Part 34

Further Applications IV:

Stochastic and Robust Optimization
Motivation

Problem: In many optimization applications, we have material/system parameters that are not known exactly, or external forces that are unpredictable. We want to take this into account when optimizing.

Example 1: We want to design an air plane that is as efficient as possible, but we only know that the viscosity of air at 10km altitude is

\[ 1.1 \cdot 10^{-4} \frac{\text{ft}^2}{\text{s}} \leq \nu \leq 1.2 \cdot 10^{-4} \frac{\text{ft}^2}{\text{s}} \]

depending on the prevailing temperature.

Example 2: We want to compute the trajectory for a rocket that takes the least amount of fuel. But this trajectory will depend on current wind conditions which we don't know exactly.

Example: We want to optimally produce an oil field but we have only incomplete knowledge of the physical structure of the oil reservoir.
**Some preliminaries**

**Definition:** Let $p$ be an uncertain parameter (such as the viscosity, the wind field, the oil reservoir) and let

$$P(p)$$

be a probability density for this parameter.

Let $F(p)$ be a function of $p$. Then we call

$$E[F] = \int F(p)P(p) \, dp$$

the expectation value of $F$ and

$$\sigma[F] = \sqrt{\int (F(p) - E[F])^2 P(p) \, dp}$$

the standard deviation of $F$ under $P$. 
Some preliminaries

Example 1 (Viscosity): If we know that

\[ 1.1 \cdot 10^{-4} \text{ft}^2 \, \text{s} \leq \nu \leq 1.2 \cdot 10^{-4} \text{ft}^2 \, \text{s} \]

then a reasonable choice would be

\[
P(\nu) = \begin{cases} 
10^5 \frac{\text{S}}{\text{ft}^2} & \text{if } 1.1 \cdot 10^{-4} \frac{\text{ft}^2}{\text{s}} \leq \nu \leq 1.2 \cdot 10^{-4} \frac{\text{ft}^2}{\text{s}} \\
0 & \text{otherwise}
\end{cases}
\]

Example 2 (Rocket in a wind field): Close to the surface, a reasonable model could be

\[
P(\vec{v}) = \frac{1}{C} e^{-\frac{||\vec{v}||^2}{(10 \text{mph})^2}}
\]
Some preliminaries

Practical approach: In practice, computing integrals like

\[ E[F] = \int F(p)P(p) \, dp \]

is difficult if the number of variables in \( p \) is large.

In that case, we can choose a uniformly distributed sample \( \{p_i\} \) and approximate

\[ E[F] \approx \frac{1}{N} \sum_{i=1}^{N} F(p_i)P(p_i) \]

Alternatively, we can choose samples \( \{p_i\} \) based on the probability distribution \( P(p) \) and approximate

\[ E[F] \approx \frac{1}{N} \sum_{i=1}^{N} F(p_i) \]
Stochastic optimization

The state equation: Let us assume that state variables $y$, control variables $q$ and system parameters $p$ are related by

$$f(y, q; p) = 0$$

and that we have additional constraints of the form

$$g(y, q) \geq 0$$

Deterministic optimization: If we knew that the parameters $p$ then the optimization problem would have the form

$$\min_{y, q} F(y, q)$$

subject to

$$f(y, q; p) = 0$$

$$g(y, q) \geq 0$$

Since we have assumed that we know $p$ this problem can be deterministically solved to find an optimal control $q$. 


**Stochastic optimization**

**Stochastic optimization**: In reality, we may not know $p$ exactly but only a probability distribution function $P(p)$. In this case, we can pose several versions of a stochastic problem.

**Version 1**: Average control – optimize the average cost over all possible values $p$ by finding a single control $q$ so that:

$$\min_{y_p, q} \quad E[F(y_p, q)]$$

subject to

$$f(y_p, q; p) = 0$$

$$g(y_p, q) \geq 0$$

**Practical implementation**: Draw samples $\{p_i\}$ from $P(p)$ and solve

$$\min_{y_i, q} \quad \frac{1}{N} \sum_{i=1}^{N} F(y_i, q_i)$$

subject to

$$f(y_i, q; p_i) = 0 \quad i=1,\ldots, N$$

$$g(y_i, q) \geq 0 \quad i=1,\ldots, N$$
Stochastic optimization

Stochastic optimization: In reality, me may not know $p$ exactly but only a probability distribution function $P(p)$. In this case, we can pose several versions of a stochastic problem.

Version 2a: Risk averse control – optimize average cost plus some safety factor over all possible values $p$:

$$\min_{y, q} \quad E[F(y, q)] + \alpha \sigma[F(y, q)]$$
subject to $f(y, q; p) = 0$
$g(y, q) \geq 0$

Practical implementation: Draw samples $\{p_i\}$ from $P(p)$ and solve

$$\min_{y_i, q} \quad \frac{1}{N} \sum_{i=1}^{N} F(y_i, q_i) + \frac{\alpha}{\sqrt{N}} \left\{ \sum_{i=1}^{N} \left[ F(y_i, q_i) - \frac{1}{N} \sum_{j=1}^{N} F(y_j, q_j) \right] \right\}^2 \frac{1}{2}$$
subject to $f(y_i, q_i; p_i) = 0 \quad i = 1, \ldots, N$
$g(y_i, q_i) \geq 0 \quad i = 1, \ldots, N$
Stochastic optimization

Stochastic optimization: In reality, we may not know $p$ exactly but only a probability distribution function $P(p)$. In this case, we can pose several versions of a stochastic problem.

Version 2b: Risky control – optimize average cost minus some safety factor over all possible values $p$:

$$\begin{align*}
&\min_{y, p, q} \quad E[F(y, q)] - \alpha \sigma[F(y, q)] \\
&\text{subject to} \quad f(y, p, q) = 0 \\
&\quad g(y, q) \geq 0
\end{align*}$$

Practical implementation: Draw samples $\{p_i\}$ from $P(p)$ and solve

$$\begin{align*}
&\min_{y, q} \quad \frac{1}{N} \sum_{i=1}^{N} F(y, q_i) - \frac{\alpha}{\sqrt{N}} \left[ \sum_{i=1}^{N} \left[ F(y, q_i) - \frac{1}{N} \sum_{j=1}^{N} F(y, q_j) \right] \right]^{1/2} \\
&\text{subject to} \quad f(y, q_i; p_i) = 0 \quad i = 1, \ldots, N \\
&\quad g(y, q_i) \geq 0 \quad i = 1, \ldots, N
\end{align*}$$
Stochastic optimization

Stochastic optimization: In reality, we may not know $p$ exactly but only a probability distribution function $P(p)$. In this case, we can pose several versions of a stochastic problem.

Version 3: Robust control – optimize the worst case cost over all possible values $p$:

$$
\min_{y_p, q} \quad \max_{p, P(p) > 0} F(y_p, q)
$$
subject to $f(y_p, q; p) = 0$
$$
g(y_p, q) \geq 0
$$

Practical implementation: Draw samples $\{p_i\}$ from $P(p)$ and solve

$$
\min_{y_i, q} \quad \max_{1 \leq i \leq N} F(y_i, q_i)
$$
subject to $f(y_i, q_i; p_i) = 0 \quad i = 1, \ldots, N$
$$
g(y_i, q_i) \geq 0 \quad i = 1, \ldots, N$$
Practical aspects

Which formulation to choose for stochastic optimization:

• If no recourse is possible if the “real” parameters happen to be unfavorable, then optimization must be robust.

  - Airfoils must be designed for the least favorable viscosity
  - Available rocket fuel must be designed for the worst possible winds

• If losses are harder to tolerate than wins, then be risk averse:

  - Investment strategies for retirement funds

• If we can mitigate unfavorable parameters, then we can choose the risky strategy (“gambling”):

  - If we are Warren Buffett
  - Production strategies for oil fields where in the worst case another hole can be drilled
Practical aspects

Stochastic optimization is typically very expensive:

- Integrals can rarely be computed analytically
- Sample sets \( \{p_i\} \) must be large enough to provide a good approximation of the integrals
- If the deterministic problem has \( M \) constraints of the form
  \[
  f(y, q; p) = 0 \\
  g(y, q) \geq 0
  \]
  then the stochastic implementation has \( MN \) constraints:
  \[
  f(y_i, q_i; p_i) = 0 \quad i = 1, \ldots, N \\
  g(y_i, q_i) \geq 0 \quad i = 1, \ldots, N
  \]

- Current research topics therefore are:
  - efficient sample generation
  - model reduction techniques
  - parametric descriptions of constraints (e.g. polynomial chaos)