MATH 676

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Finite element methods in scientific computing

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Lecture 17.75:

Generating adaptively refined meshes: “A posteriori” error estimators

Adaptive mesh refinement (AMR)

Adaptive mesh refinement happens in a loop:

SOLVE → ESTIMATE → MARK → REFINE

- SOLVE: Assemble linear system, solve it
- ESTIMATE: Compute a refinement indicator for each cell
- MARK: Determine which cells should be refined
- REFINE: Refine these cells (and possibly others)

The ESTIMATE phase

ESTIMATE phase: Compute a “refinement indicator” for each cell *K*:

$$\eta = \{\eta_{K_1}, \eta_{K_2}, \eta_{K_3}, \ldots, \eta_{K_N}\}$$

Recall from lecture 17.25:

A “reasonable” approach is to estimate the *interpolation error* of the solution, which leads to the Kelly indicator:

$$\eta_K = h_K^{1/2} \left(\int_{\partial K} |[\nabla u_h]|^2 \right)^{1/2}$$

Ideas

The “ideal” refinement indicator would be the actual error e_K:

$$\eta_K := e_K = \|u-u_h\|_K = \left(\int_K |u-u_h|^2 dx\right)^{1/2}$$

Questions/problems:

1. Would a different norm be more appropriate?
2. But we don't have the exact solution *u(x)*!
3. Does refining a cell decrease the overall error?

Ideas

Question 3: Does refining a cell decrease the overall error?

Answers:

- It depends on the norm in which we measure the error
- For *interpolation*, refining one cell only affects the error on this cell
- For the *Galerkin projection*, refining one cell affects the error *everywhere*

  This is called “error pollution”.

Ideas

Question 3: Does refining a cell decrease the overall error?

Consider the Laplace equation:

- Let T be a mesh, T^+ be any refinement
- Let V_h be the FE space on T, V_h^+ be the FE space on T^+
- Then $V_h \subset V_h^+$
- Recall

$$E := \|\nabla(u-u_h)\| = \min_{v_h \in V_h} \|\nabla(u-v_h)\|$$
$$E^+ := \|\nabla(u-u_h^+)\| = \min_{v_h \in V_h^+} \|\nabla(u-v_h)\|$$

- Consequently: $E^+ \leq E$

Ideas

Question 3: Does refining a cell decrease the overall error?

But:

- It is not necessarily true that $E^+ \le E$ if $E := \|u - u_h\|$
- The improvement may be marginal

To show that $E^+ \le \theta E, \quad \theta < 1$ one needs to refine *enough* cells.

Strategy: Dörfler marking / bulk refinement marks the cells with the largest errors so that

$$\sum_{K \in \text{marked cells}} \eta_K \ge \alpha \sum_{K \in T} \eta_K, \quad 0 < \alpha \le 1$$

A posteriori error estimation

Question 2: For evaluating the true error

$$\eta_K := e_K = \|u-u_h\|_K = \left(\int_K |u-u_h|^2 dx\right)^{1/2}$$

we would need to know the *exact* solution *u(x)*.

Answer: Yes. This is the central problem.

We need to find formulas that help us *estimate* the error only in term of $u_h(x)$.

This is called “a posteriori” error estimation.

A posteriori error estimation

A posteriori error estimation for Laplace's equation:

Consider

$$-\Delta u = f, \qquad u|_{\partial\Omega} = 0.$$

Start with Galerkin orthogonality:

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in V$$

$$(\nabla u_h, \nabla v_h) = (f, v_h) \qquad \forall v_h \in V_h \subset V$$

$$(\nabla(\underbrace{u - u_h}_{=:e}), \nabla v_h) = 0 \qquad \forall v_h \in V_h \subset V$$

A posteriori error estimation

A posteriori error estimation for Laplace's equation:

Then consider the energy norm of the error:

$$\begin{aligned}\|\nabla e\|_\Omega^2 &= (\nabla e, \nabla e)_\Omega = (\nabla e, \nabla(e - I_h e))_\Omega \\ &= \sum_K (\nabla e, \nabla(e - I_h e))_K \\ &= \sum_K (-\Delta e, e - I_h e)_K + (n \cdot \nabla e, e - I_h e)_{\partial K}\end{aligned}$$

Now use the following facts:

$$-\Delta e = -\Delta(u - u_h) = -\Delta u + \Delta u_h = \underbrace{f + \Delta u_h}_{\text{cell residual}}$$

$$(e - I_h e)|_{\partial\Omega} = ((u - u_h) - (I_h u - \underbrace{I_h u_h}_{u_h}))|_{\partial\Omega} = \underbrace{u|_{\partial\Omega}}_{=0} - \underbrace{I_h u|_{\partial\Omega}}_{=0}$$

A posteriori error estimation

A posteriori error estimation for Laplace's equation:

Thus:

$$\begin{aligned}\|\nabla e\|_\Omega^2 &= \sum_K (-\Delta e, e - I_h e)_K + (n\cdot\nabla e, e - I_h e)_{\partial K} \\ &= \sum_K (f + \Delta u_h, e - I_h e)_K + (n\cdot\nabla e, e - I_h e)_{\partial K\setminus\partial\Omega}\end{aligned}$$

The second term only integrates over interior faces. We visit each face twice (from *K* and from its neighbor *K*'):

$$\begin{aligned}\sum_K (n_K\cdot\nabla e|_K, e - I_h e)_{\partial K\setminus\partial\Omega} &= \sum_K \frac{1}{2}(n_K\cdot\nabla e|_K, e - I_h e)_{\partial K\setminus\partial\Omega} \\ &\quad + \frac{1}{2}(\underbrace{n_{K'}}_{=-n_K}\cdot\nabla e|_{K'}, e - I_h e)_{\partial K\setminus\partial\Omega} \\ &= \sum_K \frac{1}{2}(n_K\cdot(\nabla e|_K - \nabla e|_{K'}), e - I_h e)_{\partial K\setminus\partial\Omega}\end{aligned}$$

A posteriori error estimation

A posteriori error estimation for Laplace's equation:

Consider the face term:

$$\sum_K \left(n_K \cdot \nabla e|_K , e - I_h e\right)_{\partial K \setminus \partial\Omega} = \sum_K \frac{1}{2}\left(n_K \cdot (\nabla e|_K - \nabla e|_{K'}), e - I_h e\right)_{\partial K \setminus \partial\Omega}$$

Observe:

$$\nabla e|_K - \nabla e|_{K'} = \underbrace{\left(\nabla u|_K - \nabla u|_{K'}\right)}_{=0 \text{ because } u \text{ is smooth}} - \underbrace{\left(\nabla u_h|_K - \nabla u_h|_{K'}\right)}_{=: [\nabla u_h]}$$

A posteriori error estimation for Laplace's equation:

Thus:

$$\|\nabla e\|_\Omega^2 = \sum_K (\underbrace{f+\Delta u_h}_{\text{cell residual}}, e-I_h e)_K - \frac{1}{2}(\underbrace{n\cdot[\nabla u_h]}_{\text{"jump" residual}}, e-I_h e)_{\partial K\setminus\partial\Omega}$$

Using the Cauchy-Schwarz inequality on cell/face integrals:

$$\|\nabla e\|_\Omega^2 \le \sum_K \left(\|f+\Delta u_h\|_K \|e-I_h e\|_K + \frac{1}{2}\|n\cdot[\nabla u_h]\|_{\partial K\setminus\partial\Omega} \|e-I_h e\|_{\partial K\setminus\partial\Omega} \right)$$

A posteriori error estimation

A posteriori error estimation for Laplace's equation:

Thus:

$$\|\nabla e\|_\Omega^2 = \sum_K \big(\underbrace{f+\Delta u_h}_{\text{cell residual}}, e-I_h e\big)_K - \frac{1}{2}\big(\underbrace{n\cdot[\nabla u_h]}_{\text{"jump" residual}}, e-I_h e\big)_{\partial K\setminus\partial\Omega}$$

$$\leq \sum_K \left(\|f+\Delta u_h\|_K \|e-I_h e\|_K + \frac{1}{2}\|n\cdot[\nabla u_h]\|_{\partial K\setminus\partial\Omega}\|e-I_h e\|_{\partial K\setminus\partial\Omega}\right)$$

Because $e \in H^1$ we have the following basic interpolation estimates:

$$\|e-I_h e\|_K \leq C h_K \|\nabla e\|_K$$

$$\|e-I_h e\|_{\partial K\setminus\partial\Omega} \leq C h_K^{1/2} \|\nabla e\|_K$$

A posteriori error estimation for Laplace's equation:

Thus:

$$\|\nabla e\|_\Omega^2 \le \sum_K \left(\|f+\Delta u_h\|_K \underbrace{\|e-I_h e\|_K}_{\le C h_K \|\nabla e\|_K} + \frac{1}{2}\|n\cdot[\nabla u_h]\|_{\partial K\setminus\partial\Omega} \underbrace{\|e-I_h e\|_{\partial K\setminus\partial\Omega}}_{\le C h_K^{1/2}\|\nabla e\|_K} \right)$$

$$\le C \sum_K \left(h_K \|f+\Delta u_h\|_K + \frac{1}{2} h_K^{1/2} \|n\cdot[\nabla u_h]\|_{\partial K\setminus\partial\Omega} \right) \|\nabla e\|_K$$

Recall Cauchy-Schwarz inequality: $\sum_i x_i y_i \le \left(\sum x_i^2\right)^{1/2} \left(\sum y_i^2\right)^{1/2}$

Apply this to the sum over cells:

$$\|\nabla e\|_\Omega^2 \le C \left[\sum_K \left(h_K \|f+\Delta u_h\|_K + \frac{1}{2} h_K^{1/2} \|n\cdot[\nabla u_h]\|_{\partial K\setminus\partial\Omega} \right)^2 \right]^{1/2} \underbrace{\left[\sum_K \|\nabla e\|_K^2 \right]^{1/2}}_{\|\nabla e\|_\Omega}$$

A posteriori error estimation

A posteriori error estimation for Laplace's equation:

Thus:

$$\|\nabla e\|_\Omega \leq C\left[\sum_K \left(h_K\|f+\Delta u_h\|_K + \frac{1}{2}h_K^{1/2}\|n\cdot[\nabla u_h]\|_{\partial K\setminus\partial\Omega}\right)^2\right]^{1/2}$$

Interpretation: We have bounded the error in terms of only the (computable!) discrete solution!

A posteriori error estimation

A posteriori error estimation for Laplace's equation:

Last reformulation, squaring both sides:

$$\|\nabla e\|_\Omega^2 \leq C \sum_K \underbrace{\left(h_K \|f + \Delta u_h\|_K + \frac{1}{2} h_K^{1/2} \|n \cdot [\nabla u_h]\|_{\partial K \setminus \partial\Omega} \right)^2}_{=: \ \eta_K^2}$$

Nomenclature:

- We call η_K the “residual-based” error estimator for cell *K*.
- It consists of the norm of the “cell residual” and the norm of the “jump residual”/”face residual”.

A posteriori error estimation

Note: If we approximate

$$\|\nabla e\|_\Omega^2 \le C \sum_K \left(h_K \|f+\Delta u_h\|_K + \frac{1}{2} h_K^{1/2} \|n\cdot[\nabla u_h]\|_{\partial K\setminus\partial\Omega} \right)^2$$
$$\approx C' \sum_K h_K \|n\cdot[\nabla u_h]\|_{\partial K\setminus\partial\Omega}^2$$

then we get exactly the “Kelly” estimator!

Reason: For odd polynomial degrees the jump residual dominates the cell residual.

But: For even polynomial degrees, it is the other way around!

Some conclusions

Conclusion 1: It is possible to bound the (unknown) error exclusively in terms of the known, computed solution.

For example, for the Laplace equation we get:

$$\|\nabla e\|_\Omega^2 \leq C \sum_K \left(h_K \|f + \Delta u_h\|_K + \frac{1}{2} h_K^{1/2} \|n \cdot [\nabla u_h]\|_{\partial K \setminus \partial \Omega} \right)^2$$

The estimate involves “cell” and “jump” residuals.

Note: Similar estimates can often be derived for other equation as well!

Some conclusions

Conclusion 2: Estimates typically involve unknown constants.

For example, for the Laplace equation we get:

$$\|\nabla e\|_\Omega^2 \leq C \sum_K \left(h_K \|f+\Delta u_h\|_K + \frac{1}{2} h_K^{1/2} \|n \cdot [\nabla u_h]\|_{\partial K \setminus \partial \Omega} \right)^2$$

Here, the constant comes from the interpolation estimates. It is usually poorly known.

Note: For other equations, there are also stability constants, and constants from yet other estimates.

Some conclusions

Conclusion 3: If we forget about the constant, we still get good refinement criteria!

For example, for the Laplace equation we get:

$$\|\nabla e\|_\Omega^2 \leq C \sum_K \underbrace{\left(h_K \|f + \Delta u_h\|_K + \frac{1}{2} h_K^{1/2} \|n \cdot [\nabla u_h]\|_{\partial K \setminus \partial\Omega} \right)^2}_{=:\ \eta_K^2}$$

Here, the η_K are entirely computable and yield good meshes, even though

$$\|\nabla e\|_\Omega^2 \not\leq \sum_K \eta_K^2$$

Note: The same is true for other equations.

Some conclusions

Conclusion 4: That leaves the question whether we care about the H^1 norm of the error, $\|\nabla e\|_\Omega$.

Some answers:

- Sometimes we may. For the Laplace equation it has a meaning in terms of the “energy” of a physical system

- Oftentimes we don't.
- We may be much more interested in other, physical measures.

- This leads to “goal oriented” error estimates.

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