appropriate test space is then the *tangent space*,

$$v \in V_0 = \{\phi \in H^1 : \phi|_{\partial\Omega} = 0\}$$

Laplace equation

Laplace equation with non-zero boundary values on parts of the boundary:

- Strong form:

$$
\begin{aligned}
-\Delta u = f \qquad & \text{in } \Omega \\
u = g \qquad & \text{on } \Gamma_1 \subset \partial\Omega \\
\partial u / \partial n = 0 \qquad & \text{on } \Gamma_2 = \partial\Omega \setminus \Gamma_1
\end{aligned}
$$

- The corresponding weak form is then

$$J'(u)(v) = (\nabla v, \nabla u) - (v, f) = 0 \qquad \forall v$$

- The appropriate test space is again the *tangent space*,

$$v \in V_0 = \{\phi \in H^1 : \phi|_{\partial\Gamma_1} = 0\}$$

Laplace equation

From strong to weak form:

- Strong form:

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega \\ u &= g && \text{on } \Gamma_1 \subset \partial\Omega \\ \partial u / \partial n &= h && \text{on } \Gamma_2 = \partial\Omega \setminus \Gamma_1 \end{aligned}$$

- Take equation and multiply by test function:

$$(v, -\Delta u)_\Omega = (v, f)_\Omega \qquad \forall v$$

- Integrate by parts, get boundary terms:

$$(\nabla v, \nabla u)_\Omega - (v, n \cdot \nabla u)_{\partial\Omega} = (v, f)_\Omega \qquad \forall v \in V_0$$

- Split boundary term:

$$(\nabla v, \nabla u)_\Omega - (v, n \cdot \nabla u)_{\Gamma_1} - (v, n \cdot \nabla u)_{\Gamma_2} = (v, f)_\Omega \qquad \forall v \in V_0$$

Laplace equation

From strong to weak form:

- Strong form:

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega \\ u &= g && \text{on } \Gamma_1 \subset \partial\Omega \\ \partial u / \partial n &= h && \text{on } \Gamma_2 = \partial\Omega \setminus \Gamma_1 \end{aligned}$$

- Consider the boundary terms separately:

$$(\nabla v, \nabla u)_\Omega - \Big(\underbrace{v}_{=0}, n\cdot\nabla u\Big)_{\Gamma_1} - \Big(v, \underbrace{n\cdot\nabla u}_{=h}\Big)_{\Gamma_2} = (v, f)_\Omega \qquad \forall v \in V_0$$

- This leads to the final form:

$$(\nabla v, \nabla u)_\Omega = (v, f)_\Omega + (v, h)_{\Gamma_2} \qquad \forall v \in V_0$$

Laplace equation

From strong to weak form:

- Strong form:

$$\begin{aligned} -\Delta u = f & \quad \text{in } \Omega \\ u = g & \quad \text{on } \Gamma_1 \subset \partial\Omega \\ \partial u / \partial n = h & \quad \text{on } \Gamma_2 = \partial\Omega \setminus \Gamma_1 \end{aligned}$$

- Consider the boundary terms separately:

$$(\nabla v, \nabla u)_\Omega - \Big(\underbrace{v}_{=0}, n\cdot\nabla u\Big)_{\Gamma_1} - \Big(v, \underbrace{n\cdot\nabla u}_{=h}\Big)_{\Gamma_2} = (v, f)_\Omega \qquad \forall v \in V_0$$

- We call boundary conditions
  – *strong*, if a term disappears because of the test space
  – *natural*, if a term gets replaced and goes to the r.h.s.
- For Laplace: Dirichlet=strong, Neumann=natural

Laplace equation

What about Robin conditions:

- Strong form:

$$\begin{aligned} -\Delta u = f \qquad & \text{in } \Omega \\ u = g \qquad & \text{on } \Gamma_1 \subset \partial\Omega \\ bu + \partial u / \partial n = h \qquad & \text{on } \Gamma_2 = \partial\Omega \setminus \Gamma_1 \end{aligned}$$

- This yields:

$$(\nabla v, \nabla u)_\Omega - \Big(\underbrace{v}_{=0}, n\cdot\nabla u\Big)_{\Gamma_1} - \Big(v, \underbrace{n\cdot\nabla u}_{=h-bu}\Big)_{\Gamma_2} = (v, f)_\Omega \qquad \forall v \in V_0$$

- And in final form:

$$(\nabla v, \nabla u)_\Omega + (v, bu)_{\Gamma_2} = (v, f)_\Omega + (v, h)_{\Gamma_2} \qquad \forall v \in V_0$$

Laplace equation

What about equations with coefficients:

- Strong form:

$$\begin{aligned} -\nabla\cdot A\nabla u &= f && \text{in } \Omega \\ u &= g && \text{on } \Gamma_1\subset\partial\Omega \\ ??&=h && \text{on } \Gamma_2=\partial\Omega\setminus\Gamma_1 \end{aligned}$$

- This yields after integration by parts:

$$(\nabla v,\nabla u)_\Omega - \Big(\underbrace{v}_{=0}, n\cdot A\nabla u\Big)_{\Gamma_1} - \Big(v, \underbrace{n\cdot A\nabla u}_{??}\Big)_{\Gamma_2} = (v,f)_\Omega \qquad \forall v\in V_0$$

- To replace the third term, we need that the natural (Neumann) boundary condition reads

$$n\cdot A\nabla u = h \qquad on\,\Gamma_2$$

- This corresponds with the *physical* definition of flux.

More general equations

Question: Is it generally true that

- Dirichlet = strong boundary condition
- Neumann = natural boundary condition

or does this only apply to the Laplace equation?

Answer:
No! It all depends on the spaces and bilinear forms.

Example: Mixed Laplace

Consider the mixed Laplace equation:

$$\begin{aligned} A^{-1}u+\nabla p=0 &\qquad \text{in } \Omega \\ \nabla\cdot u=f &\qquad \text{in } \Omega \\ p=g &\qquad \text{on } \Gamma_1\subset\partial\Omega \\ n\cdot u=0 &\qquad \text{on } \Gamma_2=\partial\Omega \setminus \Gamma_1 \end{aligned}$$

Multiplying by test functions, integrating by parts yields

$$(v,A^{-1}u)_\Omega-(\nabla\cdot v,p)_\Omega+(v,n\,p)_{\partial\Omega}-(q,\nabla\cdot u)_\Omega=-(q,f)_\Omega \qquad \forall\, v\in U, q\in P$$

with spaces (“sort of”)

$$\begin{aligned} &U=\{v\in H(\text{div}):n\cdot v|_{\Gamma_2}=0\} \\ &P=L_2 \end{aligned}$$

Example: Mixed Laplace

Consider the mixed Laplace equation:

$$
\begin{aligned}
A^{-1}u + \nabla p = 0 \quad & \text{in } \Omega \\
\nabla \cdot u = f \quad & \text{in } \Omega \\
p = g \quad & \text{on } \Gamma_1 \subset \partial\Omega \\
n \cdot u = 0 \quad & \text{on } \Gamma_2 = \partial\Omega \setminus \Gamma_1
\end{aligned}
$$

Considering boundary terms separately:

$$
(v, A^{-1}u)_\Omega - (\nabla \cdot v, p)_\Omega + \Big(v, n \underbrace{p}_{=g}\Big)_{\Gamma_1} + \Big(\underbrace{n \cdot v}_{=0}, p\Big)_{\Gamma_2} - (q, \nabla \cdot u)_\Omega = -(q, f)_\Omega
$$

$$
\forall v \in U, q \in P
$$

This yields:

$$
(v, A^{-1}u)_\Omega - (\nabla \cdot v, p)_\Omega - (q, \nabla \cdot u)_\Omega = -(q, f)_\Omega - (v, n\, g)_{\Gamma_1} \quad \forall v \in U, q \in P
$$

Here: $p=g$ → Dirichlet → natural boundary condition
$n \cdot u=0$ → Dirichlet → strong boundary condition

Example: Mixed Laplace

What happens with conflicting boundary conditions:

$$\begin{aligned} A^{-1}u + \nabla p &= 0 && \text{in } \Omega \\ \nabla \cdot u &= f && \text{in } \Omega \\ p &= g && \text{on } \partial\Omega \\ n \cdot u &= 0 && \text{on } \partial\Omega \end{aligned}$$

After integration by parts we then have

$$(v, A^{-1}u)_\Omega - (\nabla \cdot v, p)_\Omega + \Big(\underbrace{n \cdot v}_{=0}, \underbrace{p}_{=g}\Big)_{\partial\Omega} - (q, \nabla \cdot u)_\Omega = -(q, f)_\Omega \quad \forall v \in U, q \in P$$

This yields:

$$(v, A^{-1}u)_\Omega - (\nabla \cdot v, p)_\Omega - (q, \nabla \cdot u)_\Omega = -(q, f)_\Omega \qquad \forall v \in U, q \in P$$

Here: $p=g$ does not appear at all. It will be ignored.

Example: Stokes

Consider the Stokes equations:

$$-\eta \Delta u + \nabla p = f \qquad \text{in } \Omega$$
$$\nabla \cdot u = 0 \qquad \text{in } \Omega$$

Question: What boundary conditions can we pose?

Answer: Form the bilinear form:

$$(\eta \nabla v, \nabla u)_\Omega - (v, n \cdot \eta \nabla u)_{\partial\Omega} - (\nabla \cdot v, p)_\Omega + (n \cdot v, p)_{\partial\Omega} - (q, \nabla \cdot u)_\Omega = (v, f)_\Omega \quad \forall v, q$$

Combine boundary terms:

$$(\eta \nabla v, \nabla u)_\Omega - (\nabla \cdot v, p)_\Omega - (v, n \cdot [\eta \nabla u - p I])_{\partial\Omega} - (q, \nabla \cdot u)_\Omega = (v, f)_\Omega \qquad \forall v, q$$

Example: Stokes

Consider the Stokes equations:

$$-\eta \Delta u + \nabla p = f \qquad \text{in } \Omega$$
$$\nabla \cdot u = 0 \qquad \text{in } \Omega$$

Question: What boundary conditions can we pose?

Answer: Consider the boundary term:

$$(v, n \cdot [\eta \nabla u - pI])_{\partial\Omega}$$

We can then consider the following boundary conditions:

- Prescribed velocity: $u|_{\partial\Omega} = g \quad \rightarrow \quad v|_{\partial\Omega} = 0$
- Prescribed traction force: $n \cdot [\eta \nabla u - pI]|_{\partial\Omega} = h$

Here: Dirichlet=strong, Neumann=natural

Example: Stokes

Stokes equations with Dirichlet conditions:

$$
\begin{aligned}
-\eta \Delta u + \nabla p = f \qquad & \text{in } \Omega \\
\nabla \cdot u = 0 \qquad & \text{in } \Omega \\
u = g \qquad & \text{on } \partial\Omega
\end{aligned}
$$

The complete formulation is then:

Find $u \in U_g = \{ v \in H^1(\Omega)^d : v|_{\partial\Omega} = g \}, \;\; p \in L_2(\Omega)$ so that

$$(\eta \nabla v, \nabla u)_\Omega - (\nabla \cdot v, p)_\Omega - (q, \nabla \cdot u)_\Omega = (v, f)_\Omega \qquad \forall v \in U_{0,} q \in P$$

Example: Stokes

Stokes equations with Neumann conditions:

$$
\begin{aligned}
-\eta \Delta u + \nabla p = f & \qquad \text{in } \Omega \\
\nabla \cdot u = 0 & \qquad \text{in } \Omega \\
n \cdot [\eta \nabla u - pI] = h & \qquad \text{on } \partial\Omega
\end{aligned}
$$

The complete formulation is then:

Find $u \in U = H^1(\Omega)^d,\ \ p \in L_2(\Omega)$ so that

$$
(\eta \nabla v, \nabla u)_\Omega - (\nabla \cdot v, p)_\Omega - (q, \nabla \cdot u)_\Omega = (v, f)_\Omega + (v, h)_{\partial\Omega} \qquad \forall\, v \in U, q \in P
$$

Example: Stokes

Question: What if we prescribe *u.n=g*?

Answer: Then *v.n=0.* Consider again the boundary term:

$$\left(v, n\cdot[\eta\nabla u - pI]\right)_{\partial\Omega} = \left(\underbrace{(n\otimes n)v}_{=(n\cdot v)n=0} + (I - n\otimes n)v, n\cdot[\eta\nabla u - pI]\right)_{\partial\Omega}$$
$$= \left(v, (I - n\otimes n)(n\cdot[\eta\nabla u - pI])\right)_{\partial\Omega}$$

Consequence: If we prescribe *only* the normal component of the velocity, we also need to prescribe the *tangential* component of the traction:

$$u\cdot n|_{\partial\Omega} = g$$
$$(I - n\otimes n)(n\cdot[\eta\nabla u - pI]) = h$$

Example: Stokes

Question: What if we prescribe the *physical traction force*:

$$n\cdot[2\eta\varepsilon(u)-pI]|_{\partial\Omega}=h \qquad \text{where } \varepsilon(u)=\frac{1}{2}(\nabla u+\nabla u^T)$$

Answer: Consider again the boundary term:

$$(v, n\cdot[\eta\nabla u-pI])_{\partial\Omega}$$

Consequence: You can't do this.

Example: Stokes

Question: What if we prescribe the *physical traction force*:

$$n \cdot [2\eta\varepsilon(u) - pI]|_{\partial\Omega} = h \qquad \text{where } \varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)$$

Answer: You can't do this. However, if we formulate the Stokes equations as

$$\begin{aligned} -2\eta\nabla\cdot\varepsilon(u) + \nabla p &= f \qquad \text{in } \Omega \\ \nabla\cdot u &= 0 \qquad \text{in } \Omega \end{aligned}$$

then the weak form is

$$\begin{aligned} (2\eta\varepsilon(v), \varepsilon(u))_\Omega - (\nabla\cdot v, p)_\Omega - (v, n\cdot[2\eta\varepsilon(u) - pI])_{\partial\Omega} \\ -(q, \nabla\cdot u)_\Omega = (v, f)_\Omega \qquad \forall v, q \end{aligned}$$

and we can impose this boundary condition.

Example: Stokes

Note: The two formulations of the Stokes equations,

$$\begin{aligned} -\eta \Delta u + \nabla p &= f \qquad \text{in } \Omega \\ \nabla \cdot u &= 0 \qquad \text{in } \Omega \end{aligned}$$

and

$$\begin{aligned} -2\eta \nabla \cdot \varepsilon(u) + \nabla p &= f \qquad \text{in } \Omega \\ \nabla \cdot u &= 0 \qquad \text{in } \Omega \end{aligned}$$

are equivalent because

$$2\eta \nabla \cdot \varepsilon(u) = \eta \nabla \cdot (\nabla u) + \eta \nabla \cdot (\nabla u^T) = \eta \Delta u + \eta \nabla (\underbrace{\nabla \cdot u}_{=0}) = \eta \Delta u$$

They only differ in whether we can impose:

- The *unphysical traction* $\quad n \cdot [\eta \nabla u - p I]|_{\partial\Omega} = h$
- The *physical traction* $\quad n \cdot [2\eta \varepsilon(u) - p I]|_{\partial\Omega} = h$

Summary

- Boundary conditions are complicated
- They can be resolved by looking at the boundary terms after integrating by parts
- You can impose *strong* conditions:
  – relating to the test function in these terms
  – appear as part of the function space for the solution
  – are zero in the function space for the test function
- You can impose *natural* conditions:
  – relating to the solution in these terms
  – are replaced by their boundary value
  – are brought to the right hand side of the weak form

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