Part 34

Further Applications IV:

Stochastic and Robust Optimization

Motivation

Problem: In many optimization applications, we have material/system parameters that are not known exactly, or external forces that are unpredictable. We want to take this into account when optimizing.

Example 1: We want to design an air plane that is as efficient as possible, but we only know that the viscosity of air at 10km altitude is

$$1.1 \cdot 10^{-4} \frac{\text{ft}^2}{\text{s}} \leq \nu \leq 1.2 \cdot 10^{-4} \frac{\text{ft}^2}{\text{s}}$$

depending on the prevailing temperature.

Example 2: We want to compute the trajectory for a rocket that takes the least amount of fuel. But this trajectory will depend on current wind conditions which we don't know exactly.

Example: We want to optimally produce an oil field but we have only incomplete knowledge of the physical structure of the oil reservoir.

Some preliminaries

Definition: Let *p* be an uncertain parameter (such as the viscosity, the wind field, the oil reservoir) and let

$$P(p)$$

be a probability density for this parameter.

Let *F(p)* be a function of *p*. Then we call

$$E[F] = \int F(p) P(p) \, dp$$

the expectation value of *F* and

$$\sigma[F] = \sqrt{\int (F(p) - E[F])^2 P(p) \, dp}$$

the standard deviation of *F* under *P*.

Some preliminaries

Example 1 (Viscosity): If we know that

$$1.1 \cdot 10^{-4} \frac{\text{ft}^2}{\text{s}} \leq \nu \leq 1.2 \cdot 10^{-4} \frac{\text{ft}^2}{\text{s}}$$

then a reasonable choice would be

$$P(\nu) = \begin{cases} 10^5 \dfrac{\text{s}}{\text{ft}^2} & \text{if } 1.1 \cdot 10^{-4} \dfrac{\text{ft}^2}{\text{s}} \leq \nu \leq 1.2 \cdot 10^{-4} \dfrac{\text{ft}^2}{\text{s}} \\ 0 & \text{otherwise} \end{cases}$$

Example 2 (Rocket in a wind field): Close to the surface, a reasonable model could be

$$P(\vec{v}) = \frac{1}{C} e^{-\frac{\|\vec{v}\|^2}{(10\,mph)^2}}$$

Some preliminaries

Practical approach: In practice, computing integrals like

$$E[F] = \int F(p)P(p)\,dp$$

is difficult if the number of variables in *p* is large.

In that case, we can choose a uniformly distributed sample $\{p_i\}$ and approximate

$$E[F] \approx \frac{1}{N}\sum_{i=1}^{N} F(p_i)P(p_i)$$

Alternatively, we can choose samples $\{p_i\}$ based on the probability distribution $P(p)$ and approximate

$$E[F] \approx \frac{1}{N}\sum_{i=1}^{N} F(p_i)$$

Stochastic optimization

The state equation: Let us assume that state variables *y*, control variables *q* and system parameters *p* are related by

$$f(y,q;p)=0$$

and that we have additional constraints of the form

$$g(y,q)\geq 0$$

Deterministic optimization: If we knew that the parameters *p* then the optimization problem would have the form

$$\begin{aligned} \min_{y,q} \quad & F(y,q) \\ \text{subject to} \quad & f(y,q;p)=0 \\ & g(y,q)\geq 0 \end{aligned}$$

Since we have assumed that we know *p* this problem can be deterministically solved to find an optimal control *q*.

Stochastic optimization

Stochastic optimization: In reality, me may not know p exactly but only a probability distribution function $P(p)$. In this case, we can pose several versions of a stochastic problem.

Version 1: Average control – optimize the *average* cost over all possible values p by finding a single control q so that:

$$\begin{aligned} \min_{y_p, q} \quad & E[F(y_p, q)] \\ \text{subject to} \quad & f(y_p, q; p) = 0 \\ & g(y_p, q) \geq 0 \end{aligned}$$

Practical implementation: Draw samples $\{p_i\}$ from $P(p)$ and solve

$$\begin{aligned} \min_{y_i, q} \quad & \frac{1}{N} \sum_{i=1}^{N} F(y_i, q_i) \\ \text{subject to} \quad & f(y_i, q; p_i) = 0 \qquad i = 1, \ldots, N \\ & g(y_i, q) \geq 0 \qquad i = 1, \ldots, N \end{aligned}$$

Stochastic optimization

Stochastic optimization: In reality, me may not know *p* exactly but only a probability distribution function *P(p)*. In this case, we can pose several versions of a stochastic problem.

Version 2a: *Risk averse* control – optimize *average cost plus some safety factor* over all possible values *p*:

$$\begin{aligned} \min_{y_p,q} \quad & E\left[F(y_p,q)\right] + \alpha\,\sigma\left[F(y_p,q)\right] \\ \text{subject to} \quad & f(y_p,q;p)=0 \\ & g(y_p,q)\geq 0 \end{aligned}$$

Practical implementation: Draw samples *{p_i}* from *P(p)* and solve

$$\begin{aligned} \min_{y_i,q} \quad & \frac{1}{N}\sum_{i=1}^{N} F(y_i,q_i) + \frac{\alpha}{\sqrt{N}}\left\{\sum_{i=1}^{N}\left[F(y_i,q_i) - \frac{1}{N}\sum_{j=1}^{N} F(y_j,q_j)\right]^2\right\}^{\frac{1}{2}} \\ \text{subject to} \quad & f(y_i,q_i;p_i)=0 \qquad i=1,\ldots,N \\ & g(y_i,q_i)\geq 0 \qquad i=1,\ldots,N \end{aligned}$$

Stochastic optimization

Stochastic optimization: In reality, me may not know *p* exactly but only a probability distribution function *P(p)*. In this case, we can pose several versions of a stochastic problem.

Version 2b: *Risky* control – optimize average cost *minus* some safety factor over all possible values *p*:

$$\begin{aligned} \min_{y_p,q} \quad & E\left[F(y_p,q)\right]-\alpha\,\sigma\left[F(y_p,q)\right] \\ \text{subject to} \quad & f(y_p,q;p)=0 \\ & g(y_p,q)\geq 0 \end{aligned}$$

Practical implementation: Draw samples *{p_i}* from *P(p)* and solve

$$\begin{aligned} \min_{y_i,q} \quad & \frac{1}{N}\sum_{i=1}^{N} F(y_i,q_i)-\frac{\alpha}{\sqrt{N}}\left\{\sum_{i=1}^{N}\left[F(y_i,q_i)-\frac{1}{N}\sum_{j=1}^{N}F(y_j,q_j)\right]^2\right\}^{\frac{1}{2}} \\ \text{subject to} \quad & f(y_i,q_i;p_i)=0 \qquad i=1,\ldots,N \\ & g(y_i,q_i)\geq 0 \qquad i=1,\ldots,N \end{aligned}$$

Stochastic optimization

Stochastic optimization: In reality, me may not know *p* exactly but only a probability distribution function *P(p)*. In this case, we can pose several versions of a stochastic problem.

Version 3: *Robust* control – optimize the *worst case* cost over all possible values *p*:

$$\begin{aligned} &\min_{y_p, q} && \max_{p, P(p)>0} F(y_p, q) \\ &\text{subject to} && f(y_p, q; p) = 0 \\ & && g(y_p, q) \geq 0 \end{aligned}$$

Practical implementation: Draw samples $\{p_i\}$ from *P(p)* and solve

$$\begin{aligned} &\min_{y_i, q} && \max_{1 \leq i \leq N} F(y_i, q_i) && \\ &\text{subject to} && f(y_i, q_i; p_i) = 0 && i = 1, \ldots, N \\ & && g(y_i, q_i) \geq 0 && i = 1, \ldots, N \end{aligned}$$

Practical aspects

Which formulation to choose for stochastic optimization:

- If no recourse is possible if the "real" parameters happen to be unfavorable, then optimization must be *robust*.

  - Airfoils must be designed for the least favorable viscosity
  - Available rocket fuel must be designed for the worst possible winds

- If losses are harder to tolerate than wins, then be *risk averse*:

  - Investment strategies for retirement funds

- If we can mitigate unfavorable parameters, then we can choose the *risky* strategy ("gambling"):

  - If we are Warren Buffett
  - Production strategies for oil fields where in the worst case another hole can be drilled

Practical aspects

Stochastic optimization is typically very expensive:

- Integrals can rarely be computed analytically
- Sample sets $\{p_i\}$ must be large enough to provide a good approximation of the integrals
- If the deterministic problem has M constraints of the form

$$f(y,q;p)=0$$
$$g(y,q)\geq 0$$

then the stochastic implementation has MN constraints:

$$f(y_i,q_i;p_i)=0 \qquad i=1,\ldots,N$$
$$g(y_i,q_i)\geq 0 \qquad i=1,\ldots,N$$

- Current research topics therefore are:
  - efficient sample generation
  - model reduction techniques
  - parametric descriptions of constraints (e.g. polynomial chaos)