MATH 651: Numerical Analysis II

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Homework assignment 5 – due 11/15/2019

Problem 1 (Convergence order of ODE solvers). The following two schemes for the approximation of the ODE $x'(t) = f(t, x(t))$ are both Runge-Kutta methods based on the midpoint rule:

- The explicit Runge-Kutta-2 method:

$$\begin{aligned} F_1 &= \Delta t\; f(t_k, x_k) \\ F_2 &= \Delta t\; f\left(t_k + \frac{1}{2}\Delta t, x_k + \frac{1}{2}F_1\right) \\ x_{k+1} &= x_k + F_2. \end{aligned}$$

- The implicit midpoint rule, which can be written as an implicit 1-stage Runge-Kutta method:

$$\begin{aligned} F_1 &= \Delta t\; f\left(t_k + \frac{1}{2}\Delta t, x_k + \frac{1}{2}F_1\right) \\ x_{k+1} &= x_k + F_1. \end{aligned}$$

You can probably guess the convergence order for both of these methods. Prove it rigorously.

(30 points)

Problem 2 (Regularity). We have spent a substantial amount of time talking about "regularity", i.e., about what happens if the solution of an ODE or the right hand side is not sufficient smooth. Let us examine this in practice. To this end, consider the following ODE:

$$\begin{aligned} x'(t) &= x(t) + h(t), \\ x(0) &= 1, \end{aligned}$$

and let us choose

$$h(t) = \begin{cases} 0 & \text{if } t < t^* = \frac{1}{3}, \\ 1 & \text{if } t \geq t^* = \frac{1}{3}. \end{cases}$$

Because the right hand side is discontinuous, the solution has a kink at $t = t^* = \frac{1}{3}$. (Note, however, that the right hand side still satisfies the Lipschitz condition!) It is not difficult to compute the solution of this ODE by noting that before t^*, the solution is simply an exponential and so $x(t^*) = e^{1/3}$. After that, the equation is $x'(t) = x(t) + 1$ with initial condition $x(t^*) = e^{1/3}$, which has the exact solution $x(t) = (1 + e^{-1/3})e^t - 1$.

Now compute how accurate your approximation of $x(1)$ is if you use the explicit Euler method using time steps $h = \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{1024}$. For this choice of time step sizes, none of the discrete time points end up at t^*. Compute a convergence rate from these numbers.

Then repeat this with time step sizes $h = \frac{1}{3}, \frac{1}{2} \cdot \frac{1}{3}, \frac{1}{4} \cdot \frac{1}{3}, \ldots, \frac{1}{512} \cdot \frac{1}{3}$. Now we always have a discrete time point on t^*.

Repeat all of this using the explicit fourth-order Runge-Kutta scheme. **(20 points)**

Problem 3 (More rocketry). Let's explore the world of rocketry some more. Let's say you already have your satellite in orbit around Earth. At time t (measured in seconds), its three-dimensional position (measured in meters) is $\mathbf{x}(t)$. Then Newton tells us that

$$\mathbf{x}''(t) = \frac{\mathbf{F}(\mathbf{x}(t))}{m}, \tag{1}$$

where we here consider the mass as constant. On the other hand, the forces $\mathbf{F}$ are only due to gravity. Namely, if we place the Earth at the origin of the coordinate system, then

$$\mathbf{F}(\mathbf{x}(t)) = -(6371000)^2 \frac{9.81m}{|\mathbf{x}(t)|^2} \frac{\mathbf{x}(t)}{|\mathbf{x}(t)|}. \tag{2}$$

(In other words, the negative sign times the last fraction means that the force in "inward", towards the center of the Earth; the first of the fractions implies that the force decreases with the square of the distance. The constants are chosen so that the magnitude of the acceleration is 9.81 m/s^2 at the surface of the Earth.)

Find online the current altitude and speed of the International Space Station. Then convert this into an initial position vector $\mathbf{x}_0$ and initial velocity $\dot{\mathbf{x}}_0$; you can choose $\mathbf{x}_0$ as any point you want that has the correct altitude (but remember that $\mathbf{x}$ measures distance from the Earth center!) and $\dot{\mathbf{x}}_0$ as a vector that is perpendicular to $\mathbf{x}_0$ (i.e., it is tangential to the Earth surface).

Using this model, the orbit of the International Space Station should be roughly circular. In practice, the orbit will be a bit elliptic because you may have gotten the altitude or velocity slightly wrong by a few percent. Either way, some elementary physical considerations show that the orbit must be closed, i.e., that the orbiting object returns to the same location with the same velocity after some time, and then keeps doing this over and over again. However, numerical approximations may not satisfy this property.

To test this assertion, run each of the following methods on this model:

- the explicit Euler method,
- the implicit Euler method,
- the trapezoidal (Crank-Nicolson) method,
- the Runge-Kutta method of order 4.

The implicit methods will require you to solve a nonlinear system of equations.

For each of these methods, consider the following tasks:

a) Using a time step $\Delta t = 500$ seconds, compute a few orbits of the space station and plot the resulting trajectory. What do you observe? Are the orbits closed? Illustrate your answer with plots. What happens if you make the time step smaller?

b) For each of the four methods, compute the position after 100 orbits (this should be approximately or slightly more than 6 days, i.e., 540,000 seconds). Compute $|\mathbf{x}(T_{100\text{ orbits}}) - \mathbf{x}_N)|$ where $\mathbf{x}_N$ is the numerical approximation after a number of time steps that corresponds to an end time of $T_{100\text{ orbits}}$; furthermore, because the exact orbits are closed, the exact solution will satisfy $\mathbf{x}(T_{100\text{ orbits}}) = \mathbf{x}_0$. In other words, we want to know how accurately a method predicts a closed orbit.

   Do this for time step sizes Δt equal to $\frac{T_{\text{orbit}}}{10}$, $\frac{T_{\text{orbit}}}{100}$, $\frac{T_{\text{orbit}}}{1000}$, $\frac{T_{\text{orbit}}}{10000}$, and $\frac{T_{\text{orbit}}}{100000}$ (where, obviously, T_{orbit} is the time for one orbit) and determine the convergence rate of each method.

   Note: To answer this question, you will have to think a bit about the exact value of $T_{100\text{ orbits}}$ (or, alternatively T_{orbit}). This value can be computed exactly, or you can get good approximations by just running your most accurate ODE solver for small time steps and extracting $T_{100\text{ orbits}}$ from its results; on the other hand, looking the value up online will not be useful because you will not find it with sufficient accuracy, and it will not exactly match the value that corresponds to your initial conditions. In any case, explain how you chose $T_{100\text{ orbits}}$.

(50 points)