MATH 546: Partial Differential Equations II

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Lectures:	Engineering E 206, Mondays/Wednesdays/Fridays, 11-11:50am
Office hours:	Wednesdays, 1-2pm; or by appointment.

Homework assignment 3 – due Friday 4/5/2019

Problem 1 (Weak solutions and strong solutions). In class, we have discussed that if you have a weak solution of the Laplace equation, then it is also a strong solution assuming that it is smooth enough to allow for certain operations. In other words, assume that you have a function $u \in H^1_g$ so that

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv + \int_{\Gamma_N} hv \qquad \forall v \in H^1_0,$$

and that serendipitously this function is in fact also in $C^2(\Omega)$. Then you can integrate by parts to see that for this function we have that

$$\int_\Omega (-\Delta u - f)v + \int_{\Gamma_N} \left(\frac{\partial u}{\partial n} - h\right) v \qquad \forall v \in H^1_0.$$

We argued in class that this implies that $-\Delta u - f = 0$ in a pointwise sense assuming that $f \in C^0(\Omega)$.

Complete the argument to show that also $\frac{\partial u}{\partial n} - h = 0$ pointwise assuming that $h \in C^0(\Gamma_N)$. The argument is not difficult, so pay attention to justifying why each step is correct and why each operation is in fact allowed for all of the functions you are considering in your argument. **(20 bonus points)**

Problem 2 (The biharmonic equation). The biharmonic equation is a model for the deformation of thin two-dimensional structures such as stadium roofs. In its simplest form it looks like this:

$$\begin{aligned} \Delta\Delta u &= f && \text{in } \Omega, \\ u &= g && \text{on } \partial\Omega, \\ \frac{\partial u}{\partial n} &= h && \text{on } \partial\Omega. \end{aligned}$$

(This is a fourth order differential equation, so it needs twice as many boundary conditions as the Laplace equation and so we are prescribing here both Dirichlet and Neumann conditions for the solution.)

For many of the same reasons as for the Laplace equation, it is not possible to always find strong solutions to this equation. Retrace the steps we discussed when we derived the weak formulation of the Laplace equation to obtain a weak formulation of the biharmonic equation.

Then also go through the steps of Question 1 to identify what the appropriate function spaces for the (weak) solution u and the test functions v should be. Recall how we argued what the boundary values for the test function needed to be for the Laplace equation, and think about what that would mean in the current context; specifically, you will want to incorporate *both* boundary conditions above into the spaces for u and v. **(40 points)**

Problem 3 (Singular solutions of the biharmonic equation). Recall how we showed that solutions of the Laplace equation may not always be in C^2 by considering the case of a domain with a reentrant corner (namely, a sector with an opening angle greater than π).

Do the same argument for solutions of the biharmonic equation with $f = 0$. Ideally, one would of course like to have strong solutions $u \in C^4$. Show that in a circular sector with a reentrant corner, this is not always possible by explicitly constructing a solution of the equation. For the Laplace equation, we choose $g = 0$ along the two lines adjacent to the troublesome corner. This should also work here, and you should think about whether you can also choose $h = 0$. If you can't, make sure that g and h are at the very least continuous functions. **(20 points)**

Problem 4 (Singular solutions of the Stokes equations). For both the Laplace equation and the biharmonic equation (in the previous problem), we have constructed solutions that have a singularity (i.e., for which $u \notin C^2$ or $u \notin C^4$, respectively), but at least the solution remained bounded – it was just that some derivative "blows up".

But not even this has to always be true. It's just a matter of how many derivatives you have on a variable once you're in the weak formulation. Take, for example, the Stokes equations

$$-\Delta \mathbf{u}(\mathbf{x}) + \nabla p(\mathbf{x}) = \mathbf{f}(\mathbf{x}),$$
$$-\nabla \cdot \mathbf{u}(\mathbf{x}) = 0.$$

For this equation, you derived a weak formulation in the previous homework, which will have looked something like this:

$$\int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} - \int_\Omega p(\nabla \cdot \mathbf{v}) - \int_\Omega (\nabla \cdot \mathbf{u}) q = \int_\Omega \mathbf{v} \cdot \mathbf{v},$$

for all test functions $\mathbf{v} \in H_0^1(\Omega)^3, q \in L^2(\Omega)$, where I have purposefully neglected all boundary terms. (Forgetting about boundary terms is correct if one enforces Dirichlet boundary conditions on $\mathbf{u}$, though one then also needs $\mathbf{v} \in H_0^1(\Omega)^3$; one also gets complications with non-uniqueness of the pressure component of the solution – all reasons why we don't want to deal too much with details of boundary terms here.)

The point to observe here is that neither derivatives of the pressure p nor of the test function q appear in the weak formulation. So one may suspect that these functions only need to be in L^2 and that, consequently, they can have worse singularities than the solutions of the Laplace and biharmonic equations, for which first and second weak derivatives appear in the weak formulation.

Here is a situation you may want to consider – called the "lid-driven" or simply the "driven cavity" (because as you will see in a second, a lid "drives" the motion of the fluid): Take a box (or in 2d a rectangle) filled with a viscous fluid – the "cavity". The sides of the box are fixed, but the top of the box is a plate larger than the box that is sliding across the box at a constant velocity. Since it is in contact with the fluid, the viscous friction between the top and the fluid drags the fluid along.

Try to concisely describe in formulas the boundary conditions you have for this physical situation. Now consider the 2d case and specifically the places where the sliding top is in contact with the fixed sides – i.e., the top left and top right corners of the box. At these locations, the fluid dragged along by the sliding top encounters a fixed wall, and so has to change its direction of flow.

Without having to explicitly construct a solution as a closed-form expression, can you speculate what the pressure needs to do in (the vicinity of) these corners for all boundary conditions to be satisfied exactly? Using your physical intuition of what a pressure really is (namely, a kind of *force density*), do you think that the pressure should remain bounded from above and/or below in each of these corners, or do you think that it might go to plus or minus infinity? **(20 points)**