MATH 546
Partial Differential Equations II

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Lectures:	Engineering E 206, Mondays/Wednesdays/Fridays, 10-10:50am
Office hours:	Wednesdays, 1-2pm; or by appointment.

Course outline

The syllabus description for this course is pretty short:

> Distribution theory, Green's functions, Sobolev spaces, elliptic and parabolic equations.

The description of the course in the qualifying exam handbook of our department is marginally more detailed, but not by very much:

> The core set of topics for MATH 546 are:
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> 1. Test functions.
> 2. Distributions.
> 3. Fourier transforms.
> 4. Fundamental solutions and Green's functions.
> 5. Weak derivatives, the Sobolev space $W^{k,p}$.
> 6. Molliers, partitions of unity.
> 7. The extension theorem, the trace theorem.
> 8. Sobolev embedding theorems.
> 9. Second order elliptic equations, weak solutions, elliptic regularity.
> 10. Parabolic equations, weak solutions, energy estimates, maximum principles.

That's sort of a grab-bag of what PDE theory entails, so I'll just give you an overview of what *I* think is important in this course and how this all fits in. What I care about are essentially the following topics:

- *PDE "literacy":* I care far more about you learning to speak intelligently about PDEs that you encounter, than whether or not you recall the details of a particular proof or the definition of some complicated concept. In MATH 545, you have already seen examples of the three big classes of PDEs; namely the Laplace/Poisson equation as a prototype of elliptic equations, the heat equation as the prototype for parabolic equations, the wave equation as that for second-order hyperbolic equations, and the advection equation for first-order hyperbolic equations. We have also gone through the modeling aspects and how we arrived at these equations from basic physical insight.

  I will go into this in some more detail, including modeling other situations and seeing what kinds of equations we then arrive at. In some sense, PDEs are almost always composed of certain building blocks, and if you can recognize these buildings blocks you already have an idea of how their solutions looks like, what proof techniques will work, etc. It's really just like a language: if you can recognize the words, you have an idea of what the sentence means.

  Consequently, this is what I would like to talk about:

  – Some more modeling of physical effects and the kinds of equations one then arrives at.

  - Classification of PDEs into elliptic/parabolic/hyperbolic equations and why this is an insufficient categorization in practice.
  - Classification into linear/quasilinear/nonlinear equations.
  - Classification into scalar equations and systems of equations.

- *Existence of solutions:* In MATH 545, we have simply constructed explicit solutions for a variety of equations, typically through series expansions based either on Fourier series or the method of separation of variables. This turns out to work for simple problems and simple geometries, but it completely ignores the question of whether solutions *always* exist in the first place. Indeed, this is not a trivial question and took some 50 years of mathematical development in the late 1800s and early 1900s to answer with the necessary precision.

  To do so, one needs to first understand what a derivative operator actually and really means in the context of PDEs. For example, one might think that in a second-order PDE, the solution will have to be twice continuously differentiable – but this turns out to be not correct, and to precisely define what exactly is meant will require a decent amount of work. The concepts we will have to explore to this end include the following:

  - Variational formulation of PDEs, weak solutions, test functions.
  - Weak derivatives, the Sobolev spaces $W^{k,p}$.
  - Sobolev embedding theorems.
  - The extension theorem, the trace theorem.

  In other words, trying to prove or disprove existence of solutions of common PDEs will require us to take a decent detour through many concepts of modern functional analysis and a bit of operator theory.

  Once we have the right words, asking and answering questions about the existence of solutions for elliptic and parabolic equations will turn out to not be all too difficult.

- *Regularity of solutions:* Coming back to the question of how many derivatives the solution of an equation actually has is a separate question that is typically called "regularity" – how "smooth" is a solution. There are some techniques we will investigate that help us answer this question, first at points away from the boundary of the domain on which the PDE is posed, and then also as our points of interest approach the boundary.

- *Explicit construction of solutions via Green's formula, Fourier transforms:* Having established that a solution exists, and how smooth it is, the next and most practical question is of course how one can find the solution in practice. We have seen how this works in relatively simple cases in MATH 545, but have not discussed the case of nonzero right hand sides of the equation (though we have considered the case of nonzero initial and boundary values). There are two techniques for this: Fourier transform and Green's functions ("fundamental solutions"). In reality, they are not more widely applicable in practice than the techniques you have already seen, but they allow us to explore some interesting mathematics like distribution theory.

- *First-order hyperbolic equations:* Hyperbolic equations in general are fundamentally different from elliptic or parabolic equations because no smoothing is happening, just transport. As a consequence, their solutions are typically not continuous and in particular not differentiable. If one doesn't carefully define what we consider a "solution", one also easily ends up in situations where there are infinitely many solutions to the equation.

  This is generally addressed through concepts such as viscosity (or entropy) solutions. Time permitting, I will try to explore these concepts as well.

Learning objectives

At the end of the semester, this is what you should have learned:

- To have an idea where PDEs come from, and how to translate a word description of a physical phenomenon into a mathematical equation;
- To categorize a given equation;
- To know qualitative properties of the solutions of PDEs of different categories;
- To understand how one defines solutions of PDEs, what their theoretical properties they have, and what conceptual difficulties the different kinds of equations present.
- To understand and use modern concepts of functional analysis in the context of PDEs.

Prerequisites

MATH 545 (Partial Differential Equations I).

Literature

I will be losely inspired by the book by Lawrence C. Evans: "Partial Differential Equations" (second edition, American Mathematical Society), but there are many other books about PDEs that cover similar content and that you could consider interchangeably. In any case, I will make the course self-contained and not reference particular parts or exercises in this book or any other: you can just use these books as backup material if you did not understand something or need to read up on material you may have missed.
In other words: You are not required to buy or use this book.

Webpage

Homework assignments and other course information will be posted at the course webpage
http://www.math.colostate.edu/~bangerth/teaching.html

Exams and grading

Final grades will be determined based on the following components:

- Biweekly homework and programming assignments: 50%
- Midterm exam, at a date in February/March/April still to be determined: 20%
- A final project, with presentations in the last regular week of the semester: 30%

Your minimum grade will be A, B, C, or D, for a score of 90%, 80%, 70%, and 60% over the course of the semester, respectively.

You must make arrangements in advance if you expect to miss an exam or quiz. Exam absences due to recognized University-related activities, religious holidays, verifiable illness, and family/medical emergencies will be dealt with on an individual basis. In all cases of absence from exams a written excuse is required. Ignorance of the time and place of an exam will not be accepted as an excuse for absence.

Incompletes: I will consider giving an incomplete if you have successfully completed all but a small portion of the work of the course, and are prevented from completing the course by a severe, unexpected event. Simply being behind work is not a reason for an Incomplete, though; in that case you should consider dropping the course.

S/U grades: If you are registered S/U your grade will be 'S' if your letter grade is C or above, and 'U' otherwise.

Policies *Academic integrity:* Academic integrity is integral to the success of the University and to you as a learner. Academic integrity is conceptualized as doing and taking credit for one's own work. Academic dishonesty undermines the educational experience at Colorado State University. Examples of academic dishonesty include (but are not limited to) cheating, plagiarism, and falsification. Plagiarism includes the copying of language, structure, images, ideas or thoughts of others and is related only to work submitted for credit. Cheating or any form of academic dishonesty will not be tolerated. The use of material from improperly cited or credited sources will be considered plagiarism. You are encouraged to collaborate with your classmates, unless otherwise directed, but all work intended for a grade must clearly be your work as an individual. Ignorance of the rules does not exclude any member of the CSU community from the requirements or the processes meant to ensure academic integrity.

Disabilities: Colorado State University, in compliance with state and federal laws and regulations, does not discriminate on the basis of disability in administration of its education related programs and activities. We have an institutional commitment to provide equal educational opportunities for disabled students who are otherwise qualified. Students with documented disabilities must contact the Student Disability Center (TILT Building, room 121; 970-491-6385) to make arrangements for class accommodations. It is the responsibility of the student to obtain accommodation letters from RDS and to make arrangements for the implementation of accommodations with faculty in advance. Students who believe they have been denied access to services or accommodations required by law should contact RDS (970-491-6385). Students who believe they have been subjected to discrimination on the basis of disability should contact the Office of Equal Opportunity (970-491-5836). For more information regarding disability grievance procedures, visit http://oeo.colostate.edu.