MATH 546: Partial Differential Equations II

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Lectures: Engineering E 206, Mondays/Wednesdays/Fridays, 11-11:50am
Office hours: Wednesdays, 1-2pm; or by appointment.

Homework assignment 4 – due Friday 4/26/2019

There are a number of ways in which one can define "broken Sobolev spaces" such as $H^{1/2}$, and to everyone's chagrin, these ways do not lead to the same outcome. As a consequence, when people talk about $H^{1/2}$, one has to pay attention to what definition they have used. The principal difference between these definitions is always whether the discontinuous step function is just so in, or just so not in the space. You should read this last sentence by thinking of there being a continuum of functions, starting with very nice ones (like the sine function), gradually becoming worse (functions that have a kink like $|x|$, or a cusp like $\sqrt{|x|}$), then becoming discontinuous but bounded (like the step function), to functions with singularities (like $1/x$) where the function becomes infinite, and finally functions like the delta function that is zero everywhere except at a single point where it is infinite. Function spaces generally contain all functions "up to some point" on this continuum, and the question is whether the step function is just in or just out of what's in the space $H^{1/2}$.

This week's homework is around the various kinds of definitions and whether the step function is a member of the space with a given definition. For simplicity, we'll only consider this in 1d.

Problem 1 (The space $H^{1/2}$, take 1: Via the Fourier transform). If one thinks of the space H^k as the space of functions that have k square integrable (weak) derivatives, then $H^{1/2}$ would be the space of functions that have half a derivative. This is hard to understand in terms of what such a half derivative should actually be, but we can define it as follows: Recall that the Fourier transformat satisfies the property

$$\begin{aligned}
\mathcal{F}\left[\frac{d^j}{dx^j} f(x)\right](k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{d^j}{dx^j} f(x)\right] e^{ikx}\,\mathrm{d}x \\
&= (-1)^j \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \left[\frac{d^j}{dx^j} e^{ikx}\right] \mathrm{d}x \\
&= (-1)^j \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)(ik)^j e^{ikx}\,\mathrm{d}x \\
&= (-ik)^j \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{ikx}\,\mathrm{d}x \\
&= (-ik)^j \mathcal{F}[f](k).
\end{aligned}$$

The key step here was simply the integration by parts from the first to the second line. It is not difficult to see that a similar formula holds when the domain on which f is defined is finite: then we can say that the Fourier transform

$$\mathcal{F}[f(x)] = \{a_k\}_{k=0}^{\infty}$$

simply yields the (infinite) collection of Fourier coefficients a_k so that

$$f(x) = \sum_{k=0}^{\infty} a_k e^{ikx}.$$

By a similar argument as above, one then obtains that the Fourier coefficients of the derivatives of f are given by

$$\mathcal{F}\left[\frac{d^j}{dx^j} f(x)\right] = \{(ik)^j a_k\}_{k=0}^{\infty}.$$

Because the Fourier transform is invertible, we also have

$$\frac{d^j}{dx^j} f = \mathcal{F}^{-1}\left(\mathcal{F}\left[\frac{d^j}{dx^j} f\right]\right) = \mathcal{F}^{-1}\left((-ik)^j \mathcal{F}[f]\right).$$

This formula is useful because we can now talk about what it means to take a half-derivative: we just choose $j = 1/2$ in this last formula – we can then compute $\left(\frac{d}{dx}\right)^{1/2} f$ by just doing the forward and inverse Fourier transform. Of course, this can also be done with any other j, whether it is an integer or not, and whether it is positive or not.

So let's come back to the original question: Is the step function

$$h(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0 \end{cases}$$

defined on the interval $\Omega = (-1, 1)$ in the space $H^{1/2}(\Omega)$? As usual, we will say that a function u is in $H^s(\Omega)$ if it is in $L^2(\Omega)$ and its sth derivative is square integrable.

To answer this question, you will need to figure out whether $\left(\frac{d}{dx}\right)^{1/2} h$ is square integrable. For this, you have to (i) find the Fourier series of h, and (ii) use the Plancherel identity that in the current context really just says that the L^2 norm of a function with Fourier coefficients b_k equals the sum of squares of the b_k (potentially up to a constant, depending on how exactly one defines the Fourier transform). The latter is useful because you won't have to do the awkward inverse Fourier transform.

While you're there, answer the following question:

- If your answer is that $h \in H^{1/2}$, then is h also in the spaces $H^{1/2+\varepsilon}$ for any $\varepsilon > 0$?
- If your answer is that $h \notin H^{1/2}$, then is h at least in the spaces $H^{1/2-\varepsilon}$ for any $\varepsilon > 0$?

(30 points)

Problem 2 (The space $H^{1/2}$, take 2: Slobodeckij's definition). Let's take a different definition of H^s where s may not be an integer: We say that a function $u \in L^2$ is in the space H^s if its H^s-norm is finite, where this norm is defined as follows:

$$\|u\|_{H^s(\Omega)} = \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{2s+d}} \mathrm{d}y \, \mathrm{d}x.$$

Here, d is the dimension of the domain Ω. Again, this definition (originally given by Slobodeckij) is valid for arbitrary $0 < s < 1$. (If you wanted to define the space $H^{3/2}$ in this way, you'd check that $u \in H^1$ and that $\nabla u \in H^{1/2}$.)

Answer the same questions as for Problem 1:

- Is $h \in H^{1/2}$ using this definition of the space?
- If your answer is that $h \in H^{1/2}$, then is h also in the spaces $H^{1/2+\varepsilon}$ for any $\varepsilon > 0$?
- If your answer is that $h \notin H^{1/2}$, then is h at least in the spaces $H^{1/2-\varepsilon}$ for any $\varepsilon > 0$?

(30 points)

Problem 3 (The space $H^{1/2}$, take 3). The last way we'll consider here in which one could define the space $H^{1/2}(\Omega)$ on a one-dimensional domain Ω is a bit backward because it doesn't quite give a statement one can easily check. It assumes that the one-dimensional domain Ω is (a subset of) the boundary of a two-dimensional domain $\Sigma \subset \mathbb{R}^2$, i.e., $\Omega \subset \partial\Sigma$. Since we're considering $\Omega = (-1, 1)$, we can for example choose the rectangle $\Sigma = (-1, 1) \times (0, 1)$ or the half circle $\Sigma = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| < 1 \,\text{and}\, x_2 > 0\}$ or any other similar domain that is convenient.

Then take a close look at the following statement: We define $H^{1/2}(\Omega)$ as

$$H^{1/2}(\Omega) = \left\{\varphi \in L^2(\Omega) : \text{there exists } u \in H^1(\Sigma) \text{ so that } T_\Omega u = \varphi\right\}.$$

Here, T_Ω is the trace operator that takes the boundary values of u on the part of the boundary of Σ that is Ω. In other words, $H^{1/2}(\Omega)$ is the set of all possible boundary values that functions $u \in H^1(\Sigma)$ can have. You'll note that unlike the two other definitions, this one really is specific to an index of 1/2 and can't easily be generalized to other fractional values.

As before, check whether $h \in H^{1/2}(\Omega)$ using this definition. **(20 points)**

Problem 4 (The spaces H^k and their Fourier basis). We briefly talked about this in class: Just like $\mathbb{R}^n$, one can give the spaces H^k a *basis*. There are of course many bases one could choose, but the simplest one (and in many situations the most *convenient* one) is the Fourier basis.

For this purpose, let's stay in 1d and for convenience choose $\Omega = (0, 2\pi)$. Then every function in $H^k_0(\Omega)$ can be written as

$$u(x) = \sum_{j=1}^{\infty} a_j \sin(jx) = a_1 \sin(x) + a_2 \sin(2x) + \dots$$

Prove the following statements:

1. If the coefficients a_j of a function $u(x)$ satisfy the condition $|a_j| = \mathcal{O}\left(\frac{1}{j^{1/2+\varepsilon}}\right)$ for some $\varepsilon > 0$ – that is, if there is $C < \infty$ so that $\lim_{j\to\infty} \frac{|a_j|}{1/j^{1/2+\varepsilon}} = C$ – then $u \in L^2(\Omega)$.
2. If the coefficients a_j of a function $u(x)$ satisfy the condition $|a_j| = \mathcal{O}\left(\frac{1}{j^{3/2+\varepsilon}}\right)$ for some $\varepsilon > 0$, then $u \in H^1(\Omega)$.
3. If the coefficients a_j of a function $u(x)$ satisfy the condition $|a_j| = \mathcal{O}\left(\frac{1}{j^{k+1/2+\varepsilon}}\right)$ for some $\varepsilon > 0$ and some $k \geq 0$, then $u \in H^k(\Omega)$.

This last condition in essence says that if the Fourier coefficients of a function decay rapidly enough, then the function is smooth – something that makes sense if one thinks of how one can see functions as multiples of sines and cosines with successively higher frequencies piled on top of each other.

For our purposes, this last condition also allows us to define membership in spaces H^k where k is not an integer. This is of course what we did in Problem 1. **(20 points)**

Bonus problem (The space H^{-1} and its Fourier basis). The space of functions $H^{-1}(\Omega)$ is the set of all functions $u(x)$ so that

$$\int_\Omega u(x)v(x)\,\mathrm{d}x$$

is finite for all possible $v \in H^1$. Take again $\Omega = (0, 2\pi)$ and show the following generalization of the results of the previous problem to the case $k = -1$. In other words, show the following statement:

1. If the coefficients a_j of a function $u(x)$ satisfy the condition $|a_j| = \mathcal{O}\left(\frac{1}{j^{-1+1/2\varepsilon}}\right)$ for some $\varepsilon > 0$, then $u \in H^{-1}(\Omega)$.

To get an understanding of how such functions look like, generate sequences of coefficients a_j (for example, chosen randomly in some way) that satisfy the conditions for H^1, L^2, H^{-1} and plot these functions. Do they look qualitatively different? **(20 points)**