MATH 651: Numerical Analysis II

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Homework assignment 4 – due 10/25/2019

Problem 1 (Monte Carlo integration). Monte Carlo integration is a "stochastic" method to integrate a function. It approximates the integral

$$\int_a^b f(x)\,\pi(x)\,dx \approx \frac{1}{N}\sum_{i=1}^N f(x_i)$$

where the points x_i are randomly distributed on the interval $[a,b]$ with a probability distribution $\pi(x)$ that we assume to be normalized (i.e., $\int_a^b \pi(x)\,dx = 1$). That this representation makes some sense is clear if you consider the case where $\pi = \text{const} = \frac{1}{b-a}$: In that case, the left side is simply the average value of f on the interval, and the right hand side is the interval of f over all of the point evaluations x_i.

The discussion in class made clear that Monte Carlo integration is only a win if we are in higher dimensions $d \geq 4$, but that shouldn't stop us from trying this method out even in 1d.

Use again the case where we are interested in $f(x) = \sin\frac{1}{x}$, $a = 0.05$, and $b = 1$. For an increasing number N of randomly drawn points x_i in $[a,b]$, determine how accurate the approximation is, and generate a plot like you did in previous homework. **(10 points)**

Problem 2 (Numerical differentiation). We proved in class that the symmetric two-point finite difference operator

$$D_h^{\pm} f(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h}$$

converges to the exact derivative $f'(x_0)$ like

$$|f'(x_0) - D_h^{\pm} f(x_0)| \leq Ch^2$$

as $h \to 0$.

One may ask if it is possible to get better convergence orders if one is willing to use more function evaluations. For example, one could postulate an operator

$$D_h^{[\alpha_{-1},\alpha_0,\alpha_1]} f(x_0) = \frac{\alpha_{-1} f(x_0-h) + \alpha_0 f(x_0) + \alpha_1 f(x_0+h)}{\beta h}$$

and then hope that one can choose $\alpha_{-1}, \alpha_0, \alpha_1$ and β in such a way that

$$|f'(x_0) - D_h^{[\alpha_{-1},\alpha_0,\alpha_1]} f(x_0)| \leq Ch^3$$

as $h \to 0$, or possibly of even higher order.

(a) Relatively simple considerations show that $D_h^{[\alpha_{-1},\alpha_0,\alpha_1]}$ can only be a reasonable operator if $\alpha_{-1} + \alpha_0 + \alpha_1 = 0$ and $-\alpha_{-1} + \alpha_1 = \beta$. Show why these equations have to hold by considering what happens if you apply the operator to a function f that is (i) constant, and (ii) linear.

(b) Having so reduced the number of unknowns to two (for example $\alpha_{\pm 1}$), see if you can choose these leftover unknowns so that you get at least third order convergence. To this end, you will have to do Taylor expansions as we did in class and see whether you can choose the unknowns in such a way that as many Taylor terms as possible cancel. Alternatively, you could ask that the operator is exact when applied to functions that are constant, linear, quadratic, ..., until you have enough equations to determine all coefficients. You may assume that $f \in C^\infty$. Show whether or not you can indeed get third order accuracy by using three function evaluations as above.

(c) Repeat this kind of exercise for a finite difference operator

$$D_h^{[\alpha_{-2},\alpha_{-1},\alpha_0,\alpha_1,\alpha_2]} f(x_0) = \frac{\alpha_{-2} f(x_0-2h) + \alpha_{-1} f(x_0-h) + \alpha_0 f(x_0) + \alpha_1 f(x_0+h) + \alpha_2 f(x_0+2h)}{\beta h}$$

and see what kind of convergence order you can achieve by choosing the coefficients in appropriate ways.

(d) Using the function $f(x) = e^x$ and $x_0 = 0$, demonstrate numerically that the theoretical convergence order you have derived above indeed holds for both the three-point and the five-point stencil in practice as well.

(25 points)

Problem 3 (Numerical differentiation). As part of the previous homework assignment, you have found a root of the function $f(x) = xe^x - 1$. (As a complete aside, this means that you computed $W(1)$ where $W(x)$ is the *Lambert W function*.)

Let us repeat a similar exercise except that we now want to find the *minimum* of that function. The exact minimum lies at $x^* = -1$, as can easily be verified. To find it numerically, one would solve the problem $g(x) = f'(x) = 0$ for which we can use a Newton method that computes $x_{k+1} = x_k - \frac{g(x_k)}{g'(x_k)}$. Here, this equates to iterating $x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)}$.

(a) Use Newton's method to compute the root x^* to at least six digits of accuracy, starting with $x_0 = -0.5$ and using exact derivatives of f.

(b) Let us pretend that $f(x)$ was an expensive function for which we do not have a formula, and that we can only compute f' and f'' in Newton's method by a finite difference approximation using the symmetric first and second derivative operators. Using these, compute approximate roots $x^{h,*}$ for a variety of values ϵ to at least six digits of accuracy. Demonstrate numerically that $x^{h,*} \to x^*$ as $h \to 0$ and plot the error $|x^{h,*} - x^*|$ as a function of h for a number of values $h = 1, \ldots, 10^{-6}$.

(20 points)

Problem 4 (Numerical solution of a ODE). Consider the following scalar ordinary differential equation (ODE):

$$x'(t) = \frac{1}{2}x(t), \qquad x(0) = 1.$$

The solution of this equation is $x(t) = e^{\frac{1}{2}t}$. Compute approximations to $x(4)$ using the

- first order Taylor expansion method,
- second order Taylor expansion method,
- implicit Euler method,
- trapezoidal (Crank-Nicolson) method,

each with step sizes $\Delta t = 2, 1, \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{32}$. Compute their respective errors $e = |x_N - x(4)|$ where x_N is the approximation to $x(4)$ at the end of the last time step, and compute the convergence rates. Compare the accuracy of all these methods for the same step size Δt. **(15 points)**

Problem 5 (Numerical solution of a second-order ODE). A rocket that is shot up vertically experiences upward acceleration from its engines, and downward acceleration due to gravity. Its height therefore satisfies Newton's law

$$d''(t) = \frac{F(t)}{m(t)}, \tag{1}$$

where $d(t)$ denotes the distance from the earth's center. Assume that the rocket is initially at rest at $d(0) = 6371000$. After ignition, the engines produce a constant thrust for 10 minutes before shutting down:

$$T(t) = \begin{cases} 12 & \text{for } t < 600, \\ 0 & \text{for } t \geq 600. \end{cases}$$

On the other hand, gravity generates the force

$$G(t) = -(6371000)^2 \frac{10m(t)}{d(t)^2}.$$

(The factors here are chosen in such a way that at the surface – i.e., at $d = 6371000$ meters from the center of the Earth – the gravity equals 10 meters per second square, i.e., approximately the correct value. Furthermore, as is indeed the case, gravity decreases with the square of the distance.) The total force is then $F(t) = T(t) + G(t)$. The mass of the rocket decreases while fuel is burnt in the engines according to

$$m(t) = \begin{cases} 1 - \frac{0.9t}{600} & \text{for } t < 600, \\ 0.1 & \text{for } t \geq 600. \end{cases}$$

Rewrite this second order ordinary differential equation as a system of two first order equations. Then numerically approximate the altitude of the rocket for times between $t = 0$ and $t = 36000$ using the explicit Euler method. Try to determine the altitude at $t = 36000$ up to an accuracy of 1000 meters. **(30 points)**