DSCI 320: Optimization Methods in Data Science

Homework assignment 1 – due Tuesday 9/20/2019

Problem 1 (Optimization problems in your field). Optimization problems are usually posed in the following way: let x be a vector of variables that describe the quantities that are subject to optimization (i.e. the *design variables* u introduced in the first class) and auxiliary variables (i.e. the *state* variables y); then the problem is to find that vector x for which

$$\begin{aligned} f(x) &\to \min!, \\ g(x) &= 0, \\ h(x) &\geq 0, \end{aligned}$$

with an objective function $f(x)$, a function $g(x)$ that describes equalities that need to hold at the solution, and $h(x)$ inequalities. Both g and h can be vector-valued, and in this case the (in)equalities have to hold for each element $g_1(x), g_2(x), \ldots, h_1(x), h_2(x), \ldots$.

Pick an optimization problem in an area you find interesting. Describe as best as you can:

- What are the variables that make up x?
- What are the functions f, g, h (i.e. what do they mean) and, if possible, their form as a formula?
- What can you say about the classification of the problem, i.e. is it convex/nonconvex, smooth/nonsmooth, etc, according to the criteria discussed in class?

(30 points)

Problem 2 (Fitting data 1). Assume you are given the following time series:

t_i	0	1	2	3
y_i	1.1	1.9	2.8	3.2

Consider the problem of fitting a line $y(t) = at + b$ through this data set. One way to do so is to ask for that set of parameters $x = \{a, b\}$ for which the sum of squares deviation $f(x) = \sum_{i=1}^{4} |y_i - y(t_i)|^2$ is minimal. Note that the right hand side depends on x through the equation for $y(t)$.

Plot this function $f(x)$ for the values of t_i, y_i above. Describe whether this function $f(x)$ is linear/nonlinear, convex/nonconvex, smooth/nonsmooth, whether derivatives can be computed or not, and whether the design variables a, b are discrete or continuous.

From the plot of $f(x)$ obtain (using your eyes, no minimum finder) a reasonable guess for those values a, b for which $f(x)$ is minimal, and plot the resulting line $y(t) = at + b$ along with the data points above. **(20 points)**

Problem 3 (Fitting data 2). Repeat all parts of the previous problem but replace the objective function by the one that tries to minimize the sum of absolute values $f(x) = \sum_{i=1}^{4} |y_i - y(t_i)|$ instead of squares. Comment in particular on the smoothness of $f(x)$. **(10 points)**

Problem 4 (Fitting data 3). Repeat the previous problem a final time, but replace the objective function by the one that tries to minimize the *maximal* deviation, $f(x) = \max_{1 \leq i \leq 4} |y_i - y(t_i)|$. Comment again on the smoothness of $f(x)$. Can you say something about the uniqueness of the minimum?

(10 points)

Problem 5 (Convexity, derivatives). Let $x = \{x_1, x_2\} \in \mathbb{R}^2$ and $f(x) = \|x\|_{l_2}^2 = x_1^2 + x_2^2$. Prove that $f(x)$ is a strictly convex function. Compute the gradient $\nabla f(x)$ at all points x and infer from that where $f(x)$ has a minimum using the necessary condition for minima of convex differentiable functions.

(20 points)

Problem 6 (Convexity). Let $x = \{x_1, x_2\} \in \mathbb{R}^2$ and $f(x) = \|x\|_{l_1} = |x_1| + |x_2|$. Unlike the function in the previous problem, this function is not differentiable everywhere. Plot this function if you have trouble imagining how it might look.

Prove that $f(x)$ is a convex function. **(10 points)**