DSCI 320: Optimization Methods in Data Science

Homework assignment 3 – due Friday 10/18/2019

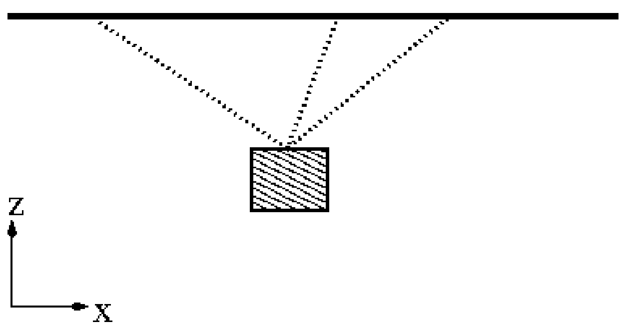
Problem 1 (Line search). In the previous homework assignment, you had considered the function

$$f(x) = x \arctan x - \frac{1}{2}\ln(1+x^2)$$

and showed Newton's method only converges with step length $\alpha_k = 1$ if one starts close enough to the solution.

Implement the backtracking line search algorithm to find a step length α_k that guarantees that the step $x_{k+1} = x_k + \alpha_k p_k$ is reasonable. Demonstrate that the method now converges from any starting point, e.g., $x_0 = 10^6$ or $x_0 = -10^6$ (i.e., far outside the convergence radius you determined previously). Do so by showing tables that list x_k and α_k for each iteration k. What convergence rate do you observe for the last few iterations? We have always argued that quadratic convergence only holds if $\alpha_k = 1$, and that that is going to be the case once we're close enough to the solution (because in that case, the step to the minimum of the quadratic approximation satisfies the Wolfe conditions) – is this indeed the case here? **(30 points)**

Problem 2 (A modeling exercise). Consider the following system of three springs suspended from the ceiling at positions $(x, z) = (-20\text{cm}, 0\text{cm})$, $(5\text{cm}, 0\text{cm})$ and $(15\text{cm}, 0\text{cm})$ and that hold in place a body:



Each spring has a rest length of $L_0 = 20\text{cm}$, and extending (or compressing) spring $i, i = 1 \dots 3$ to a length L_i requires an energy of $E_{\text{spring},i} = \frac{1}{2}D(L_i - L_0)^2$ where the spring constant for all three springs equals $D = 300\frac{N}{m}$. On the other hand, the potential energy of the body is $E_{\text{pot}} = mgz$ where the body's mass is $m = 500g$, the gravity constant is $g = 9.81\frac{\text{m}}{\text{s}^2}$, and z is the vertical coordinate of the body's position.

Express the total energy in the system (spring energies plus potential energy) as a function of the body's position (x, z). Make sure to convert all the quantities above into common units kg, m, and s. Nature likes to do things so that the energy is minimal, so implement a numerical method to find the location at which this energy is minimal. In your answer, show this location as well as the value of the total energy function at this location. Is there only a single energy minimum? **(30 points)**

Problem 3 (To converge or not to converge). Consider the function $f(x) = x_1^4 - x_1^2 + x_2^2$. It's minima lie at $x^* = (\pm\frac{1}{2}\sqrt{2}, 0)^T$. Explain in words, graphs, or numbers what is going to happen if we started Newton's method at the point $x_0 = (0, 2)^T$ and did all computations exactly (i.e. not in floating point arithmetic), and explain why this leads to a somewhat unsatisfactory result.

(10 points)

Problem 4 (BFGS for quadratic functions). Consider $f : \mathbb{R}^4 \to \mathbb{R}$ given by $f(x) = \frac{1}{2}x^T Ax$ with

$$A = \begin{pmatrix} 9 & 1 & 1 & 1 \\ 1 & 9 & 1 & 1 \\ 1 & 1 & 9 & 1 \\ 1 & 1 & 1 & 9 \end{pmatrix}$$

Its minimum lies at $x = (0, 0, 0, 0)$.

Implement the BFGS algorithm for this problem, using full step length $\alpha_k = 1$ in each step, starting at $x_0 = (1, 2, 3, 4)^T$ and using $B_0 = I$. How many steps do you need to converge to the minimum? How many steps would you need if you used the exact Newton matrix instead of the BFGS approximation?

(30 points)

If you have comments on the way I teach – in particular suggestions how I can do things better, if I should do more or less examples, powerpoint slides vs whiteboard, etc – or on other things you would like to critique, feel free to hand those in with your homework as well. I want to make this as good a class as possible, and all comments are certainly much appreciated!